UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics **Paper: I (Advanced Analysis)** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt FIVE questions in all selecting at least TWO questions from each Section.

SECTION 1

Q1	a) Prove that the unit interval I = [0,1] is not denumerable.	10
	b) Prove that $c^{\aleph_0} = c$	10
Q2	a) For cardinal number α, β and γ, prove that (i) $\alpha\,(\beta + \gamma) = \alpha\beta + \alpha\gamma$ (ii) If $\alpha \leq \beta$, then $\alpha\gamma \leq \beta\gamma$	10
	b) If A and B are similar well-ordered set then show that there exists only one similarity mapping of A onto B.	10
Q3	a) Define inequality relations between ordinal numbers and for and ordinal number λ, show that $\lambda + 1$ is the immediate successor of λ.	10
	b) If $S(A)$, the collection of all initial segments of elements in a well-ordered set A is ordered by set inclusion, then show that A is similar to $S(A)$.	10
Q4	a) Show that the axiom of choice is equivalent to Zermelo's postulate.	10
	b) Show that for the set D_m consisting of all positive divisors of integer m ordered by divisibility, the $sup(a,b)$ and $inf(a,b)$ exist for any pair $a, b \in D_m$.	10

P.T.O.

SECTION 2

Q5	a) Prove that Lebesgue outer measure is a translation invariant.	10
	b) Let $\{E_n\}$ be a decreasing sequence of measurable sets and $m(E_1) < \infty$, then show that $m(\bigcap_{i=1}^{\infty} E_i) = m(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n)$.	10
Q6	a) Let A be Lebesgue measurable set with $m(A) > 0$. Then show that there exists $E \subset A$ such that E is not Lebesgue measurable.	10
	b) Define L^p-Spaces. Prove that $f(x) = \frac{1}{\sqrt[3]{x}}$ belongs to $L^1[0,8]$ but does not belong to $L^3[0,8]$	10
Q7	a) If $\{f_n\}$ is a sequence of extended real- valued measurable function with same domain, then show that $\mathrm{Inf}_{n\in\mathbb{N}} f_i$ is measurable for each n.	10
	b) Let f be an extended real-valued function defined on Borel set D. Then show that f is Borel function if and only if for each open set $V \in \overline{\mathbb{R}}, f^{-1}(V)$ is a Borel set.	10
Q8	a) Let f be bounded function defined on [a,b] which is Rieman integrable. Then show that f is measurable function and $R\int_{\underline{a}}^{b} f = \int_{a}^{b} f$	10
	b) Evaluate Lebesgue integral of $f(x) = \frac{1}{x^2+1}$ on [-1,1]	10
Q9	a) State and prove Dominated Convergence Theorem.	10
	b) If f and g are non-negative measurable functions on E, then show that $\int_E f + g = \int_E f + \int_E g$	10

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics **Paper: II (Methods of Mathematical Physics)** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt FIVE questions in all selecting at least TWO questions from each Section.

Q. No		Marks
	Section-I	
1(a)	Use Lagrange's method to find the complete integral for the first order linear PDE $xz_x + yz_y = z$.	**10**
1(b)	Find the integral surface of the quasilinear PDE $(y^2 - z^2)z_x - xyz_y = xz$ containing the curve $x = y = z,\ x > 0$.	**10**
2(a)	Show that $$\beta F(\alpha;\beta;x) = \beta F(\alpha - 1;\beta;x) + xF(\alpha;\beta + 1;x)$$	**10**
2(b)	Find the general solution of the inhomogeneous linear second order PDE $(D^2 - D'^2 - 3D + 3D')z = e^{x-2y}$.	**10**
3(a)	Use the method of Frobenius to obtain two linearly independent series solutions about the regular singular point $x_0 = 0$ of the DE $$xy'' + (1 - x)y' - y = 0.$$	**10**
3(b)	Find the general solution of the ODE $$y'' + 2xy = 0$$ with $x_0 = 0$ as the center of expansion.	**10**
4(a)	Determine the eigenvalues and eigenfunctions of the system $y'' + \lambda y = 0$ with the boundary conditions $y(0) = y(\pi),\ \ y'(0) = 2y'(\pi)$.	**10**
4(b)	Find the steady state solution of the equation $$\frac{\partial u}{\partial t} = \frac{1}{K}\frac{u^2 u}{\partial x^2} + p(x), \quad 0 < x < a, \quad t > 0$$ subject to the conditions $u(0,t) = u_1,\quad u(a,t) = u_2,\ \ t > 0,\ \ u(x,0) = 0,\ \ 0 < x < a.$	**10**
5(a)	Derive the one dimensional heat equation through conduction of heat in a cylinder of finite length.	**10**
5(b)	Show that $$J^2{}_{1/2}(x) + J^2{}_{-1/2}(x) = \frac{2}{\pi x}.$$	**10**

P.T.O.

Section-II

6(a)	State and prove fundamental theorem of calculus of variation.	10
6(b)	Find the Green's function for the boundary value problem $$u'' + \lambda u = 0, \quad u(0) = u(1), \quad u'(0) = u'(1).$$	10
7(a)	Find the Fourier sine transform of the function $$f(x) = xe^{-ax}.$$	10
7(b)	Find the extremal of the function $$\int_0^{\pi/2} (y'^2 + z'^2 + 2yz)dx$$ where the boundary conditions are $y(0) = 0, \quad y(\pi / 2) = 1, \quad z(0) = 0, \quad z(\pi / 2) = -1.$	10
8(a)	Solve by using Fourier transform method $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \; u(x,0) = p(x), \; u_t(x,0) = q(x) \; and \; u_0 > 0$$ and $u, u_x \to 0 \; as \; x \to \pm\infty.$	10
8(b)	If $f(x)$ is piecewise smooth and absolutely integrable function then $\lim_{\lvert k\rvert\to\infty} F(k) = 0.$	10
9(a)	Solve the DE by Laplace transform method $$y''(t) + ty'(t) - y(t) = 0, \quad y(0) = 0, \quad y'(0) = 1.$$	10
9(b)	Solve the problem by using Laplace transform method $u_{tt}(x,t) = a^2 u_{xx}(x,t), \quad t > 0, \quad x > 0, \quad u(x,0) = u_t(x,0) = 0,$ $u(0,t) = f(t)$ and $\lim_{x\to\infty} u(x,t) = 0.$	10

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics **Paper: III (Numerical Analysis)** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt FIVE questions in all selecting at least TWO questions from each Section.

Section I

Q1.

a) State and prove Newton Raphson method to find an approximate root of the non-linear equation $f(x) = 0$.

b) Prove that Newton Raphson method is a quadratically convergent method.

c) Write an algorithm for Newton Raphson method to find an approximate root of the non-linear equation $f(x) = 0$.

(05+07+08)

Q2.

a) Find the root correct to three decimal places between 0 and 1 of the equation $1 + cost - 4t = 0$, using a numerical technique.

b) Apply Runge Kutta method of order two on $\frac{du}{dx} = Sin(xu)$, $u_0 = 0.1$, $h = 0.1$ to find $u(0.3)$.

(10+10)

Q3.

a) Write an algorithm for Heun's method for the solution of initial value problem.

b) Solve the following linear system by LU decomposition using Doolittle's method

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 3 \\ 2 & 2 & 2 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} -2 \\ 3 \\ -6 \end{pmatrix}$$

(10+10)

Q4.

Solve $\frac{dy}{dx} = x^2 + y^2 + 2$ using Milne's method for $x = 0.4$. The values for $x = 0.1, 0.2,$ 0.3 should be obtained by Taylor's series method of order two.

