UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Examination – 2020

Roll No. ....

Subject: Mathematics Paper: I (Advanced Analysis) Time: 3 Hrs. Marks: 100

NOTE: Attempt FIVE questions in all selecting at least TWO questions from each Section.

SECTION 1		
Q. 1	a) Show that the relation $\approx$ of being equipotent is an equivalence relation in any collection of sets.	10
	b) Define cardinality and show that $\mathbb{N}\times\mathbb{N}\approx\mathbb{N}$.	10
Q. 2	a) Prove that $[0,1]\approx(0,1)$	10
	b) State and prove the principle of Transfinite Induction.	10
Q. 3	a) Show that the axiom of choice is equivalent to Zermelo's postulate.	10
	b) If $S(A)$, the collection of all initial segments of elements in a well-ordered set A is ordered by set inclusion, then show that A is similar to $S(A)$.	10
Q. 4	a) Show that every non-empty set can be well-ordered.	10
	b) Let X be a partially ordered set. Then show that X contains a maximal chain.	10

P.T.O

SECTION 2		
Q. 5	a) The Lebesgue outer measure is a translation invariant.	10
	b) Define Borel set. Give an example of a non-Borel set.	10
Q. 6	a) Show that step function f with measurable domain is measurable.	10
	b) For measurable set E, $1 \leq p \leq \infty$ and $f, g \in L^p(E)$, state and prove Minkowski's inequality.	10
Q. 7	a) Let E be measurable se, $1 \leq p \leq \infty$ and q be the conjugate of p. For $f \in L^p(E)$ and $g \in L^q(E)$, show that $f.g$ is integrable and $\int_E \|f.g\| \leq \|\|f\|\|_p \cdot \|\|g\|\|_q$.	10
	b) Let f be a bounded function defined on $[a,b]$ which is Riemann integrable. Then show that f is a measurable function	10
Q. 8	a) State and prove the bounded convergence theorem.	10
	b) Show that every set with a positive Lebesgue measure has a non-measurable set.	10
Q. 9	a) If f and g are non-negative measurable functions on E, then show that $\int_E f+g = \int_E f + \int_E g$	10
	b) Let f be a non-negative measurable function on E. Then show that $\int f = 0$ if and only if f=0 a.e. on E.	10

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Examination – 2020

Roll No. ....

Subject: Mathematics **Paper: II (Methods of Mathematical Physics)** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt FIVE questions in all selecting at least TWO questions from each Section.

SECTION – I

Q.1(a)	Use Lagrange's method to obtain a complete integral for the partial differential equation $$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=(x+y)z$$	(10)
(b)	Find a particular integral for the partial differential equation $$(D^2-DD'-2D'^2)z=(y+1)e^x,\quad where\ D\equiv\frac{\partial}{\partial x}\ and\ D'\equiv\frac{\partial}{\partial y}.$$	(10)
Q.2(a)	Discuss the classification of a singular point. Determine the singular points of the given differential equation. Classify each singular point as regular/irregular. $$x^3(x^2-25)(x-2)^2\frac{d^2y}{dx^2}+3x(x-2)\frac{dy}{dx}+7(x+5)y=0.$$	(10)
(b)	Find the power series solution of the given differential equation about the ordinary point $x=0$ $$(x^2+1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0.$$	(10)
Q.3(a)	Show that $$xJ_v'(x)=vJ_v(x)-xJ_{v+1}(x),$$ where $J_v(x)$ is a Bessel function of order v.	(10)
(b)	Using the concept of hypergeometric function, Prove that $$F(\alpha-1,\beta-1;\gamma;x)-F(\alpha,\beta-1;\gamma;x)=\frac{(1-\beta)x}{\gamma}F(\alpha,\beta;\gamma+1;x)$$	(10)
Q.4(a)	Prove that eigen values of a periodic SL system are always real.	(10)
(b)	Using the Rayleigh quotient, discuss the sign of the eigen values of the SL problem $\frac{d^2y}{dx^2}+\lambda y=0,\quad y(0)=0,\quad y(a)=0.$	(10)
Q.5(a)	Formulate and solve the problem of steady flow of heat in a circular disc where the boundary of the disc is maintained at temperature $f(\theta)$.	(10)
(b)	Solve the problem $$\frac{\partial^2y}{\partial t^2}=c^2\frac{\partial^2y}{\partial x^2}+x^2;\ y(x,0)=\frac{\partial y}{\partial t}(x,0)=0;y(0,t)=y(5,t)=0;\ 0\le x\le 5$$	(10)

SECTION – II

Q.6(a)	Evaluate $\mathcal{L}\{\sin\sqrt{t}\}$ and deduce the value of $\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$.	(10)
(b)	Formulate and solve the problem of geodesics.	(10)
Q.7(a)	If $\mathcal{L}\{f(t)\} = F(s)$, then prove that $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$ provided that $\lim\limits_{t\to 0}\frac{f(t)}{t}$ exists.	(10)
(b)	Construct Green's function associated with the problem $$u'' + \lambda u = 0,\quad u(0) = 0,\quad u(1) = 0.$$	(10)
Q.8(a)	Find the shortest distance between the points *A(1,-1,0)* and *B(2,1,-1)* in the plane *15x-7y+z-22=0*.	(10)
(b)	Assuming the asymptotic behavior for u and its derivatives, and for g *i.e.* $g, u, u_x, u_{xx}, u_{xxx} \to 0$ as $x \to \pm\infty$, solve by the Fourier transform method $$u_{xxxx} = \frac{1}{a^2}u_{tt},\quad u(x,0) = f(x),\quad u_t(x,0) = ag'(x).$$	(10)
Q.9(a)	Prove that $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{3s+4}}\right\} = \frac{1}{\sqrt{3\pi t}}e^{-\frac{4}{3}t}$.	(10)
(b)	A uniform cable is fixed at its ends at the same level in space and is allowed to hang under gravity. Find the final shape of the cable.	(10)

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Examination – 2020

Roll No. ....

