UNIVERSITY OF THE PUNJAB

Part-II A/2018
Examination:- M.A./M.Sc.

Roll No. ....

Subject: Mathematics
PAPER: I (Advanced Analysis)

TIME ALLOWED: 3 hrs.
MAX. MARKS: 100

NOTE: Attempt FIVE questions in all selecting at least TWO questions from each section.

Section-I

1. (a) Show that the interval $[0, 1]$ is an uncountable set. (10)

   (b) Let $s(\lambda)$ be the set of ordinals less than the ordinal λ. Then show that $\lambda = ord(s(\lambda))$. (10)

2. (a) Let A be a denumerable subset of an uncountable set B, then show that removing A from B does not change the cardinality of B. (10)

   (b) Let A and B be well-ordered sets. Then define anti-lexicographical ordering for $A \times B$. Also, Show that $\omega 2 = \omega + \omega$ but $2\omega = \omega \neq \omega 2$, where ω denotes ordinal number of N. (10)

3. (a) State Zorn's lemma. Let X be a partially ordered set. Then show that X contains a maximal chain. (10)

   (b) Prove that every element of a well-ordered set has a unique immediate successor except the last element. (10)

4. (a) Let A be a well-ordered set and $B \subseteq A$. Let $f : A \to B$ be a similarity mapping of A onto B. Then for every $a \in A$. $a \leq f(a)$. (10)

   (b) Let A and T be the sets of algebraic and transcendental numbers, respectively. Then show that A is denumerable and T is uncountable. (10)

Section-II

5. (a) Define measurable set and show that the interval $\{x \in \mathbb{R} : -\infty < x \leq a\}$ is measurable. (10)

   (b) Let E be a measurable set, $1 \leq p < \infty$ and q the conjugate of p. If the function $f \in L^p(E)$ and $g \in L^q(E)$, then show that $\int_E |f.g| \leq \|f\|_p \|g\|_q$. Moreover if $f \neq 0$, then $\int_E f.f^* = \|f\|_p$ and $\|f^*\|_q = 1$, where $f^* = \|f\|_p^{(1-p)} sign(f)|f|^{(p-1)}$ and $sign(f) = \frac{|f|}{f}$. (10)

6. (a) Let the function f be defined on a measurable set E. Then f is measurable if and only if for each open set O, the inverse image of O under f, $f^{-1}(O) = \{x \in E : f(x) \in O\}$ is measurable. (10)

   (b) Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E of finite measure. Let f be a measurable real valued function such that $f_n(x) \to f(x)$ for each $x \in E$. Then, given $\epsilon > 0$ and $\sigma > 0$, there is a measurable set $A \subset E$ with measure $m(A) < \sigma$ and a natural number p such that for all $x \notin A$ and all $n \geq p$, $|f_n(x) - f(x)| < \epsilon$. (10)

PTO

7. (a) If E_1 and E_2 are measurable sets, then show that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

(10)

(b) Let f be a bounded function defined on a closed, bounded interval $[a, b]$. If f is Riemann integrable over $[a, b]$, then show that it is also Lebesgue integrable over $[a, b]$ and the two integrals are equal. (10)

8. (a) Let A be any subset of $\mathbb{R}$ and $\{E_i\}_{i=1}^{\infty}$ be a countable collection of pairwise disjoint measurable subsets of $\mathbb{R}$. Then show that

$$m^*\left(A \bigcap \left(\bigcup_{i=1}^{\infty} E_i\right)\right) = \sum_{i=1}^{\infty} m^*\left(A \bigcap E_i\right).$$

(10)

(b) Show that the function $f = \frac{1}{\sqrt[3]{x}}$ belongs to $L^1([0, 64])$ but does not belong to $L^3([0, 64])$. (10)

9. (a) State and prove Bounded Convergence Theorem. (10)

(b) Define Cantor set and show that it is a measurable set with measure zero. (10)

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Part-II A/2018
Examination:- M.A./M.Sc.

Roll No. ....

Subject: Mathematics
PAPER: II (Methods of Mathematical Physics)

TIME ALLOWED: 3 hrs.
MAX. MARKS: 100

NOTE: Attempt FIVE questions in all, selecting at least TWO questions from each section.

Q. No		Marks
	Section-I	
1(a)	Discuss the Lagrange's method of solution for first order quasilinear PDEs.	10
1(b)	Find the integral surface of the PDE $yz_x + xz_y = z$ with the initial data $z(x,0) = x^3, \quad z(0,y) = y^3.$	10
2(a)	Define Hypergeometric functions. Prove that $F_{21}(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tz)^{-\beta}dt.$	10
2(b)	Find the general solution of the inhomogeneous linear second order PDE $(D^2 - DD' - 2D'^2)z = (y+1)e^x.$	10
3(a)	Determine the eigenvalues and eigenfunctions of the system $\frac{d^2y}{dx^2} + (1+\lambda)y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$ Also check the orthogonality of the eigenfunctions.	10
3(b)	Derive the one dimensional wave equation through transverse vibrations of a stretched string of finite length.	10
4(a)	Find the steady state solution of the equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \quad 0 < x < a, \quad t > 0,$ subject to $u_x(0,t) = 0, \quad u_x(a,t) = 0, \quad t > 0,$ $u(x,0) = f(x), \quad 0 < x < a.$ Discuss the uniqueness of the solution.	10
4(b)	Use method of Frobenius to obtain two linearly independent series solutions about regular singular point $x_0 = 0$ for $2xy'' - y' + 2y = 0$.	10
5(a)	Find the series solution of $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2y = 0$ with center of expansion at $x_0 = 1$.	10