(20)

Q5.

a) Use a numerical technique, find five approximations to the following system of equations:

$$11x + 2y + z = 15, \quad x + 10y + 2z = 16, \quad 2x + 3y - 8z = 1$$

b) Define dominant eigenvalue and dominant eigenvectors. Use the Rayleigh quotient to compute the eigenvalue λ of A corresponding to the eigenvector $\boldsymbol{x}$.

$$A = \begin{pmatrix} 3 & 2 & -3 \\ -3 & -4 & 9 \\ -1 & -2 & 5 \end{pmatrix}, \boldsymbol{x} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

(10+10)

P.T.O.

Section II

Q6.

a) Write an algorithm for Weddle's Rule to approximate the integral of $f(x)$ over the interval $[a, b]$ using n subintervals.

b) Using the following data, apply Lagrange's formula to find an approximate polynomial and $f(2)$.

x	-2	0	3	4
$f(x)$	25	1	-20	-23

(10+10)

Q7.

Derive 9 points Simpson's Rule. Apply 9 points Simpson's Rule to approximate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \tan(\frac{\pi x}{2})dx$. Also find the absolute error.

(20)

Q8.

a) From the values in the table given below, find the value of $Sec31^o$ using numerical differentiation

θ^o	31	32	33	34
$tan\theta$	0.6008	0.6249	0.6494	0.6745

b) Find a Newton polynomial matching the following data points:

$$(-2, -6), \quad (-1, 0), \quad (1, 0), \quad (2, 6), \quad (4, 60)$$

(10+10)

Q9.

a) Define difference operators. Prove that $\Delta\nabla = \nabla\Delta = \Delta - \nabla = \delta^2$

b) Define difference equation. Solve:

$$y_{t+2} - 9y_{t+1} + 20y_t = 4^t(t^2 + 1)$$

(10 + 10)

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics **Paper: IV-VI (Opt. i) [Mathematical Statistics]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions selecting at least TWO questions from each section. All questions carry equal marks.

		SECTION-I	Marks
Q.1	**(a)**	(i) Let A and B be two not-mutually exclusive events. If C is any other non-empty given event, then prove the following conditional probability, $$P(A \cup B/C) = P(A/C) + P(B/C) - P(A \cap B/C)$$ (ii) If A and B are mutually exclusive and $P(C) \neq 0$, then show that the above conditional probability can be reduced as, $$P(A \cup B/C) = P(A/C) + P(B/C)$$	(10)
	(b)	What is the probability that a positive integer selected at random from first 200 positive integers is divisible by either 8 or 12 or 14?	(10)
Q.2	**(a)**	Show that for a very large value of N, the Hypergeometric distribution tends to the Binomial distribution.	(10)
	(b)	What is the probability that a fifth six will first appear on the eleventh roll? How many rolls should we expect to obtain seven sixes.	(10)
Q.3	**(a)**	Prove that if X and Y are independent Gamma variates, with parameters l and m respectively, then $\frac{X}{X+Y}$ is a $\beta_1(l, m)$ variate.	(10)
	(b)	Find the mean and the variance of a random variable Z, where Z assumes the values of absolute differences of the upper faces of two dices one of which is green and the other is red. Further, if green is labelled by X and red is labelled by Y, then find (i) $P(X + Y > 13)$ (ii) $P(X + Y < 8)$	(10)
Q.4	**(a)**	Prove that the Normal distribution is symmetrical. That is, mean, mode and median are equal.	(10)
	(b)	In a Statistics class test, the marks obtained were normally distributed with a mean of 66 and a standard deviation of 4. What proportion of the class would be expected to score between 58 and 72 points? How many scores will it take for 77% marks of the overall class?	(10)

P.T.O.

SECTION-II

Q.5	**(a)**	Define partial correlation coefficients and establish if $r_{12.3} = 1$, then $r_{13.2} = 1 = r_{32.1}$	(10)
	(b)	List all the six partial regression coefficients and write their formulas. Prove that the standard equation of the regression plane passing through the origin (as the mean point) can be written as $\frac{x_1}{s_1} = \left(\frac{r_{12} - r_{13}r_{23}}{1 - r_{23}^2}\right)\frac{x_2}{s_2} + \left(\frac{r_{13} - r_{12}r_{23}}{1 - r_{23}^2}\right)\frac{x_3}{s_3}$	(10)
Q.6	**(a)**	Given the joint density $f(x,y) = \begin{cases} \alpha x + \alpha y & for \quad 0 < x < 1 \; and \; 0 < y < 1 \\ 0 & elsewhere \end{cases}$ Find α, $\mu_{Y/x}$ and $\mu_{X/y}$.	(10)
	(b)	Find the moment generating function of random variable X, whose moments about origin are given by $\mu'_r = 2^r(r+1)!$	(10)
Q.7	**(a)**	If the joint probability density of X and Y is given by $f(x,y) = \begin{cases} 16xy, & for \quad 0 < x < 2 \; and \; 0 < y < 2 \\ 0 & elsewhere \end{cases}$ Find the probability density function of $Y_1 = XY$, $Y_2 = X + Y$	(10)
	(b)	Find the moment generating function of Uniform distribution and discuss its coefficeients of skewness and kurtosis.	(10)
Q.8	**(a)**	Establish the probability function for χ^2-distribution.	(10)
	(b)	Establish the probability function for F-distribution. Also find its Mode.	(10)
Q.9	**(a)**	Find the coefficient of skewness and and kurtosis for the χ^2-distribution.	(10)
	(b)	Prove that all even order moments about origin of t-distribution with n degree of freedom is given by $\dfrac{n^r\Gamma(r+\frac{1}{2})\Gamma(\frac{n}{2}-r)}{\Gamma(\frac{1}{2})\Gamma(\frac{n}{2})}$	(10)

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics **Paper: IV-VI (Opt. ii) [Computer Applications]** **Time: 3 Hrs.** **Marks: 50**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

Section I

Q1. Write a program to find the smallest element in an array of twenty elements using Function Subprogram. (10)

Q2. a) Write a program to find the complex roots of a quadratic equation.

b) Write a FORTRAN expression corresponding to the following Mathematical expression:

$$\frac{\sqrt{x^2} + Sinx + e^{x^3+e^x}}{|x - \sin x| + \log x^{2x}}$$

(05+05)

Q3. Define loops, arrays and Format in FORTRAN Programming. (10)

Q4. Write a program to find the area of a triangle when two sides and angle between them are given, using Subroutine Subprogram.

(10)

Section II

Q5. Write a program to find the value of the following integral $\int_4^{10} \frac{1}{x}$ using Trapezoidal Rule.

Compare the approximate value with the exact value; find the absolute error and relative error. (10)

Q6. Write a program to solve the following system of equations using Jacobi iterative method:

$8x - 3y + 2z = 20,\ 4x + 11y - z = 33,\ 6x + 3y + 12z = 35$ (10)

Q7. From the following table write a program to find $f(0.5)$ using Lagrange interpolating formula.

P.T.O.

Table

x	-1	0	2	3
$f(x)$	-1	0	8	27

(10)

Q8. Write a program to solve the following system of differential equation by Runge Kutta method of order two:

$$\frac{dy}{dx} = y - \frac{2x}{y}, y(0) = 1, \text{ over the interval } [0,2].$$

(10)

Q9. Write the Mathematica statements for the following:

1. Evaluate $\frac{d}{dx}\left(\log_3^{x^2+4x}\right)$.
2. Evaluate $\int_0^1 \sec x dx$.
3. Plot the graph of $\tan x$, $Sec(x)$, $-\frac{\pi}{4} < x < \frac{+\pi}{4}$.
4. Find the conjugate of $z = x + iy$
5. Find the sum of the series $2+4+6+8+\cdots+2^n$.

