UNIVERSITY OF THE PUNJAB

Roll No. ....

M.A./M.Sc. Part – I Annual Exam – 2019

Subject: Mathematics (Old & New Course) **Paper: I (Real Analysis)** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.

SECTION I

1. (a) Suppose S is a non-empty set of real numbers which is bounded below and let $\alpha > 0$, then prove that (10 marks)

$$Inf(\alpha S) = \alpha Inf(S).$$

(b) Prove that if $x > -1$, then $(1+x)^n \geq 1 + nx$. (10 marks)

2. (a) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$. (10 marks)

(b) A sequence of real numbers is convergent if and only if it is a Cauchy sequence. (10 marks)

3. (a) Prove the following by applying the definition of limit of a function, (10 marks)

$$Lim_{x\to\infty} \left(\frac{(-1)^x x}{x^2+1} \right) = 0.$$

Also find the value of the corresponding δ.

(b) Show that if a function $f : (a,b) \to \mathbb{R}$ is uniformly continuous, then we can extend it to a function $\tilde{f}$ that is also uniformly continuous on $[a,b]$. (10 marks)

4. (a) Show that every continuous function on a closed and bounded interval $I \subset \mathbb{R}$ is uniformly continuous. (10 marks)

(b) Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$, then f is bounded on some neighborhood of c. (10 marks)

5. (a) Let $I \subseteq R$ be an interval and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I, then the function g inverse to f is strictly monotone and continuous on $J := f(I)$. (10 marks)

(b) Let $f : [0,\infty) \to \mathbb{R}$ is differentiable everywhere. Assume that

$$lim_{x\to\infty} f(x) + f'(x) = 0.$$

Show that $lim_{x\to\infty} f(x) = 0$.. . (10 marks)

P.T.O.

SECTION II

6. (a) If Suppose f is a bounded function on $[a, b]$, then show that to every$\epsilon > 0$ there corresponds $\delta > 0$ such that (10 marks)

$$U(P, f) < \int_a^b f dx + \epsilon.$$

(b) Give an example to illustrate that all of the hypotheses in Dini's Theorem are essential. (10 marks)

7. (a) Consider the sequence $\{f_n\}$ defined by $f_n(x) = \frac{1}{1+x^n}$, for $x \in [0, 1]$. Find $f(x) = lim_{n\to\infty} f_n(x)$. Show that for $0 < a < 1$, $\{f_n\}$ converges uniformly to f on $[0, a]$. Show that $\{f_n\}$ does not converge uniformly to f on $[0, 1]$. (10 marks)

(b) Show that the function

$$f(x) = \begin{cases} n^2x, & 0 \leq x \leq \frac{1}{n}, \\ -n^2x + 2n, & \frac{1}{n} \leq x \leq \frac{2}{n}, \\ 0, & \frac{2}{n} \leq x \leq 1. \end{cases}$$

8. (a) Suppose ϕ is a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$. Let α is a monotonically increasing function on $[a, b]$ and $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by $\beta(y) = \alpha(\phi(y))$, $\quad g(y) = f(\phi(y))$. Then $g \in \mathcal{R}(\beta)$ and (10 marks)

$$\int_A^B g d\beta = \int_a^b f d\alpha$$

(b) Test the convergence of $\int_0^\infty \frac{dx}{1+x^4 \sin^2 x}$. (10 marks)

9. (a) Prove that a function f is of bounded variations can be expressed as a difference of two monotone increasing functions. (10 marks)

(b) The function f is defined on $]0, 1]$ by

$$f(x) = (-1)^{n+1} n(n+1), \frac{1}{n+1} \leq x < \frac{1}{n}, n \in \mathbb{N}.$$

Show that $\int_0^1 f(x)dx$ does not converge. (10 marks)

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Subject: Mathematics (Old & New Course) **Paper: II (Algebra)** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.

SECTION-I

Q. 1 .... *20 marks*

(a) Prove that center of a group G is normal in G. [10]

(b) Find normal subgroups of the Quaternion group Q_8. [10]

Q. 2 .... *20 marks*

(a) Show that $Aut(C_5) \cong C_4$. [10]

(b) Let H be a normal subgroup of a group G. Then $\frac{G}{H}$ is abelian if and only if $G' \subseteq H$. [10]

Q. 3 .... *20 marks*

(a) Let G be a group. Then show that [10]

$$\frac{G}{Z(G)} \cong Inn(G),$$

where $Z(G)$ denotes center of G and $Inn(G)$ denotes the group of inner automorphisms of G.