PTO

5(b)	Find the value of $J_{-3/2}(x)$.	10
	Section-II	
6(a)	Find the equation of the path in space down which a particle will fall from one point to another point in the shortest possible time.	10
6(b)	Find the Green's function associated with the boundary value problem $\frac{d}{dx}\left\{(1-x^2)\frac{du}{dx}\right\} - \frac{h^2}{1-x^2}u + \lambda r(x)u = 0,$ with $u(-1)$ and $u(1)$ both are finite.	10
7(a)	Find the Fourier transform of $g(x) = \frac{1}{(x^2+a^2)^2}.$	10
7(b)	Use calculus of variation to prove that a straight line is the shortest distance between two points in a plane.	10
8(a)	Show that the bounded solution of the heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t},$ given $u(0,t) = u_0(a\ constant)$ and $u(x,0) = 0,$ by taking Fourier transform can be written as $u(x,t) = u_0\left(1 - erf\frac{x}{2\sqrt{kt}}\right).$	10
8(b)	Find Fourier transform of $xe^{-\alpha x^2}, \quad \alpha > 0.$	10
9(a)	Verify that $L^{-1}\{\tan^{-1}(a/s)\} = \frac{1}{t}\sin at.$	10
9(b)	Use Laplace transform method to solve the problem $y''' - y'' + 4y' - 4y = 2\cos 2t - \sin 2t,$ $y(0) = 1, \quad y'(0) = 4, \quad y''(0) = 1.$	10

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Part-II A/2018
Examination:- M.A./M.Sc.

Roll No. ....

Subject: Mathematics
PAPER: III (Numerical Analysis)

TIME ALLOWED: 3 hrs.
MAX. MARKS: 100

NOTE: Attempt any FIVE questions, selecting at least TWO questions from each section.

Section I

Q1.

Find $y(0.1), y(0.2), y(0.3)$, from $\frac{dy}{dx} = x^2 + y^2$; $y(0.0) = 0, h = 0.1$ using Runge Kutta method of order two and $y(0.4)$ using Adams-Bashforth method.

[20]

Q2.

Derive the modified Euler's method. Apply modified Euler's method on $\frac{dy}{dx} = x + y, x_0 = 0, y_0 = 1, h = 0.1$ to find $y(0.3)$. Also find the relative error.

[20]

Q3.

a) Write an algorithm for Regula-Falsi Method to find an approximate root of the non-linear equation $f(x) = 0$.
b) Define dominant eigenvalue and eigenvector. Compute the dominant eigenvalue and dominant eigenvector of matrix A :

$$A = \begin{bmatrix} 1 & 2 & --2 \\ -2 & 5 & --2 \\ -6 & 6 & --3 \end{bmatrix}$$

[10+10]

Q4.

a) Find a root correct to three decimal places using Bisection method of the equation $xe^x - 3 = 0$.
b) Find an approximate solution correct to three decimal places using an iterative method of the system of linear equations:

$$3x + 2y + z = 10$$
$$x + 4y - z = 6$$
$$x + 2y + 5z = 20$$

(10+10)

Q5.

a) Use the secant method to find correct to four decimal places the root between 0.4 and 0.6 of the equation $sinx = 5x - 2$.
b) Write an algorithm for Runge Kutta method of order four for the solution of an initial value problem.

[10+10]

Section II

Q6.

Derive Boole's Rule. Apply nine points Boole's Rule to evaluate $\int_0^\pi \frac{sinx}{1+x} dx$

[20]

Q7.

a) Evaluate $\int_{0.1}^{0.7} tan^{-1}(x)dx$ using seven point 3/8th Simpson's rule
b) Write an algorithm for Trapezoidal Rule to approximate the integral of $f(x)$over the interval $[a, b]$ using n subintervals.

[10+10]

Q8.

Derive five points Lagrange's interpolation formula. Find a polynomial of degree 3 or less, such that

$f(0) = 3,\quad f(1) = -1,\quad f(3) = -1,\quad f(4) = -3$

[20]

Q9.

Solve
$y_{n+4} - A^n y_n = 0$, A is an integer.
$y_{t+2} - 9y_{t+1} + 20y_t = 3^t(t^2 + 1)$

[10+10]

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Part-II A/2018
Examination:- M.A./M.Sc.

Roll No. ....

Subject: Mathematics
PAPER: IV-VI (opt.vi) [Fluid Mechanics]

TIME ALLOWED: 3 hrs.
MAX. MARKS: 100

NOTE: Attempt any FIVE questions in all, selecting at least TWO questions from each section.