(10)

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics **Paper: IV-VI (Opt. iii) [Group Theory]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

		SECTION-I	Marks
Q.1	(a)	Prove that a finite group whose order is divisible by a prime p contains a Sylow p-subgroup.	(10)
	(b)	Let A and B be two cyclic groups of order n and m, respectively. Then show that direct product of A and B is cyclic group of order nm if and only if $\gcd(n,m)=1$. Also, find the number of elements of order 3 in a group $G = Z_9 \oplus Z_3$.	(10)
Q.2	(a)	Show that the number k of Sylow p-subgroups of a finite group G is congruent to 1 mod p.	(10)
	(b)	Show that for any prime divisor p of the order n of a goup G, G has a unique Sylow p-subgroup H if and only if H is normal in G.	(10)
Q.3	(a)	Show that the center of a group G is a characteristic subgroup of G.	(10)
	(b)	What is meant by the holomorph of a group G? Find the holomorph of the group $G = \langle a, b: a^3 = b^2 = (ab)^2 = 1\rangle$.	(10)
Q.4	(a)	Let H be characteristic subgroup of a normal subgroup of a group G. Then prove that H is normal subgroup of G.	(8)
	(b)	State and prove Orbit Stabalizer Theorem.	(8)
	(c)	Discuss the simplicity of A_4.	(4)

P.T.O.

SECTION-II

Q.5	(a)	State and prove Zassenhaus Butterfly Lemma.	(10)
	(b)	Define the normal series of a group G. What is a subnormal subgroup of a group? Give an example of a group in which a subnormal subgroup is not necessarily a normal subgroup of the group?	(10)
Q.6	(a)	Show that a group G is solvable if and only if it has a normal series with abelian factors.	(10)
	(b)	Define a nilpotent group and its nilpotency class, with illustration of an example. Is every solvable group also nilpotent and vice versa? Justify your answer.	(10)
Q.7	(a)	Construct lower and upper central series of dihedral group D_8 of order eight .	(6)
	(b)	Show that every term $\xi_i(G)$ in upper central series of a group G is characteristic subgroup of G.	(6)
	(c)	Let G be a nilpotent group and H be its proper subgroup. Then show that H is properly contained in its normalizer.	(8)
Q.8	(a)	Define the Frattini subgroup of a group. Write the Frattini subgroups of A_4 and of the group Q_8 of quaternions.	(10)
	(b)	Define partial complement of a subgroup and prove that a normal subgroup H of a group G is contained in Frattini subgroup of G if and only if H has no partial complement in G.	(10)
Q.9	(a)	Define special linear group and projective linear group. Prove that general linear group is not simple.	(10)
	(b)	Write a note on the following: i) General Linear Groups ii) Representation of a group	(10)

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics **Paper: (IV-VI) (Opt. iv) [Rings & Modules]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions selecting at least TWO questions from each section.

Section I

Q. 1. a) Let R be an integral domain, R is Unique Factorization Domain if and only if R is a factorization domain and every irreducible element is prime.

b) Prove that

$R = \{a + b\sqrt{-5} : a, b \in Z\}$ is not a unique Factorization Domain. 10+10

Q. 2. a) Let R be an integral domain, $p \in R\backslash\{0\}$. Then p is prime if and only if $R/_{pR}$ is an integral domain.

b) By giving an example, justify that an irreducible element in $\mathbb{Z}[\sqrt{-3}]$ may not be prime. 10+10

Q. 3. a) Prove that $\mathbb{Z}[i]$, the ring of Gaussian integers, is Euclidean domain.

b) If R is an integral domain, show that

(i) s|t if and only if $tR \subseteq sR$

(ii) u is a unit of R if and only if $uR = R$

(iii) the set of all units of R is an abelian group with respect to multiplication. 10+10

Q. 4. a) Let $f(x) = x^3 - 2 \in \mathbb{Q}[x]$. Show that the splitting field of f(x) over $\mathbb{Q}$ is $K = \mathbb{Q}(\sqrt[3]{2}, \sqrt{3}i)$ and find $[K: \mathbb{Q}]$.

b) If L is finite extension of K and K is a finite extension of F, then prove that L is finite extension of F and $[L: F] = [L: K][K: F]$. 10+10

Q. 5. a) Let D be an integral domain, let F be a field such that $F \subseteq D$. Suppose unity 1 of F is also unity of D. Then D can be regarded as a vector space over F. Show that D is a field if $[D: F] =$ finite.

b) Find the splitting fields of the following polynomials over $\mathbb{Q}$.

i. $x^4 - 1$
ii. $x^4 - x^2 - 2$

10+10

P.T.O.

Section II

Q. 6. a) Let M be R- module and N is a submodule of M. Then prove that the set

$$\frac{M}{N} = \{x + N : x \in M\}$$

Is a left R- module under addition and scalar multiplication defined by

$(x + N) + (y + N) = x + y + N$ and $r(x + N) = rx + N$ for all $x, y \in M, r \in R$.

b) Let $M = \mathbb{Z} \oplus \mathbb{Z}$ be the $\mathbb{Z}$ -module (under component wise addition and scalar multiplication) then prove or disprove that the following subsets of M are $\mathbb{Z}$ -submodules of M

i. $N_1 = \{(m, n) \in M \mid n - m \text{ is even}\}$

ii. $N_2 = \{(m, n) \in M \mid n - m \text{ is odd}\}$ 10+10

Q. 7. a) Let M and N be two R- modules and $f: M \to N$ be an onto module homomorphism. Then prove that

$$\frac{M}{Ker\, f} \cong N$$

b) Let M be an R- module and $N = \{m \in M \mid rm = 0,$ for some non zero $r \in R\}$. Show that if R is an integral domain then N is an R – submodule of M.

10+10

Q.8. a) Prove that every FG – R-module is a homomorphic image of a free module.

b) The submodules of the quotient module $M/_N$ are of the form $U/_N$, where U is a submodule of M containg N. 10+10

Q. 9. a) Let M be an R-module and let $\{x_1, x_2, \dots \dots . x_s\}$ be a finite subset of M. Prove that the following statements are equivalent:

i. every element $m \in M$ is uniquely expressed in the form $x = \sum_{i=1}^{s} r_i x_i$.

ii. every m_i is torsion free and $M = Rx_1 \oplus Rx_2 \oplus \dots \dots . \oplus Rx_s$. 10+10

b). Show that any two cyclic R- modules are isomorphic if and only if they have same order ideal.

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics **Paper: IV-VI (opt.vii) [Quantum Mechanics]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

SECTION I

1. (a) Show that eigen vectors belonging to two different eigenvalues are orthogonal. (10 marks)

   (b) Write a note on wave-particle duality. (10 marks)

2. (a) Define Hermitian operator and show that eigen values of hermitian operators are always real. (10 marks)

   (b) Derive the operator equation $\frac{d}{dx}x^n = nx^{n-1} + x^n\frac{d}{dx}$ and show that $\left[\frac{d}{dx}, x^n\right] = nx^{n-1}$. (10 marks)

3. (a) Find the solution of the following Schrodinger equation in three dimensional rectangular box

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z) = E\psi(x,y,z).$$

   Show that E_{211}, E_{121} and E_{112} are the corresponding degenerate eigen values of energies. (10 marks)

   (b) Show that the uncertainty relation $\Delta x\Delta p > \hbar$ forces us to reject the semiclassical Bohr model for the hydrogen atom. (10 marks)

P.T.O.