(b) Let D_3 be the Dihedral group of order 6. Then [10]

(i) List all elements of D_3.

(ii) Find the order of each element of D_3.

(iii) List all subgroups of D_3.

Q. 4 .... *20 marks*

(a) Let H be a normal subgroup of a group G and K a subgroup of G. Then show that [10]

$$\frac{HK}{H} \cong \frac{K}{H \cap K}.$$

(b) Define a characteristic subgroup of a group G. Prove that center of a group G is a characteristic subgroup of G. [10]

Q. 5 .... *20 marks*

(a) Show that the number n_p of sylow p-subgroups of a group G is of the form $1 + kp$, $(k = 0, 1, 2, 3, \cdots)$ and is a divisor of the order of G. [10]

(b) Show that a group of order 15 is cyclic. [10]

P.T.O.

SECTION-II

Q. 6. .... *20 marks*

(a) Let R be a commutative ring with unity and I be an ideal of R. Prove that R/I is a field if and only if I is a maximal ideal in R. [10]

(b) Let $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ be a commutative ring with unity and $I = \{0, 3\}$ be an ideal of $\mathbb{Z}_6$. Find $\frac{\mathbb{Z}_6}{I}$, and show that [10]

$$\frac{\mathbb{Z}_6}{I} \cong \mathbb{Z}_3.$$

Q. 7. .... *20 marks*

(a) If I and J are ideals of a ring R, then prove that IJ is an ideal of R. [10]

(b) Find all the maximal and prime ideals of $\mathbb{Z}_{12}$. [10]

Q. 8. .... *20 marks*

(a) State and prove Caley-Hamilton theorem. [10]

(b) Let V be a finite dimensional vector space over a field F and $T : V \to V$ be a homomorphism. Then prove the Rank-Nullity therorem: [10]

$$\text{Dim}(V) = \text{Rank}(T) + \text{Nullity}(T).$$

Q. 9. .... *20 marks*

(a) Find a real orthogonal matrix P, if possible, for which $P^{-1}AP$ is diagonal, where [10]

$$A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}.$$

(b) If $\text{Dim}(U) = m$ and $\text{Dim}(W) = n$ then show that $\text{Dim}(Hom(U, W) = mn$. [10]

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M.A./M.Sc. Part – I Annual Exam – 2019

Roll No. ....

Subject: Mathematics (Old & New Course)

Paper: III (Complex Analysis and Differential Geometry)

Time: 3 Hrs. Marks: 100

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.

SECTION I

1. (A) Find all z such that $sinhz = -i$.

   (B) For any complex numbers z_1, z_2 and $z = x+iy$, prove that (a) $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$ and (b) $(|x| + |y|)/\sqrt{2} \leq |z| \leq |x| + |y|$.

   [10+10=20]

2. (A) Prove that an entire and bounded function is a constant function. Use this statement to prove that $sinz$ is bounded. What about the function $tanz$.

   (B) Find power series expansion of the function $\frac{1}{z^2-z-2}$ in all regions of the plane.

   [10+10=20]

3. (A) Prove that $\lim\limits_{z\to 0} (\bar{z}/z)$ does not exist. Does the $\lim\limits_{z\to 0} (z/\bar{z})$ exists.

   (B) State an prove fundamental theorem of algebra and show that the function $f(z) = z^4+z^2+1$ has exactly four zeros.

   [10+10=20]

4. (A) What is an analytic function? How is it related to Cauchy-Riemann equations? Prove that the Cauchy-Riemann equations can be written in the polar coordinates as $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

   (B) Let C be the circle $|z| = 1$. Then prove $\left|\int_C (x^2 + iy^2)dz\right| \leq 2\pi$.

   [10+10=20]

5. (A) Evaluate the integral $\int_0^{\pi} \frac{1}{5+3\cos\theta}d\theta$.

   (B) Discuss nature of the singular points of the function $\frac{z^2+z^4}{sin\pi z}$ and compute residues only at poles.

   [10+10=20]

SECTION II

6. (A) Find the arc-length s as a function of ϑ along the epicycloid given by $x(\vartheta) = (r_0 + r_1)\cos\vartheta - r_1\cos\left(\frac{r_0+r_1}{r_1}\vartheta\right)$, $y(\vartheta) = (r_0 + r_1)\sin\vartheta - r_1\sin\left(\frac{r_0+r_1}{r_1}\vartheta\right)$ and determine its intrinsic equations.

   (B) Prove that a plane curve is uniquely determined except as the position in plane, when its signed curvature is given as function of its arc length. What happens if we replace signed curvature with curvature.