Section I

Q.1(a)	Define the followings: Dynamic viscosity, Kinematic viscosity, Compressible fluid, Ideal fluid and Newton's law of viscosity.	(10)
(b)	A flat plate having dimensions of $2m \times 2m$ slides down an inclined plane at an angle of one radian to the horizontal at a speed of 6 m/s. The inclined plane is lubricated by a thin film of oil having a viscosity of 30×10^{-3} $Pa.s$. The plate has a uniform thickness of 20 mm and a density of 40,000 kg/m^3. Determine the thickness of lubricating oil film.	(10)
Q.2(a)	What is material derivative? Explain its both parts in detail.	(10)
(b)	The velocity field for a certain fluid flow in cylindrical is given by $(r^2 z\cos\theta, rz\sin\theta, z^2 t)$. Is this flow steady or unsteady? Determine the acceleration components.	(10)
Q.3(a)	Discuss the flow when a fluid is moving subject to external impulsive forces.	(10)
(b)	Derive the Euler's equation of motion and write the same in cylindrical form.	(10)
Q.4(a)	The velocity potential for certain two-dimensional incompressible fluid flow is $\phi = A\tan^{-1}\left(\frac{y}{x}\right)$, find the velocity components, speed, stagnation points, stream function.	(10)
(b)	Define equipotential lines and streamlines. Show that the equipotential lines are orthogonal to the streamlines.	(10)
Q.5(a)	Define two dimensional vortex, discuss the velocity potential, stream function and complex velocity potential for a two dimensional vortex.	(10)
(b)	Find the Cartesian equation of the streamlines when the fluid is streaming from three equal sources situated at the corners of an equilateral triangle.	(10)

Section II

Q.6(a)	State and prove Milne-Thomson circle theorem.	(10)
(b)	Discuss the flow past a circular cylinder with circulation Γ and show that the direction of the lift is upward if the cylinder rotates in the clockwise direction.	(10)
Q.7(a)	Describe Karman' s vortex street and evaluate the expression for the velocities with which the both rows of vortices advance	(10)
(b)	The velocity distribution for the steady, laminar and fully developed flow between two fixed parallel plates is given by $u = -\frac{1}{2\mu}\frac{dp}{dx}(hy - y^2)$, *where* $\frac{dp}{dx}$ is the pressure gradient, h is the gap width between the plates and y is the distance measured upward from the lower plate. Determine the volumetric flow rate, maximum velocity and the shearing stress at both the plates.	(10)
Q.8(a)	Show that the velocity distribution for the Hagen-Poiseuille flow depends upon the external pressure for its existence.	(10)
(b)	What is the Stokes' first problem? Write its mathematical formulation and solve it for the velocity field.	(10)
Q.9(a)	State and prove Blausius Theorem	(10)
(b)	Discuss the pulsating flow of viscous fluid between two parallel plates when both the plates are rest and $\frac{\partial p}{\partial x} = P_x \cos\omega t$: Find the expression for the velocity field	(10)

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Part-II A/2018
Examination:- M.A./M.Sc.

Roll No. ....

Subject: Mathematics **TIME ALLOWED: 3 hrs.**
PAPER: IV-VI (Opt. ii) [Computer Applications] **MAX. MARKS: 50**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

Section 1

1. **[5+5 marks]** What are the outputs of the following programs?

a)
```
Program Alpha
Real:: u=2.5, v=5.9
u=u+v
v=u-v
u=u-v
Print*, u
Print*, v
```

b)
```
Program Mat
Integer, dimension (2,2)::M, a, b
Do a = 1,2
Do b = 1,2
M(a,b) = a*a + b*b -2*a*b
Print*, M(a,b)
End Do
End Do
```

2. (a) **[5 marks]** What is the difference between function subprogram and subroutine subprogram? Give examples.
   (b) **[5 marks]** What are different forms of Do loop structure? Give examples.

3. (a) **[5 marks]** Write a Fortran 90 program to find octal equivalent of a decimal number.
   (b) **[5 marks]** Write a Fortran 90 program to find whether three point (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear or not.

4. (a) **[5 marks]** Write a Fortran 90 program to find transpose of a given 2×2 matrix.
   (b) **[5 marks]** Write a function subprogram for $f(x)$ which is defined as

$$f(x) = \begin{cases} x^2 - 3x, & x < 1, \\ 0, & x = 1, \\ x - 2, & x > 1. \end{cases}$$

Section 2

5. **[10 marks]** Write a Fortran 90 program to find Lagrange interpolating polynomial for the the following data $(1, e), (2, e^2), (3, e^3), (4, e^4)$.

6. **[10 marks]** Write a Fortran 90 program to implement improved Euler formula to solve the initial value problem $y' = \frac{y}{x-2y}$ $y(1) = 5$.

7. **[10 marks]** Write a Fortran 90 program to find the roots of $x - ln(x) = 0$ using Regula Falsi method.

8. **[10 marks]** Write a Fortran 90 program to implement Simpson's $\frac{1}{3}$ formula to evaluate the integral

$$\int_4^5 x ln(x) dx$$

9. Write the **MATHEMATICA** statements for the following.
   (a) **[2 marks]** Find the conjugate of $\sqrt{x} - iy^2$.
   (b) **[2 marks]** Draw graph of $x^2 Tan(x)$, $0 \leq x \leq \pi$. Draw frame as well.
   (c) **[2 marks]** Evaluate $\int_2^3 \int_1^x e^{x+y} dy dx$.
   (d) **[2 marks]** Expand Sin(x+iy) assuming that x and y are complex.
   (e) **[2 marks]** Solve the differential equation $y' = 2xy$, $y(0) = 1$.