4. (a) Consider a particle incident on a potential step as shown in the figure 1. Obtain the solution of Schrödinger equation for (i) $V_2 > E > V_1$ (ii) $E > V_2$. (10 marks)

(b) Show that the de Broglie wavelength of an electron of kinetic energy $E(ev)$ is $\lambda_c = \frac{12.3\times10^{-8}}{\sqrt{E}}$ cm and that of proton is $\lambda_p = \frac{0.29\times10^{-8}}{\sqrt{E}}$ cm. (10 marks)

SECTION II

5. (a) For a system with an angular momentum $l = 1$, find the eigen values and eigen vectors of $L_x L_y + L_y L_x$. (10 marks)

(b) Starting from the radial part of the Schrödinger equation, determine the scattering amplitude and total scattering cross-section by using partial wave analysis. (10 marks)

6. Explain the importance of perturbation theory. Obtain first order correction in energy for time-dependent non-degenerate perturbation theory. (20 marks)

7. (a) Prove the following relations for the angular momentum operator (10 marks)
$[\hat{L}^2,\ \hat{L}_z] = 0$, $\mathbf{L} \times \mathbf{L} = \iota\hbar\mathbf{L}$.

(b) Use transformation equations relating spherical coordinates $(r,\ \theta,\ \phi)$ with cartesian coordinates $(x,\ y,\ z)$ to obtain the expression $\hat{L}_z = -\iota\hbar\frac{\partial}{\partial\phi}$. (10 marks)

8. (a) Show that $\hat{L}_x$ is Hermitian. (10 marks)

(b) Prove that the Pauli exclusion principle does not hold for bosons. (10 marks)

9. (a) Compute the expression for ionization potential for hydrogen, helium and Lithium atoms. (10 marks)

(b) Using the Born approximation, calculate the phase shifts δ_1 for scattering in a centrally symmetric field. (10 marks)

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics **Paper: IV-VI (Opt. v) [Number Theory]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

SECTION-I Marks

Q.1 **(a)** Show that square of any integer is either a multiple of 4 or one more than a multiple of 4. (10)

(b) Let $a > 1$ and let m, n be distinct integers. Show that, (10)

$$gcd(a^{2^m}+1, a^{2^n}+1) = \begin{cases} 1, & if\ a\ is\ even \\ 2, & if\ a\ is\ odd \end{cases}$$

Q.2 **(a)** Let m be a positive integer, a and b any integers. Prove that the linear congruence $ax \equiv b(\bmod m)$ is solvable if and only if $d/b,\ d = (a, m)$. In case; prove that the given congruence has d mutually incongruent solutions. (10)

(b) Show that if $b_1, b_2, b_3, \cdots, b_t$ form a Reduced Residue System $(mod\ n)$, then $t = \varphi(n)$. (10)

Q.3 **(a)** Find the last digit in the decimal representation of $1997^{1998^{1999}}$. (10)

(b) Define Euler's phi function. Prove that if $d|n$ then, $\varphi(d)|\varphi(n)$. (2+8)

Q.4 **(a)** Define multiplicative arithmetic functions. Let n be an integer > 1. Show that $\sigma(n)$ is odd if and only if n is a perfect square or twice a perfect square. (10)

(b) (i) For any integer n, the exponent e of any prime p, such that $p^e \mid n!$ is at most $\sum_{i=1}^{\infty}[\frac{n}{p^i}]$ (5+5)

(ii) Find $E_{100}(13)$. That is, exponent of 13 in 100!.

P.T.O.

Q.5 (a) Show that if c is any odd integer, then $c^{2^{n-2}} \equiv 1 (mod\ 2^n),\ \forall n \geq 3.$ (10)

(b) Let a be primitive root modulo m and b, k be integers, then show that $Ind\ b^k \equiv k\ Ind\ b \pmod{\varphi(m)},\ k > 0$. Hence solve the congruence $4x \equiv 19\ (mod\ 23)$. (10)

SECTION II

Q.6 (a) Let p be an odd prime and a an integer co-prime to p. If m denotes the number of integers that leave negative least residues in the set (10)

$$\left\{a, 2a, 3a, \cdots, \frac{(p-1)a}{2}\right\},$$

then show that

$$\left(\frac{a}{p}\right) = (-1)^m.$$

(b) State and prove the Gauss law of quadratic reciprocity. (10)

Q.7 (a) Prove that every algebraic number has a unique minimal polynomial. Further, if θ is algebraic of degree n, then any $\alpha \in F[\vartheta]$ can be expressed uniquely as (10)

$$\alpha = \sum_{i=0}^{n-1} a_i \vartheta^i,\ a_i \in F.$$

(b) Prove that all integral bases for a field $K = R(\theta)$ have the same discriminant. (10)

Q.8 (a) If K is finite extension over F, E over K, then show that E is finite extension over F. (10)

(b) For $\alpha, \beta \in R[\theta]$, Show that $N\alpha\beta = N\alpha N\beta$, where N is the norm of the algebraic number. (10)

Q.9 (a) If ζ is a primitive *3rd* root of unity, the show that the discriminant of the corresponding cyclotomic polynomial follow the formula, $(-1)^{\frac{p-1}{2}} p^{p-2},\ p = 3$. Justify your answer using Vandermonde Determinant. (10)

(b) Prove that the totality of numbers algebraic over a field F forms a field. (10)

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics **Paper: IV-VI (opt.vi) [Fluid Mechanics]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

Section I

Q.1(a)	Differentiate the followings: **(i)** Hydrostatic pressure and Stress. **(ii)** Uniform and non-uniform flow. **(iii)** Laminar and turbulent flow.	(10)
(b)	A flat plate having dimensions of 2m × 2m slides down an inclined plane at an angle of 1 radian to the horizontal at a speed of 6 m/s. The inclined plane is lubricated by a thin film of oil having a viscosity of 30×10^{-3}Pa.s. the plate has a uniform thickness of 20 mm and a density of 40,000 kg/m^3. Determine the thickness of the lubricating oil film.	(10)
Q.2(a)	Find the relationship between Eulerian and Lagrangian methods.	(10)
(b)	For a two-dimensional flow, the velocity components at a point in a fluid may be expressed in Eulerian coordinates by $u = x + y + 2t$ and $v = 2y + t$. Determine the Lagrange coordinates as functions of the initial positions x_0 and y_0 and the time t.	(10)
Q.3(a)	State and prove Kelvin's minimum energy theorem.	(10)
(b)	Find Bernoulli's equation for steady inviscid flow under conservative forces.	(10)
Q.4(a)	For $u = -\omega y,\ v = \omega x,\ w = 0$, show that the surfaces intersecting the streamlines orthogonally exist and are planes through z-axis, although the velocity potential does not exist.	(10)

P.T.O.

(b)	Find the velocity potential and complex velocity potential when the fluid particles are moving in circles about a central point.	(10)
Q.5(a)	Find the equipotential lines and streamlines for the fluid flow whose complex velocity potential is given by $z = A\cos w$, where A is a real constant.	(10)
(b)	Discuss the flow when a fluid is moving subject to external impulsive forces.	(10)

Section II

Q.6(a)	Find Navier-Stokes equations for incompressible laminar flow.	(10)
(b)	Consider the velocity field $v = a(x^2 - y^2),\ v = -2axy, w = 0.$ Show that it is a solution to the Navier-Stokes equations of motion for an incompressible steady viscous flow with negligible body force.	(10)
Q.7	Discuss steady laminar flow between two rotating coaxial cylinders find expression for velocity also calculate average velocity and volumetric flow rate.	(20)
Q.8(a)	Derive the equation of a circular cylinder with circulation moving in liquid	(10)
(b)	What is the Stokes' second problem? Write its mathematical formulation and solve it for the velocity field.	(10)
Q.9(a)	State and prove Blausius Theorem	(10)
(b)	The velocity distribution for the steady, laminar and fully developed flow between two fixed parallel plates is given by $u = -\frac{1}{2\mu}\frac{dp}{dx}(hy - y^2),\ where\ \frac{dp}{dx}$ is the pressure gradient, h is the gap width between the plates and y is the distance measured upward from the lower plate. Determine the volumetric flow rate, maximum velocity and the shearing stress at both the plates.	(10)

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics

Time: 3 Hrs. Marks: 100

Paper: IV-VI (Opt.viii)[Special Theory of Relativity and Analytical Dynamics]