   [10+10=20]

7. (A) Derive Serret-Franet formulae and give the matrix form. Verify these formulae for the helix $\mathrm{x}(u) = (a\cos u, a\sin u, bu)$.

   (B) A number κ is a principal curvature if and only if κ is a solution of the equation $(EG - F^2)\kappa^2 - (EN - 2FM + GL)\kappa + (LN - M^2) = 0$, where E, F, G and L, M, N are first and second fundamental coefficients.

   [10+10=20]

8. (A) Find the first and second fundamental form for the Monge's patch $\mathrm{x}(u, v) = (u, v, h(u, v))$. Show that the surface area on the Monge's patch is given by $A(\mathrm{x}(u, v)) = \iint\limits_R \sqrt{1 + h_u^2 + h_v^2}$

   (B) Verify Codazzi-Mainardi equations for the Monge's patch $\mathrm{x}(u, v) = (u, v, h(u, v))$.

   [10+10=20]

9. (A) Prove that circular cylinder given by a level surface $\{(x, y, z)|x^2 + y^2 = 1\}$ is a smooth surface consisting of an atlas of two charts. Find this atlas.

   (B) Derive Gauss-Weingarten equations and verify these equations for the surface $\mathrm{x}(u, \theta) = (u\cos\theta, u\sin\theta, g(u))$.

   [10+10]

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Subject: Mathematics (Old & New Course) **Paper: IV (Mechanics)** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.

SECTION I

1. (a) If $\vec{A} = (3x+y)\hat{i} - x\hat{j} + (y-2)\hat{k}$ and $\vec{B} = 2\hat{i} - 3\hat{j} + \hat{k}$, evaluate $\oint_C (\vec{A} \times \vec{B}) \times d\vec{r}$ around the circle in the $xy-$plane having center at the origin and radius 2 traversed in the positive direction. (10 marks)

   (b) Evaluate $\iint_S \vec{A} \cdot \hat{n}\, dS$ over the entire surface S of the region bounded by the cylinder $x^2 + z^2 = 9$, $x = 0$, $y = 0$, $z = 0$ and $y = 8$, if $\vec{A} = 6z\hat{i} + (2x+y)\hat{j} - x\hat{k}$. (10 marks)

2. (a) Verify the divergence theorem for $\vec{A} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$. (10 marks)

   (b) Evaluate $\iint_S \vec{r} \cdot \hat{n}\, dS$, where S is the surface of the cube bounded by $x = -1$, $y = -1$, $z = -1$, $x = 1$, $y = 1$, $z = 1$. (10 marks)

3. (a) Derive an expression for $\nabla\Phi$ in orthogonal curvilinear coordinates. (10 marks)

   (b) Prove that a spherical coordinate system is orthogonal. (10 marks)

4. (a) Evaluate $\iiint_V \sqrt{x^2+y^2}\, dxdydz$, where V is the region bounded by $z = x^2 + y^2$ and $z = 8 - (x^2 + y^2)$. (10 marks)

   (b) A quantity $A(p,q,r)$ is such that in the coordinate system x^i, $A(p,q,r)B_r^{qs} = C_p^s$, where B_r^{qs} is an arbitrary tensor and C_p^s is a tensor. Prove that $A(p,q,r)$ is a tensor. (10 marks)

5. (a) Derive the transformation law for the Christoffel symbols of the first kind. (10 marks)

   (b) Find the contravariant components of a tensor in spherical coordinates if its covariant components in rectangular coordinates are $2x - z$, x^2y, yz. (10 marks)

SECTION II

6. (a) An xyz coordinate system rotates with angular velocity $\vec{\omega} = \cos t\hat{i} + \sin t\hat{j} + \hat{k}$ with respect to a fixed XYZ coordinate system having the same origin. If the position vector of a particle is given by $\vec{r} = \sin t\hat{i} - \cos t\hat{j} + t\hat{k}$, find the apparent velocity and the true velocity at any time t. (10 marks)

   (b) Prove that the total angular momentum of a system of particles about any point O equals the angular momentum of the total mass assumed to be located at the center of mass plus the angular momentum about the center of mass. (10 marks)

7. (a) Find the center of mass of a uniform solid hemisphere of radius a. (10 marks)

   (b) The instantaneous velocities of particle at points $(a,0,0)$, $(0,a/\sqrt{3},0)$, $(0,0,2a)$ of a rigid body are $(u,0,0)$, $(u,0,v)$, $(u+v,-\sqrt{3}v,v/2)$ respectively with respect to a rectangular coordinate system. Find the magnitude and direction of spin of the body and the point at which the central axis cuts the $XZ-$plane. (10 marks)