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Part-II A/2018

Examination:- M.A./M.Sc.

Roll No. ....

Subject: Mathematics **TIME ALLOWED: 3 hrs.**

PAPER: IV-VI (Opt. iii) [Group Theory] **MAX. MARKS: 100**

NOTE: Attempt any FIVE questions by selecting at least TWO from each section.

SECTION 1		
Q1.	a) Define decomposable groups and prove that every cyclic group whose order has at least two prime divisors is decomposable. b) Define Sylow p-subgroups of a finite group. Find Sylow 3-subgroups of A_4.	10 10
Q2	a) State Fundamental Theorem of finite abelian group. Also classify all abelian groups of order p^4. b) Let P be Sylpw p-subgroup and K be any normal subgroup of a group G. Then show that $P \cap K$ and $\frac{PK}{K}$ are Sylow p-subgroup of K and $\frac{G}{K}$	10 10
Q3	a) Prove that intersection of any class of characteristic subgroups is characteristic. b) Define normal product of two groups. Determine the normal product of cyclic group of order 4(C_4) by cyclic group of order 2(C_2).	10 10
Q4	a) Prove that two permutations in S_n are conjugate if and only if they are of same type. Also determine all the conjugacy classes of S_5 b) State and prove Orbit - Stabalizer Theorem. c) Prove that A_4 is not simple.	8 8 4
SECTION II		
Q5	a) Define composition series and prove that every finite group has a composition series. Also show that any two composition series of a group are isomorphic. Write at least two composition series of D_4. b) Let G be a group with normal series, H be a subnormal subgroup of G with a normal series. Then show that there exists a normal series of H whose factors are isomorphic to subgroup of the factors of normal series of G.	12 8
Q6	a) Let N be normal subgroup of G. Then show that G is solvable if and only if N and G/N are solvable. b) Show that S_n $(n > 4)$ is not solvable. c) Prove that a group of order 24 is solvable.	8 4 8
Q7	a) Construct lower and upper central series of A_4 . b) Show that every term $\xi_i(G)$ in upper central series of a group G is characteristic subgroup of G. c) Let G be a nilpotent group and H be its proper subgroup. Then show that H is properly contained in its normalizer.	6 6 8
Q8	a) Prove that the Frattini subgroup of a finite group G is nilpotent. b) Define partial complement of a subgroup and prove that a normal subgroup H of G is contained in Frattini subgroup of G if and only if H has no partial complement in G.	10 10
Q9	a) Prove general linear group $GL(n,q)$ is not simple. Also show that $\lvert SL(n,q)\rvert = (q^n-1)(q^n-q)......(q^n-q^{n-1})/(q-1)$ b) Write a note on i) General Linear Groups ii) Representation of a group	10 10

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Part-II A/2018
Examination:- M.A./M.Sc.

Roll No. ....

Subject: Mathematics
PAPER: (IV-VI) (Opt. iv) [Rings & Modules]

TIME ALLOWED: 3 hrs.
MAX. MARKS: 100

NOTE: Attempt any FIVE questions selecting at least TWO questions from each section.

Section I

Q. 1. a)A polynomial $p \in K[x]$ where K is a field and $\deg(p) = n \geq 0$. Show that p has at most n distinct roots in K. 10+10

b)Let $Z_{30} = \{0,1,2,\ldots\ldots,29\}$ be a ring, $I = \{0,15\}$, $J = \{0,10,20\}$ and $K = \{0,6,12,18,24\}$ are ideals of Z_{30}. Prove that $Z_{30}$is an internal direct sum of I, J, K.

Q. 2. a)If $R$is an integral domain, then show that every prime is irreducible. 10+10

b)Suppose that a ring R is the internal direct sum of its ideals $I_1, I_2 \ldots\ldots\ldots, I_n$. Then every element has unique expression $r = r_1 + r_2 \ldots\ldots\ldots + r_n, r_i \in I_i$: $i = 1,2,\ldots\ldots\ldots,n$.

Q. 3. a) Prove or disprove that $\mathbb{Z}[i]$, the ring of Gaussian integers, is Euclidean domain. 10+10

b)Show that $\left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R}\right\} \subseteq M_2(\mathbb{R})$ together with the usual matrices addition and multiplication forms a field isomorphic to $\mathbb{C}$.

Q. 4. a) Let K be an extension of a field F, and a ∈ K be algebraic of degree n over F. Show that $F(a_1, a_2, a_3, \ldots\ldots, a_n)$is a finite extension of F. 10+10

b) If L is finite extension of K and $K$is a finite extension of F, then prove that $L$is finite extension of F and $[L:F] = [L:K][K:F]$.

Q. 5. a) Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$, where $\mathbb{Q}$ is the field of rational numbers, also find the degree of extension of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$. 10+10

b) Define an algebraic extension of a field and prove that every finite extension of a field is algebraic.

Section II

Q. 6. a) Let M and Nbe two R-modules, $f: M \to N$ and $g: N \to M$ be two module homomorphisms such that$gof = I_M$ (identity map on M). Show that $N = Kerg \oplus Imf$. 10+10

b) Let $\mathbb{Q}$ be the set of rational numbers. Then show that $\mathbb{Q}$ is not $FG - \mathbb{Z}$- module.