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

SECTION I

1. (a) Explain Michelson Morley Experiment in detail. What was the objective of this experiment? (10 marks)

   (b) An observer sees a clock as showing 1 hour to be half an hour. If he sees an object lying at an angle of $\pi/4$ as having a length of $2m$. What is the rest length of the object? (10 marks)

2. (a) Write a comprehensive note on Lorentz and Poincare groups. (10 marks)

   (b) Prove that if the sum of two velocity four-vectors is a velocity four-vector, the angle between them is $2\pi/3$. While if their difference is a velocity four-vector, the angle between them is $\pi/3$. (10 marks)

3. (a) Determine the formula for minimum kinetic energy required to produce a particle of mass M. (10 marks)

   (b) A proton of rest mass 938.2 MeV/c^2 strikes another proton and produces two new particles of rest masses 1346 MeV/c^2 and 3154 MeV/c^2 in addition to the original protons. What is the least energy that the incoming proton must have had? (10 marks)

4. (a) Derive the equation of continuity from Maxwell field tensor $F^{\alpha\beta}$. (10 marks)

   (b) If an object moves relative to an observer at a speed $c/2$, what should be the angle between the velocity vector and the line joining the objectand the observer, so that no Dopler shift is observed? (10 marks)

P.T.O.

5. (a) Find the components of Maxwell field tensor and transformation of the field components. Show that $F^{\mu\nu}{}_{,\nu} = \mu_0 J^\mu$, $F_{[\mu\nu,\rho]} = 0$ represent Maxwell's equations in four-vector formalism. (10 marks)

(b) Two particles of equal rest-mass have a head-on collision in the laboratory frame, where one article has twice the momentum of the other. Determine the center of mass frame. (10 marks)

SECTION II

6. (a) Deive Lagrange's equation of motion for non-holonomic systems. (10 marks)

(b) AB is a straight frictionless wire fixed at point A on a vertical axis OA such that AB rotates about OA with constant angular velocity ω. A bead of mass m is constrained to move on the wire. Set up the Lagrangian. Write Lagrange's equations. (10 marks)

7. (a) Work out the Hamilton's equations of motion from variational principle. (10 marks)

(b) A bead of mass m slides freely on a frictionless circular wire of radius a that rotates in a horizontal plane about a point on the circular wire with a constant angular velocity ω. Find the reaction of the wire on the bead. (10 marks)

8. (a) State and explain Hamilton's principle of least action. (10 marks)

(b) Show that the transformation $P = m\omega q \cot Q$ and $P = \frac{m\omega q^2}{2\sin^2 Q}$ is canonical. Also, find the generator of the transformation. (10 marks)

9. (a) Apply the Hamilton-Jacobi method to study the motion of Harmonic oscillator. (10 marks)

(b) A particle of mass m moves in a force field whose potential in spherical coordinates is $V = -(K\cos\theta)/r^2$. Write the Hamilton-Jacobi equation describing its motion. (10 marks)

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics **Paper: IV-VI (opt. x) [Operations Research]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

Section – I

Q1.	Wild West produces two types of cowboy hats. A Type 1 hat requires twice as much labor time as a Type 2. If all the available labor time is dedicated to Type 2 alone, the company can produce a total of 400 Type 2 hats a day. The respective market limits for Type 1 and Type 2 are 150 and 200 hats per day, respectively. The profit is \$8 per Type 1 hat and \$5 per Type 2 hat. Determine the number of hats of each type that maximizes profit. Construct LP model and solve it by using graphical method.	20
Q2	a) Write a note on simplex method.	8
	b) Solve the following by simplex method $\text{Max: } z = 8x_1 + 6x_2 + 3x_3 - 2x_4$ Subject to $x_1 + 2x_2 + 2x_3 + 4x_4 \leq 40$ $2x_1 - x_2 + x_3 + 2x_4 \leq 8$ $4x_1 - 2x_2 + x_3 - x_4 \leq 10$ $x_1, x_2, x_3, x_4 \geq 0$	12
Q3	A company manufactures two products, A and B. The unit revenues are \$2 and \$3, respectively. Two raw materials, M1 and M2, used in the manufacture of the two products have daily availabilities of 8 and 18 units, respectively. One unit of A uses 2 units of M1 and 2 units of M2, and 1 unit of B uses 3 units of M1 and 6 units of M2. a) Determine the dual prices of M1 and M2 and their feasibility ranges. b) Suppose that 2 additional units of M1 can be acquired at the cost of 25 cents per unit. Would you recommend the additional purchase?	20
Q4	a) Write a note on M-technique.	8
	b) Consider the problem $\text{Max: } z = x_1 + 5x_2 + 3x_3$ Subject to $x_1 + 2x_2 + x_3 = 6$ $2x_1 - x_2 + 0x_3 = 8$ $x_1, x_2, x_3 \geq 0$ The variable x_3 plays the role of a slack. Thus no artificial variable, R, is needed in the first constraint. In the second constraint, an artificial variable, R, is needed. Solve the problem using x_3 and R as the starting variables.	12
Q5	a) Write a note on assignment model.	8
	b) Find the starting basic solution of the following transportation problem by Vogel approximation and find the optimal solution by UV multipliers method.	12

0	2	1	6
2	1	5	9
2	4	3	5
5	5	10	

P.T.O.

SECTION II

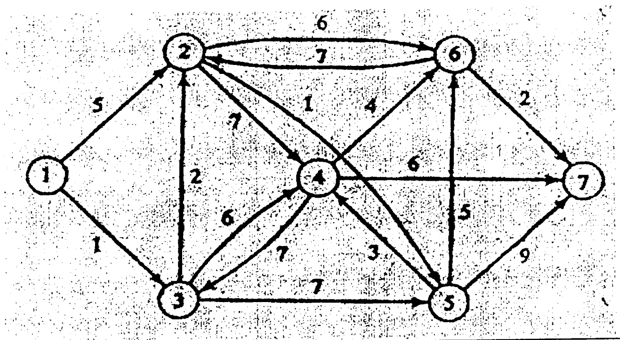
Q6 (20)

a) Write a note on Maximal flow algorithm.

b) The network given below gives the distances in miles between pairs of cities 1,2,..., and 7. Use Dijkstra's algorithm to find shortest route between

I. Cities 1 and 7

II. Cities 2 and 7



Q7

a) Write a note on revised simplex method (8)

b) Solve the following bounded LP model (12)

$$\text{Min: } z = 6x_1 - 2x_2 - 3x_3$$

Subject to

$$2x_1 + 4x_2 + 2x_3 \le 8$$
$$x_1 - 2x_2 + 3x_3 \le 7$$
$$0 \le x_1 \le 2,\ 0 \le x_2 \le 2, 0 \le x_3 \le 1$$

Q8 (20)

Solve the following LP model by branch and bound algorithm. For convenience, always select x_1 as the branching variable at node 0.

Max: $z = 2x_1 + 2x_2$ Subject to

$2x_1 + 5x_2 \le 27$

$6x_1 + 5x_2 \le 16$

$x_1, x_2 \ge 0$

and integers.

Q9 (20)

A 4-ton vessel can be loaded with one or more of three items. The following table gives the unit weight w_i in tons and the unit revenue in thousands of dollars r_i for item i. The goal is to determine the number of units of each item that will maximize the total return.

Item i	w_i	r_i
1	2	31
2	3	47
3	1	14

Because the unit weight w_i and the maximum weight W are integers, the state x_i assumes integer values only.