8. (a) Work out the principal moments of inertia at the center of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. (10 marks)

   (b) A rigid body which is symmetric about an axis has one point fixed on this axis. Discuss the rotational motion of the body, assuming that there are no forces acting other than the reaction force at the fixed point. Also calculate the precession frequency in case of the earth rotating about its axis. (10 marks)

9. (a) Find the moment of inertia of a rectangular plate of sides a and b about a diagonal. (10 marks)

   (b) Work out the relationship between the time rate of change of angular momentum of a rigid body relative to axes fixed in space and in the body respectively. Also, using the principle of angular momentum, formulate the Euler's equations of motion. (10 marks)

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M.A./M.Sc. Part – I Annual Exam – 2019

Roll No. ....

Subject: Mathematics (Old & New Course)
Paper: V (Topology and Functional Analysis)

Time: 3 Hrs. **Marks: 100**

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.

SECTION-I

Q.1	(a)	Let A be a subset of a topological space (X, τ) then Prove that : (i) $(A^0)^c = \overline{A^c}$ (ii) $\bar{A} = A \cup F_r(A)$ (iii) $A^0 = A \backslash F_r(A)$.	(10)
	(b)	Prove that the set of rational numbers $\mathbb{Q}$, as a subspace of $\mathbb{R}$, does not have discrete topology.	(10)
Q.2	(a)	Prove that a function $f: X \to Y$, is continuous on X if and only if for any subset A of X, $f(\bar{A}) \subseteq \overline{f(A)}$.	(10)
	(b)	Show that the mapping $(-1, 1) \cong \mathbb{R}$.	(10)
Q.3	(a)	Prove that every metric space is a Tychonoff space.	(10)
	(b)	Prove that $A = \{(x,y) \epsilon \mathbb{R}^2 : x^2 + y^2 = 1\}$ is compact but $B = \{(x,y) \epsilon \mathbb{R}^2 : y^2 = x\}$ is not compact in $\mathbb{R}^2$, with respect to the usual topology.	(10)
Q.4	(a)	A space X is a T_0 –space if and only if, for any a,b in X, $a \neq b$ implies $\overline{\{a\}} \neq \overline{\{b\}}$.	(10)
	(b)	Let X be an infinite set with co-finite topology τ on X. Then show that (i) (X, τ) is compact (ii) (X, τ) is connected.	(10)

SECTION-II

Q.5	(a)	Show that the space l^∞ with respect to the norm defined by $\|x\| = \sup_{i=1}^{\infty} \|x_i\|$, is a Banach space.	(10)
	(b)	Define the distance from a point x to a subset M of (X, d) and show that $\|d(x,M) - d(y,M)\| \leq d(x,y)$. Also show that the function $f: X \to \mathbb{R}$, defined by $f(x) = d(x,M)$, is uniformly continuous.	(10)
Q.6	(a)	Give an example of Cauchy sequence of real valued continuous functions defined on a [-1,1] that converges to a discontinuous function.	(10)
	(b)	(i) Prove that the closure $\bar{C}$ of a convex subset C of a normed space X, is a convex set. (ii) Find the norm of the linear functional $f(x) = \int_{-1}^{0} x(t)dt - \int_{0}^{1} x(t)dt$ on $C[-1,1]$ under the integral norm.	(10)
Q.7	(a)	Prove that the dual space of norm space l^3 is isomorphic to $l^{\frac{3}{2}}$.	(10)
	(b)	If the closed unit ball $\bar{B}(0,1) = \{x \in X : \|\|x\|\| \leq 1\}$ in a normed space X, is compact then prove that X has a finite dimension.	(10)
Q.8	(a)	Prove that the normed space l^5 is not separable.	(10)
	(b)	For any $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ define $f_a: \mathbb{R}^n \to \mathbb{R}$ by $f_a(x) = \sum_{i=1}^{n} a_i x_i$, $x \in \mathbb{R}$ then prove that (i) f_a is linear functional (ii) f_a is bounded (ii) $\|f_a\| = \|a\|$.	(10)
Q.9	(a)	State and prove parallelogram law in a normed space X. Give example of two continuous functions defined on a closed interval that do not satisfy parallelogram law.	(10)
	(b)	Let M be a convex subset of a Hilbert space H, and $\{x_n\}$ be a sequence of M such that $\|x_n\| \to \inf_{x \in M} \|x\|$. Show that $\{x_n\}$ converges in H. Give an illustrative example in $\mathbb{R}^n$.	(10)