Q. 7. a) Let M be an R- module and I be a right ideal of R. If $A = \{y \in M : ay = 0 \ \forall\, a \in I\}$. Then show that $A$is submodule of M. 10+10

b) Let M be a module over an integral domain R and let Tdenote the set of torsion elements of M. Show that $T$is a submodule of M the quotient module $M/_T$ is torsion free.

8a) Let R be a ring. Show that $R[x]$ is a $FG - R$-module if and only if $R = \{0\}$. 10+10

b) The extended homomorphism in freely generated module is unique.

Q. 9.Let M be an R-module and let $\{m_1, m_2, \ldots\ldots. m_s\}$ be a finite subset of M. Prove that the following statements are equivalent:

i. $\{m_1, m_2, \ldots\ldots. m_s\}$generates M freely.
ii. $\{m_1, m_2, \ldots\ldots. m_s\}$ is linearly independent and generates M.
iii. every element $m \in M$ is uniquely expressed in the form $m = \sum_{i=1}^{s} r_i m_i$.
iv. everym_i is torsion free and $M = Rm_1 \oplus Rm_2 \oplus \ldots\ldots. \oplus Rm_s$. 20

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Part-II A/2018
Examination:- M.A./M.Sc.

Roll No. ....

Subject: Mathematics
PAPER: IV-VI (opt.vii) [Quantum Mechanics]

TIME ALLOWED: 3 hrs.
MAX. MARKS: 100

NOTE: Attempt any FIVE questions in all, selecting at least TWO questions from each section.

SECTION I

1. (a) Consider a virus of size $10\AA$. Suppose that its density is equal to that of water (g/cm^3) and that the virus is located in a region that is approximately equal to its size. What is the minimum speed of the virus. (10 marks)

   (b) Define black body radiations and Describe its construction and functions in detail. (10 marks)

2. (a) Show that eigen vectors belonging to two different eigen values are orthogonal. (10 marks)

   (b) Consider the wave function

$$\Psi(x,t) = [Ae^{\iota px/\hbar} + Be^{-\iota px/\hbar}]e^{-\iota p^2 t/2m\hbar}.$$

   Find the probability current corresponding to this wave function. (10 marks)

3. (a) If $\hat{H} = \frac{\hat{p}^2}{2m} + V$, then find $[\hat{x},\ [\hat{x},\ \hat{H}]]$ and prove that (10 marks)

$$[\hat{A},\ \hat{B}^3] = 3\hat{B}^2[\hat{A},\ \hat{B}].$$

   (b) Define Hermitian operator and show that eigen values of hermitian operators are always real. (10 marks)

4. (a) Consider the Hamiltonian for a one dimensional system of two particles of masses m_1 and m_2 subjected to a potential that depends only on the distance between the particles $x_1 - x_2$, (10 marks)

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(x_1 - x_2),$$

   then write the Schrödinger equation using the new variables x and X, where

$$x = x_1 - x_2 \text{ (relative distance) and } X = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}.$$

   (b) Write down any two de Broglie postulates. (10 marks)

SECTION II

5. (a) A one-dimensional harmonic oscillator is characterized by the potential $V(x) = \frac{1}{2}kx^2$, where k is a real positive constant. It can be shown that the frequency is $\omega = \sqrt{k/m}$, where m is the mass of the oscillator. (a) Solve the stationary Schrödinger equation for this potential and find the stationary eigen states for this system. (10 marks)

   (b) Consider a particle with mass m in a one dimensional harmonic potential. At $t = 0$, the normalized wave function is $\psi(x) = \left(\frac{1}{\pi\sigma^2}\right)^{1/4} e^{-x^2/2\sigma^2}$, where $\sigma^2 \neq \frac{\hbar}{m\omega}$ is a constant. Find the probability that the momentum of the particle at $t > 0$ is positive. (10 marks)

6. (a) Prove the following relations for the angular momentum operator: (a) $[L^2, L_z] = 0$; (b) $\mathbf{L} \times \mathbf{L} = \iota\hbar\mathbf{L}$ (10 marks)

   (b) What is the probability that a measurement of L_x will equal zero for a system with angular momentum of one and is in the state $\frac{1}{\sqrt{14}}(1 \ \ 2 \ \ 3)^T$? (10 marks)

7. (a) Using Born approximation, evaluate the differential scattering cross section for scattering of particles of mass m and incident energy E by the repulsive spherical well with potential

$$V(r) = \begin{cases} V_0, & 0 < r < a, \\ 0, & r > a, \end{cases}$$

   where there exist explicit dependence of E and θ. (10 marks)

   (b) Calculate the hard sphere potential of the form

$$V(r) = \begin{cases} 0, & r > r_0, \\ \infty, & r < r_0, \end{cases}$$

   where $k_0 r_0 \ll 1$. (a) Assume only s-wave scattering and calculate $\delta_0(k)$, $f(k)$, $d\sigma/d\Omega$ and σ_T. (10 marks)

8. (a) Derive the formulas for the first and second order energy corrections for a time-independent perturbations. Also, derive the first-order corrections to the eigen states. Assume that there is no degeneracy. (10 marks)