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics **Paper: IV-VI (opt. ix) [Electromagnetic Theory]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

SECTION I

1. (a) Work out the potential energy for a continuous charge distribution. (10 marks)

   (b) What is the magnitude of the point charge chosen so that the electric field 75 cm away has value $2.30N/C$. (10 marks)

2. (a) Work out the relationship between capacitance and resistance. (10 marks)

   (b) A point charge Q is located at the origin. The potential at $(1,0,0)$ is 20 V and at $(0,2,0)$ is 10 V. Find the value of charge. (10 marks)

3. (a) Discuss the conservation of charge and then formulate the equation of continuity. (10 marks)

   (b) Work out the vector potential A in cylindrical coordinates and use it to evaluate the magnetic induction for a long straight wire. (10 marks)

4. (a) Calculate the self-inductance of a toroid. (10 marks)

   (b) The inductance of a closely wound N turns coil is such that an emf of 3 mV is induced when the current changes at the rate of 5 A/sec. The steady current of 8 A produces a magnetic flux of 40 μ wb through the turn. Calculate the inductance of the coil. How many turns does the coil have? (10 marks)

5. (a) A point charge Q is located at the origin; the potential at $(1,0,0)$ is $20V$ while the potential at $(0,2,0)$ is $10V$. Find Q? (10 marks)

   (b) Discuss the Gauss's Law in Dielectrics? (10 marks)

P.T.O.

SECTION II

6. (a) Formulate the Maxwell's equations inside matter in general form. (10 marks)

   (b) Prove that the electric and magnetic energy densities for a plane electromagnetic wave in free space are equal. (10 marks)

7. (a) Verify that the flux of Poynting vector through any closed surface gives the energy flow through the volume enclosed by that surface. (10 marks)

   (b) Discuss the propagation of plane electromagnetic waves in conducting media. (10 marks)

8. (a) Formulate the Fresnel's equations when the incident wave is polarized with its $\vec{E}$ vector parallel to the plane of incident. (10 marks)

   (b) Define a waveguide. Show that the transverse electromagnetic waves cannot be transmitted inside a hollow conducting waveguide but can be transmitted in a co-axial line. (10 marks)

9. (a) Calculate the reflection and refraction at the interface between two non-conductors if $k_{m1} = k_{m2} = 1$ for the case when the incident wave is polarized with its $\vec{E}$ vector parallel to the plane of incidence. (10 marks)

   (b) Prove that a plane electromagnetic wave in free space has only transverse components of electric and magnetic field vectors. (10 marks)

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019 (Clash)

Roll No. ....

Subject: Mathematics **Paper: IV-VI (opt.xi) [Theory of Approximation & Splines]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions in all, selecting atleast TWO from each section.

SECTION I

Q1. (a) Show that the coefficients A and B of least-squares line $y=Ax+B$ can be computed as

$$C=\sum_{k=1}^{N}(x_k-\bar{x})^2$$

$$A=\frac{1}{C}\sum_{k=1}^{N}(x_k-\bar{x})(y_k-\bar{y}) \qquad (10)$$

$$B=\bar{y}-A\bar{x}$$

Where $\bar{x}$ and $\bar{y}$ are data means.

(b) Show that if the transformation t defined as $t(x)=Ax$, for all $x\in R^2$ is an isometry then A is orthogonal. (10)

Q2. (a) Find the power fits $y=\frac{A}{x}$ and $y=\frac{B}{x^2}$ for the following data and use $E_2(f)$ to determine which curve fits best. (10)

x_k	0.5	0.8	1.1	1.8	4.0
y_k	7.1	4.4	3.2	1.9	0.9

(b) Establish the Pade approximation

$$\ln(1+x)\approx R_{3,2}(x)=\frac{30x+21x^2+x^3}{30+36x+9x^2} \qquad (10)$$

Q3. (a) Determine the image of the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$, under the transformation of reflection through the angle $\frac{\pi}{8}$. (10)

(b) Find the Chebyshev polynomial $P_3(x)$ that approximates the function $f(x)=e^x$ over [-1,1]. (10)

P.T.O.

Q4. (a) Prove that the Euclidean congruence is an equivalence relation. (10)

(b) The Chebyshev approximation polynomial $P_N(x)$ of degree $\leq N$ for $f(x)$ over [-1, 1] can be written as a sum of $\{T_j(x)\}$:

$f(x) \approx P_N(x) = \sum_{j=0}^{N} c_j T_j(x)$. Show that the coefficient $\{c_j\}$ are obtained as

$$c_0 = \frac{1}{N+1}\sum_{k=0}^{N} f(x_k) \text{ and } c_j = \frac{2}{N+1}\sum_{k=0}^{N} f(x_k)T_j(x_k) \text{ for } j = 1,2,3,\ldots,N$$ (10)

SECTION II

Q5. (a) Construct the cubic B-Spline S (t) for $t \in [i, i+1]$. (10)

(b) Consider the following cubic Hermite interpolatory function

$$S(t) = (1-\theta)^2\,(1+2\theta)\,f_i + (1-\theta)^2\,\theta\,h_i\,d_i + \theta^2(3-2\theta)\,f_{i+1} - \theta^2(1-\theta)\,h_{i+1}\,d_{i+1},$$

$$t_i \leq t \leq t_{i+1},\; i = 0,1,2,3,\ldots,n-1$$

Apply second derivative continuity at knots and derive tridiagonal system of *(n-1)* equations in *(n+1)* unknowns $d_i s$. (10)

Q6. (a) Define uniform B-Spline. Show that $N_0^3(t)$ is a spline of degree 2. (10)

(b) Show that the Bernstein Bezier quadratic rational form represents the conic section. (10)

Q7. (a) Find the natural cubic spline *S(x)* that passes through (0, 0.0), (1, 0.5), (2, 2.0) and (3,1.5) with the boundary conditions $S''(0) = 0,\ S''(3) = 0$.

(10)

(b) Determine Bernstein Bezier cubic form $P(\theta)$ for $\theta = \frac{1}{4}$. (10)

Q8. (a) Calculate new control points for Bernstein Bezier cubic form for the interval $[\frac{1}{3}, \frac{2}{3}]$ using subdivision algorithm. (10)

(b) Prove that Bernstein Bezier rational cubic form preserves the convex hull property. (10)

Q9. (a) Find out new control points of Bernstein Bezier quartic form from Bernstein Bezier cubic form. Show it geometrically as well. (10)

(b) Determine a function $P:[0,2] \rightarrow R$ such that $P(0) = 2,\ P(1) = 3,\ P(2) = 2$, and P is linear over $[0,1[$ and quadratic over $[1,2]$

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics **Paper: IV-VI (opt.xi) [Theory of Approximation & Splines]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

SECTION I

Q1. (a) Prove that an affine transformations map ellipses to ellipses, parabolas to parabolas, and hyperbolas to hyperbolas. (10)

(b) Find the least-squares parabola for the four points (-3, 3), (0, 1), (2, 1) and (4, 3). (10)

Q2. (a) For the given set of data, find the least-squares curve for $f(x) = C\,e^{Ax}$ (10)

x_k	y_k
1	0.6
2	1.9
3	4.3
4	7.6
5	12.6

(b) Establish the Pade approximation

$$\cos(x) \approx R_{4,4}(x) = \frac{15120 - 6900x^2 + 313x^4}{15120 + 660x^2 + 13x^4} \qquad (10)$$

Q3. (a) Determine the image of circle $x^2 + y^2 = 16$, under the transformation of stretching along x-axis by factor 2
(10)

(b) Find the Chebyshev polynomial $P_3(x)$ that approximates the function $f(x) = e^x$ over [-1, 1].
(10)

Q4. (a) Show that an affine transformation preserves ratios of lengths along parallel straight lines. (10)

(b) The Chebyshev approximation polynomial $P_N(x)$ of degree $\le N$ for $f(x)$ over [-1, 1] can be written as a sum of $\{T_j(x)\}$:

$f(x) \approx P_N(x) = \sum_{j=0}^{N} c_j T_j(x)$. Show that the coefficient $\{c_j\}$ are obtained as

$$c_0 = \frac{1}{N+1}\sum_{k=0}^{N} f(x_k) \text{ and } c_j = \frac{2}{N+1}\sum_{k=0}^{N} f(x_k)T_j(x_k) \text{ for } j = 1,2,3,\ldots,N \qquad (10)$$

P.T.O.

SECTION II

Q5. (a) State and Prove de Casteljau Algorithm for Bernstein Bazier curve of degree *n*. (10)

(b) Consider the following cubic Hermite interpolatory function

$$S(t) = (1-\theta)^2\ (1+2\theta) f_i + (1-\theta)^2 \theta h_i d_i + \theta^2 (3-2\theta) f_{i+1} - \theta^2 (1-\theta) h_{i+1} d_{i+1},$$
$$t_i \le t \le t_{i+1},\ i = 0,1,2,3,\ldots,n-1$$

(10)

Apply second derivative continuity at knots and derive tridiagonal system of *(n-1)* equations in *(n+1)* unknowns $d_i s$.

Q6. (a) Show that $B_i^j(\theta) = (1-\theta) B_i^{j-1}(\theta) + \theta B_{i-1}^{j-1}(\theta),\ i = 0,1,2,3,\ldots,j$ (10)

(b) Show that the Bernstein Bazier quadratic rational form represents the conic section. (10)

Q7. (a) Find the clamped cubic spline *S(x)* that passes through (0, 0.0), (1, 0.5), (2, 2.0) and (3, 1.5) with the endpoint conditions $S'(0) = 0.2,\ S'(3) = -1.$ (10)

(b) Determine the error bound of cubic Hermite interpolating function S(t) defined on $a = t_0 < t_1 < \ldots < t_n = b$ such that $S(t_i) = f(t_i),\ S^{(1)}(t_i) = f^{(1)}(t_i), i = 0,1,2,\ldots,n$ and $f \in C^4[a,b]$. (10)

Q8. (a) Calculate the control points corresponding to the sub-interval $[0,\frac{1}{3}]$ and $\left[\frac{1}{3},1\right]$ for Bernstein Bazier cubic form using subdivision algorithm. (10)

(b) Prove that Bernstein Bazier rational cubic form preserves the convex hull property. (10)

Q9. (a) Find out new control points of Bernstein Bazier quartic form from Bernstein Bazier cubic form. Show it geometrically as well. (10)

(b) Determine a function $P:[0,2] \to \Re$ such that *P* is cubic over $[0,1[$ and quadratic over $[1,2]$. Moreover P, P', P'' are continuous and $P(0) = 5,\ P(1) = 6,\ P(2) = 5,\ P'(1) = \frac{3}{2}$.

(10)

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics **Paper: (IV-VI) (Opt. xii) [Advanced Functional Analysis]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.

Section-I

Q1 (a) Let $X = (X, \|\cdot\|)$ be a normed space. Prove that there exists a Banach space $\hat{X}$ and an isometry A from X onto a subspace W of $\hat{X}$ which is dense in $\hat{X}$. Furthermore, $\hat{X}$ is unique up to an isometry.

(b) Consider the set of all integers $\mathbb{Z}$ and the set of all rational numbers $\mathbb{Q}$ in $\mathbb{R}$ with usual metric. Determine which set is everywhere dense and which set is nowhere dense. Also describe the category of each of these sets in $\mathbb{R}$. Properly justify your answers.

(10+10)

Q2 (a) Let M be an orthonormal set in a Hilbert space H. Show that M is total in H if and only if for all $x \in H$ the Parseval identity holds.

(b) Prove that an orthonormal set M in a separable Hilbert space H is at most countable.

(10+10)

Q3 (a) Let B be a nonempty subset of a Hilbert space H. Prove that $\overline{\operatorname{span} B} = H$ if and only if $B^{\perp} = \{0\}$.

(b) Prove that in every Hilbert space $H \neq \{0\}$ there exists a total orthonormal set.

(10+10)

Q4 (a) Show that a subspace Y of a Hilbert space H is closed in H is and only if $Y = Y^{\perp\perp}$.

(b) Let M and N be two orthogonal subspaces of a Hilbert space H. Then show that

$$\operatorname{span}(M \cup N) = M + N = \{m + n \mid m \in M, n \in M\}$$

(10+10)

Section-II

Q5 (a) Let X be a complex vector space and p be a sublinear functional on X. Furthermore, let f be a linear functional which is defined on a subspace Z of X such that

$$|f(z)| \leq p(z) \qquad \text{for all } z \in Z.$$

Then prove that f has a linear extension $\tilde{f}$ from Z to X satisfying

$$|\tilde{f}(x)| \leq p(x) \qquad \text{for all } x \in X.$$

P.T.O.

(b) State and prove "Open Mapping Theorem".

(10+10)

Q6 (a) Let X be a normed space and let $x_0 \neq 0$ be any element of X. Then prove that there exists a bounded linear functional $\tilde{f}$ on X such that

$$\|\tilde{f}\| = 1 \qquad \text{and} \qquad \tilde{f}(x_0) = \|x_0\|$$

(b) Let X be a locally convex topological linear space and X_0 be a closed subspace of X. Suppose $x_1 \in X - X_0$. Then show that there exists a continuous linear functional $\tilde{f}$ on X such that $\tilde{f}(x_1) = 1$ and $\tilde{f}(x) = 0$ if $x \in X_0$.

(10+10)

Q7 (a) Show that the dual space of l^p is l^q; here, $1 < p < +\infty$ and q is the conjugate of p, that is,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

(b) Show that every Hilbert space X is reflexive.

(10+10)

Q8 (a) Show that the normed space X of all polynomials with norm defined by

$$\|\mathrm{P}\| = \min_j |\alpha_j|$$

is not complete, where P is a polynomial in X and $\alpha_0, \alpha_1, \cdots$ be the coefficients of P.

(b) Let (x_n) be a weakly convergent sequence in a normed space X, say, $x_n \xrightarrow{w} x$ Then prove that the weak limit x of (x_n) is unique and the sequence $(\|x_n\|)$ is bounded.

(10+10)

Q9 (a) If the dual space X' of a normed space X is separable, then show that X itself is separable.

(b) Let Y be subspace of $X = C[0,1]$ which consists of all functions $f \in X$ which have a continuous derivative. Then show that the differential operator

$$T: Y \longrightarrow X$$

defined as $T(f) = f'$, is closed.

(10+10)

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics **Paper: IV-VI (opt. xiii) [Solid Mechanics]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

SECTION I

1. (a) The displacement field in a material is given by $u_x = A(3x - y)$, $u_y = Axy^2$ where A is a small constant. Evaluate the strains. What is the rotation ω? Sketch the deformation and any rigid body motions of a differential element at the point $(1, 1)$. (10 marks)

   (b) The displacement field in a material is given by $u_x = A(3x - y)$, $u_y = Axy^2$ where A is a small constant. Sketch the deformation and rigid body motions at the point $(0, 2)$ by using a pure shear strain superimposed on the rotation. (10 marks)

2. (a) Show that, in a state of plane strain ($\varepsilon_{zz} = 0$) with zero body force,

$$\frac{\partial e}{\partial x} - 2\frac{\partial \omega_z}{\partial y} = \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}$$

   where e is the volumetric strain (dilatation), the sum of the normal strains: $e = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$. (10 marks)

   (b) Explain three-dimensional stress-strain relations. (10 marks)

3. (a) Consider now a two dimensional infinitesimal element of width and height Δx and Δy and unit depth. Looking at the normal stress components acting in the x-direction, and allowing for variations in stress over the element surfaces, the stresses are as shown in Fig. 1. Derive the corresponding equation of motion. (10 marks)

   (b) The elementary beam theory predicts that the stresses in a circular beam due to bending are

$$\sigma_{xx} = My/I, \quad \sigma_{xy} = \sigma_{yx} = V(R^2 - y^2)/3I, \quad (I = \pi R^4/4)$$

   and all the other stress components are zero. Do these equations satisfy the equations of equilibrium? (10 marks)

4. (a) Show that the contour lines for the warped cross section are hyperbolas having the principal axes of the ellipse as asymptotes. (10 marks)

   (b) State and prove generalized Hook's law. (10 marks)

SECTION II

5. (a) Explain seismic wave types and their properties. (10 marks)

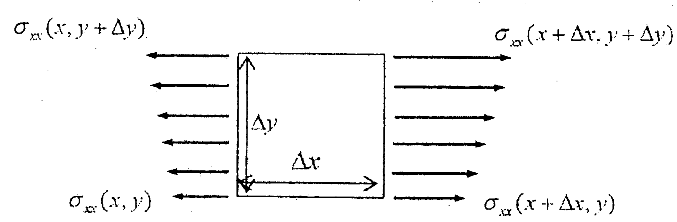


Figure 1:

P.T.O.