   (b) Consider a particle of mass m subjected to the Hamiltonian

$$H = \begin{cases} \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2, & 0 \leq r \leq a, \\ \frac{p^2}{2m}, & r > a, \end{cases}$$

   where $r = \sqrt{x^2 + y^2}$. Use the second order perturbation to find the corrections to the ground state energy. (10 marks)

9. (a) Consider two angular momenta of magnitudes j_1 and j_2. The total angular momentum of this system is then $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$, where $\mathbf{J}_1$ and $\mathbf{J}_2$ are commuting operators. Let $|m_1 m_2\rangle$ be the common eigen states of the observables $\{\mathbf{J}_1^2, \mathbf{J}_2^2, J_{1z}, J_{2z}\}$. Let $|JM\rangle$ be the common eigen states of $\{\mathbf{J}_1^2, \mathbf{J}_1^2, \mathbf{J}^2, J_z\}$. Find all the possible values for m_1 and m_2. (10 marks)

   (b) Let $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ be the total angular momentum of two spin 1/2 particles ($S_1 = S_2 = 1/2$). Calculate the Clebsch-Gordan coefficients $\langle m_1 m_2 | S m_s \rangle$ by successive applications of $S_\pm = S_x \pm \iota S_y$ on the vectors $|S m_s\rangle$. Work separately in the two subspaces $S = 1$ and $S = 0$. (10 marks)

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Part-II A/2018
Examination:- M.A./M.Sc.

Roll No. ....

Subject: Mathematics **TIME ALLOWED: 3 hrs.**
PAPER: IV-VI (Opt.viii)[Special Theory of Relativity and Analytical Dynamics] **MAX. MARKS: 100**

NOTE: Attempt any FIVE questions in all, selecting at least TWO questions from each section.

SECTION I

1. (a) Show that the composition of two Lorentz transformations is also a Lorentz transformation. (10 marks)

   (b) A proton (of rest mass 938.2 MeV/c^2) strikes another proton and produces two new particles of rest-masses 1346 MeV/c^2 and 3154 MeV/c^2 in addition to the original protons. What is the least energy that the incoming proton must have had? (10 marks)

2. (a) Formulate the energy-momentum transformation and show that $p^2 - E^2/c^2$ is Lorentz invariant. (10 marks)

   (b) Using the relation $\tanh\phi = v/c$, show that
   (i) $\sinh\phi = (v/c)(1 - v^2/c^2)^{-1/2}$ and $\cosh\phi = (1 - v^2/c^2)^{-1/2}$
   (ii) the Lorentz transformation can be written as $x' = (\cosh\phi)x - (c\sinh\phi)t$, $y' = y$, $z' = z$, $t' = (\cosh\phi)t - \frac{1}{c}(\sinh\phi)x$. (10 marks)

3. (a) Derive general Lorentz transformation (3-dimension). What are the corresponding expressions for time dilation and length contraction. (10 marks)

   (b) Prove that the speed of light, i.e., c is electromagnetic in nature. (10 marks)

4. (a) Determine the formula for minimum kinetic energy required to produce a particle of mass M. (10 marks)

   (b) Two observers A and B, see a rocket taking off at distances d and d' at times t and t' respectively, having set their clocks at zero when they were coincident. If their direction of relative motion is in line with the rocket, find the speed of B relative to A. (10 marks)

5. (a) Define the Maxwell field tensor in terms of the electromagnetic four-vector potential and work out the components of Maxwell field tensor. Also, derive equation of continuity from Maxwell field tensor. (10 marks)

   (b) Define timelike, spacelike and lightlike vectors. For the Minkowski metric, identify the following vectors:

$$X^\alpha = (0,0,1,0),\ Y^\alpha = (2,1,0,1),\ Z^\alpha = (1,1,0,1),$$

   as timelike, spacelike or lightlike. (10 marks)

SECTION II

6. (a) What are generalized coordinates and generalized momenta? For generalized momentum p_j, establish the relation $\dot{p}_j = \frac{\partial L}{\partial q_j}$. (10 marks)

   (b) Two equal masses m connected by a massless rigid rod of length l forming a dumb-bell is rotated in the xy-plane. Find the Lagrangian and obtain Lagrange's equations of motion. (10 marks)

7. (a) If F and G are any pair of functions of (q,p) or (Q,P), then show that the Poisson bracket of the functions F and G is invariant under canonical transformation. (10 marks)

   (b) Using Lagrange's method of undetermined multiplier, find the equation of motion and force of constraint in the case of a simple pendulum. (10 marks)

8. (a) A bead of mass m slides on a frictionless wire under the influence of gravity. The shape of wire is parabolic and it rotates about the z-axis with constant angular velocity ω. Taking $z^2 = \alpha\rho$ as the equation of the parabola, obtain the Hamiltonian of the system. Is $E = H$? (10 marks)

   (b) If L is the Lagrangian for a system of n degrees of freedom satisfying Lagrange's equations, show by direct substitution that

$$L' = L + \frac{dF(q_1, q_2, \ldots, q_n, t)}{dt}$$

   also satisfies Lagrange's equation where F is any arbitrary but differentiable function of its arguments. (10 marks)

9. (a) State and prove Hamilton-Jacobi theorem. (10 marks)

   (b) If H is the Hamiltonian, prove that if f is any function depending on position, momenta and time, then

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [H, f].$$

   (10 marks)

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Part-II A/2018
Examination:- M.A./M.Sc.