(b) Work out the spherically symmetric waves in the infinite medium. Also, discuss the outgoing and ingoing waves related with the problem of explosion and implosion. (10 marks)

6. (a) Show that if $C'_{i'j'k'l'} = \lambda_1 \delta_{i'j'}\delta_{k'l'} + \mu_1 \delta_{i'k'}\delta_{j'l'} + \mu_2 \delta_{i'l'}\delta_{j'k'}$, then under coordinate rotation, $C_{ijkl} = \lambda_1 \delta_{ij}\delta_{kl} + \mu_1 \delta_{ik}\delta_{jl} + \mu_2 \delta_{il}\delta_{jk}$. In other words, C_{ijkl} remains the same under coordinate rotation, i.e., it is an isotropic tensor. (10 marks)

(b) Consider an infinite string with an elastic spring located at $x = 0$ and let a step pulse be incident on this discontinuity. Determine the reflected and transmitted wave system. (10 marks)

7. (a) Discuss the dispersion relation of elastic waves in a cubic crystal which are propagated (a) in the crystal plane, that of a cube face, (b) in the crystal direction, that of a cube diagonal. (10 marks)

(b) Show that the ratio of the mechanical impedances determines the nature of reflection and transmission at the interface. (10 marks)

8. (a) A cylindrical rod of length l collides with a semi-infinite rod of the same outside diameter. The impact rod has a longitudinal cylindrical hole drilled in the impact end. The depth of the hole is $\frac{l}{4}$ and the diameter is one half the rod diameter. Construct the characteristic plane representation of the wavefront propagation of this impact problem. (10 marks)

(b) Determine the displacement constants u_1 and u_2 due to two dimensional center of compression. (10 marks)

9. (a) Write a comprehensive note on the geophysical applications of the elasto-dynamics. (10 marks)

(b) Consider the reflection-transmission expressions

$$\frac{\partial^2 u_1}{\partial x_1^2} = \frac{1}{c_1^2}\frac{\partial^2 u_1}{\partial t^2}, \quad \frac{\partial^2 u_2}{\partial x_2^2} = \frac{1}{c_2^2}\frac{\partial^2 u_2}{\partial t^2},$$

where c_1 and c_2 are the bar velocities of the respective materials in the form $\sigma_t = K_t \sigma_i, \sigma_r = K_r \sigma_i$, where K_t and K_r appropriately defined transmission and reflection co-efficients. Plot K_t and K_r as a function of $\frac{Z_2}{Z_1}$ for various ratios of $\frac{A_1}{A_2}(\frac{1}{4}, \frac{1}{2}, 1, 2)$. (10 marks)

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Exam – 2019

Roll No. ....

Subject: Mathematics **Paper: (IV-VI) (Opt. xiv) [Theory of Optimization]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.

Section-A

Q1) a) State Khun Tucker conditions. **[10]**

b) For function f(x)

$$\text{Minimize } f(x) = x_1 + x_2, \text{ w.r.t } x_1, x_2$$
$$\text{Subject to } c(x) = x_1^2 + x_2^2 - 2 = 0,$$

Find lagrangian and optimality condition, also establish minimum. **[10]**

Q2) a) For any function f(x),

$$\text{minimize } f(x) \text{ w.r.t } x \in R^n, \text{ such that}$$
$$c_j(x) = 0, \quad j = 1, 2, 3 \ldots\ldots m.$$
$$c_k(x) \geq 0, \quad k = 1, 2 \ldots. i.$$

Find Lagrangian of given problem also state Khun Tucker conditions and sufficient condition. **[10]**

b) What are necessary and sufficient conditions for finding the minimum of a function f(x). **[10]**

Q3) a) State and prove weak duality theorem. **[10]**

b) Use dual simplex method to solve following LP problem **[10]**

$$\max z = 2x_1 - x_2 \text{ s.t}$$
$$x_1 + x_2 - x_3 \geq 5$$
$$x_1 - 2x_2 + 4x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0$$

Q4) a) State Kuhn Tucker theorem. **[10]**

b) Minimize $f = x_1^2 + 2x_2^2 + 3x_3$ subject to $g_1 = x_1 - x_2 - 2x_3 \leq 12$, $g_2 = x_1 + x_2 - 3x_3 \leq 8$ using Kuhn Tucker condition. **[10]**

Q5) For given primal problem find dual problem, **[20]**

$$\text{Primal max } z = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$$

$$\text{s.t} \quad a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \leq b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \leq b_3$$
$$a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 \leq b_4, \qquad x_1, x_2, x_3 \geq 0$$

P.T.O.

Section-B

Q6) a) Briefly explain what Euler equations are. **[10]**

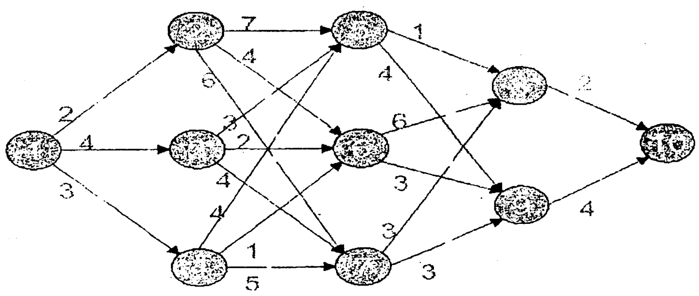
b) Find extremal of functional, $J = \int_{t_0}^{t_f} (2x^2(t) + 24\dot{x}(t).t)dt$ using Euler equations. **[10]**

Q7) a) A collection of n jobs is to be processed in arbitrary order by a single machine. Job I has processing time pi and when it completes a reward ri is obtained. Find the order of processing that maximizes the sum of the discounted rewards. **[10]**

b) What is the discounted, cost criterion of dynamic programming for an infinite horizon problem. **[10]**

Q8) a) Define Principle of optimality and the optimality equations in different cases. **[10]**

b) Consider the' stage coach problem' in which a traveler wishes to minimize the length of a journey from town 1 to town 10 by first traveling to one of 2, 3 or 4 and then on wards to one of 5, 6 or 7 then onwards to one of 8 or 9 and the finally to 10. Thus there are 4'stages'. The arcs are marked with distances between towns: Find the minimal distance. **[10]**



Q9) a) What is Bellman's equation, define briefly. **[10]**

b) An investor receives annual income from a building society of x_t pounds in year t. He consumes u_t and adds $x_t - u_t$ to his capital, $0 \le u_t \le x_t$. The capital is invested at interest rate $\theta \times 100\%$, and so his income in year t+1 increases to

$$x_t + 1 = a(x_t, u_t) = x_t + \theta(x_t - u_t).$$

He desires to maximize his total consumption over h years, $C = \sum_{t=0}^{h-1} u_t$. **[10]**