Roll No. ....

Subject: Mathematics **TIME ALLOWED: 3 hrs.**
PAPER: IV-VI (opt.xi) [Theory of Approximation & Splines] **MAX. MARKS: 100**

NOTE: Attempt any FIVE questions, select at least TWO questions from each section.

Section I

Q1. (a) Show that composition of reflection and rotation is reflection. (10)

(b) Find the equation after stretching the circle $x^2+y^2=1$, parallel to $x-$axis by factor 2 and parallel to $y-$axis by factor 3. (10)

Q2. (a) Determine the image of $\frac{x^2}{4}+\frac{y^2}{9}=1$ under the affine transformation t defined as $t(\underline{x}) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \underline{x}, \forall \underline{x} \in I\!R^2$. (10)

(b) Determine $(t_2 \circ t_1^{-1})$ if t_1 and t_2 are defined as (10)
$t_1(\underline{x}) = \begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $t_2(\underline{x}) = \begin{bmatrix} -1 & -2 \\ 1 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Q3. (a) Find the power fits $y = A/x$ and $y = B/x^2$ for the following data and use $E_2(f)$ to determine which curve fits best. (10)

x_k	0.7	0.9	1.1	1.6	3.0
y_k	8.1	4.9	3.3	1.6	0.5

(b) Consider the date points $(-1, 6.62)$, $(0, 3.94)$, $(1, 2.17)$, $(2, 1.35)$ and $(3, 0.89)$. Determine the least-squares curve $f(x) = Ce^{Ax}$ for the given data. (10)

Q4. (a) Establish the Pade approximation: (10)
$e^x \approx R_{2,2} = \frac{12+6x+x^2}{12-6x+x^2}$.

(b) For $f(x) = sin(x)$ on $[-1, 1]$, find the Lagrange-Chebyshev polynomial approximation $P_2(x)$. (10)

Section II

Q5. (a) Define a function $P(\theta)$, $\theta \in [0, 2]$
$s.t.$ $P(0) = 2$, $P(1) = 3$, $P(2) = 3$, (10)
where P is linear over $[0, 1]$ and is quadratic over $[1, 2]$.

(b) (i) For the control point form, (5)
$$\underline{P}(\theta) = (1-\theta)^2(2\theta - k\theta + 1)\underline{b}_0 + k(1-\theta)^2\theta\underline{b}_1 + k(1-\theta)\theta^2\underline{b}_2 + \theta^2(-2\theta + k\theta + 3 - k)\underline{b}_3.$$
show that $k = 3$ is the B. B. cubic form.
Show that $\underline{P}(\theta)$ satisfies the convex hull property.
Does this property also hold for $\theta \in I\!R - [0, 1]$ as well?

(ii) Determine the barycentric coordinates of $\underline{v}(v_x, v_y)$ with respect to $\underline{v}_0 = (1, 0)$, $\underline{v}_1 = (2, 0)$, $\underline{v}_2 = (2, 1)$. (5)

PTO

Q6.	(a)	Determine the quadratic Hermite vector form defined on $[0,\ 1]$, s.t. $\underline{P}(0) = \underline{F}_0$, $\underline{P}(1) = \underline{F}_1$ and $\underline{P}'(0) = \underline{T}_0$.	(10)
	(b)	Consider the following B. B. rational quadratic form $\underline{P}(\theta) = \frac{(1-\theta)^2 w_0 \underline{b}_0 + 2(1-\theta)\theta w_1 \underline{b}_1 + \theta^2 w_2 \underline{b}_2}{(1-\theta)^2 w_0 + 2(1-\theta)\theta w_1 + \theta^2 w_2}$, $0 \le \theta \le 1$, $\underline{b}_i \in I\!R^2$, $i = 0, 1, 2$. Determine whether $\underline{P}(\theta)$ represents the conic section?	(10)
Q7.	(a)	Discuss the cubic Hermite spline defined over $[a,\ b]$ $s.t.\ a = t_0 < t_1 < t_2 < \ldots < t_n = b$, interpolating $(t_i,\ f_i)$, $i = 0, 1, \ldots, n$. Determine the system in matrix form corresponding to clamped end conditions.	(10)
	(b)	Determine $\underline{b}_i^*$, $i = 0, 1, 2, 3$ for B. B. cubic form $\underline{P}(\theta) = \sum_{i=0}^{3} B_i^n(\theta)\underline{b}_i$, $\theta \in [\alpha,\ \beta]$ when $\alpha = \frac{1}{3}$, $\beta = 1$.	(10)
Q8.	(a)	Discuss the cubic Hermite spline S defined over $[a,\ b]$ $s.t.\ a = t_0 < t_1 < t_2 < \ldots < t_n = b$, interpolating the data points $(t_i,\ f_i)$, $i = 0, 1, \ldots, n$. Determine the system in matrix form corresponding to periodic end conditions.	(10)
	(b)	Determine whether the following function is a cubic spline $S(x) = \begin{cases} 2x + 1, & x \in]-\infty,\ -1[, \\ x^3 + 3x^2 + 5x + 2, & x \in [-1,\ 1[, \\ -2x^3 + 12x^2 - 4x + 5, & x \in [1,\ 2[, \\ 20x - 11, & x \in [2,\ \infty[. \end{cases}$	(10)
Q9.	(a)	Determine the error bound of cubic Hermite interpolating function S defined on $a = t_0 < t_1 < t_2 < \ldots < t_n = b$ s.t. $S(t_i) = f(t_i)$, $S^{(1)}(t_i) = f^{(1)}(t_i)$, $i = 0, 1, 2, \ldots, n$, and $f \in C^4[a,\ b]$.	(10)
	(b)	Consider $S(x) = \begin{cases} x^3 + 2x + 1, & x \in]-\infty,\ -1[, \\ -3x^2 - x, & x \in [-1,\ 1[, \\ 2x^3 - 9x^2 + 5x - 2, & x \in [1,\ 2[, \\ x^3 - 3x^2 - 7x + 6, & x \in [2,\ \infty[. \end{cases}$ Represent the spline S in terms of the following truncated power function form $S(x) = \sum_{i=0}^{3} a_i x^i + \sum_{i=0}^{2} c_i (x - x_i)_+^3$.	(10)

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Part-II A/2018
Examination:- M.A./M.Sc.

Roll No. ....

Subject: Mathematics **TIME ALLOWED: 3 hrs.**
PAPER: IV-VI (opt. xiii) [Solid Mechanics] **MAX. MARKS: 100**

NOTE: Attempt any FIVE questions in all, selecting at least TWO questions from each section.

SECTION I

Q1(a) Work out the relationship between the components of stress and the components of strain. [10]

(b) A cylinder with its axis horizontal has the gravity stress represented by $\sigma_x = -\rho g y$, $\sigma_y = -\rho g y$, $\tau_{xy} = 0$. Its ends are confined between smooth fixed rigid planes which maintain the condition of plane strain. Sketch the forces acting on the surfaces, including the ends. [10]

Q2(a) Use Hook's law to show that the stress invariant $\Theta = \tau_{ii}$ and the strain invariant $\vartheta = e_{ii}$ are connected by the relation $\Theta = 3k\vartheta$, where k is the modulus of compression. [10]

(b) Show that $\phi = \frac{q}{8c^3}\left\{x^2\left(y^3 - 3c^2y + 2c^3\right) - \frac{1}{5}y^3\left(y^2 - 2c^2\right)\right\}$ is a stress function and find what problem it solves when applied to the region included in $y = \pm c, x = 0$ on the side x positive. [10]

Q3(a) Determine the value of the constant C in the stress function $\phi = Cr^2(\cos 2\theta - \cos 2\alpha)$ required to satisfy the conditions

$$\sigma_\theta = 0, \tau_{r\theta} = s \text{ on } \theta = \alpha$$
$$\sigma_\theta = 0, \tau_{r\theta} = -s \text{ on } \theta = -\alpha,$$

corresponding to uniform shear loading on each edge of a wedge, directed away from the vertex. Verify that no concentrated force or couple acts on the vertex. [10]

(b) Work out the differential equations of equilibrium for two dimensional problems. [10]

Q4(a) Find graphically the principal strains and their directions from rosette measurements $\epsilon_\phi = 2\times10^{-3}$, $\epsilon_{\alpha+\phi} = 1.35\times10^{-3}$, $\epsilon_{\alpha+\beta+\phi} = 0.95\times10^{-3}$ in.per.in. where $\alpha = \beta = 45^\circ$. [10]

(b) Work out the strain components in polar coordinates. [10]

SECTION II

Q5(a) Describe reflection phenomena in plates by considering plane waves to be incident on the straight boundary of a semi infinite plane $y > 0$. [10]

(b) Discuss the propagation of spherically symmetric waves in the infinite medium. [10]

Q6(a) Explain in detail, the experimental results on waves in plates. [10]

(b) Derive the governing equation of motion for an inhomogeneous rod where the modulus varies as $E = E_0(1 + \varepsilon x^2)$. Assume the density remains constant. [10]

Q7(a) Consider a thin rod of circular cross-section imbedded in an elastic medium such that a lateral resistive pressure $f_r = -ku_r$ is developed, where u_r is the radial displacement of the surface. Derive the equation of motion for the system and sketch the general form of the frequency spectrum. [10]

(b) Write a note on the geophysical applications of the elasto-dynamics. [10]

Q8(a) Derive the group velocity versus wave-number curve for the Bernoulli-Euler beam. [10]

(b) Obtain the expression for the impedance of a semi-infinite beam subjected to the end force $F_0 e^{-i\omega t}$ and free to rotate at the end. [10]

Q9(a) Consider beams of incident, reflected and refracted S-H waves and examine the average transmission of energy, in particular for the case $\frac{c_T^B}{c_T}\sin\theta_0 > 1$. [10]

(b) A thin layer of substances whose elastic constants are very small and can be neglected is adhered to the surface of an elastic half space. The mass density of the substance is ρ_s. A plane harmonic longitudinal wave is incident on the covered surface. Determine the influence of mass layer on the amplitude of the reflected waves. [10]