Subject: Mathematics **Paper: III (Numerical Analysis)** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt FIVE questions in all selecting at least TWO questions from each Section.

Section I

Q.1. **(10+10)**

a) Find an iterative formula to find $(N)^{1/3}$, where N is a positive number and hence, find $(7)^{1/3}$ correct to four decimal places.

b) Find real root of the equation $2x - 3\sin(x) - 5 = 0$ up to 4 decimal places by Regula falsi method.

Q.2. **(8+8+4)**

a) Solve the following system of equations using Crout's factorization method

$$5x+3y+4z=3$$
$$2x+5y+3z=-2$$
$$3x+6y-4z=4$$

b) Use the Power method to approximate the dominant eigenvalue and corresponding eigenvector of the matrix $A = \begin{vmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{vmatrix}$ with $x^{(0)} = (-1,2,1)^t$. Perform five iterations only.

c) For matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, compute $\|A\|_1$ and $\|A\|_2$.

Q.3. **(10+10)**

a) Derive the Runge-Kutta method of order two for solving initial value problems.

b) Apply Runge-Kutta method of order 4 to find approximate value of y for x=0.2 with step-length h= 0.1, if it satisfies the differential equation:

$$\frac{dy}{dx} = x + y^2 \quad , \quad y(0) = 1.$$

Q.4. **(10+10)**

a) Apply Gauss Jacobi's iterative technique to find approximate solutions of the following system. Perform only five iterations.

$$10x_1 - x_2 + 2x_3 = 6$$
$$-x_1 + 11x_2 - x_3 + 3x_4 = 25$$
$$2x_1 - x_2 + 10x_3 - x_4 = -11$$
$$3x_2 - x_3 + 8x_4 = 15$$

b) Prove that fixed point iterative method is linearly convergent.

Q.5. **(10+10)**

a) Using Adams Bashforth Predictor-Corrector formulae, find $y(0.4)$, if y satisfies $\frac{dy}{dx} = 0.5\,x\,y,\ \ y(0)=1, y(0.1)=1.0025, y(0.2)=1.0101, y(0.3)=1.0228.$

b) Using Taylor Series method of order 3, find the solution of $\frac{dy}{dx} = y + x^3;\ \ y(1) = 1$, at $x = 1.1$ and at $x = 1.2$ correct to four decimal places.

Section – II

Q.6. **(10+10)**

a) Find y(5.3) using Gauss's forward interpolation formula from the following data :

X	1	3	5	7	9	11
Y	3	14	19	21	23	28

b) Derive Gregory Newton's backward interpolation formula.

Q.7. **(10+10)**

a) Derive the Gauss's Legendre Quadrature two points formula.

b) Apply Weddle's rule to evaluate $\int_0^{\frac{\pi}{2}} \sin x\, dx$ by taking n=12. Also compare the result with the exact value.

Q.8. **(10+10)**

a) Find Y'(5) using Stirling's formula from the following data

X	1	3	5	7	9	11
Y	3	14	19	21	23	28

b) From the following table, find the maximum value of y

x	-1	1	2	3
y	-21	15	12	3

Q.9. **(10+10)**

a) Solve the following difference equation:

$$y_{n+2} + 4y_{n+1} - 2y_n = Sin^2 n$$

b) Solve the following system of linear difference equations

$$3V_{n+1}-2V_n+U_n=n^2$$
$$4U_{n+1}-3U_n+V_n=3^n$$

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Examination – 2020

Roll No. ....

Subject: Mathematics **Paper: IV-VI (Opt. i) (Mathematical Statistics)** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt FIVE questions in all selecting at least TWO questions from each Section.

		SECTION-I	Marks
Q.1	(a)	For any three events A, B and C in a sample space S. Prove that $$P(A \cap \bar{B}/C) + P(A \cap B/C) = P(A/C)$$ provided $P(C) \neq 0$.	(10)
	(b)	A corona virus test machine results are 85% effective in detecting a disease when it is present. However the test yields 3% false positive result for healthy person tested. If 0.8% of the population has actually disease what is the probability that a person has disease even that his test result is negative.	(10)
Q.2	(a)	For a negative Binomial distribution show that mean is smaller than variance.	(10)
	(b)	Find $P(\mu - \sigma < x < \mu + \sigma)$ for exponential distribution $$f(x) = \frac{1}{3}e^{-\frac{x}{3}}, \qquad x > 0.$$ Also compare the result given by Chebyshev's theorem.	(7+3)
Q.3	(a)	Define Gamma distribution and prove that mode of a gamma distribution is that value of X for which probability density function is maximum.	(10)
	(b)	Suppose a certain type of small data processing firm is so specialized that some have difficulty making a profit in their first year of operation. The probability density function that characterizes the proportion Y that makes a profit is given by $$f(y) = \begin{cases} ky^4(1-y)^3, & 0 \le y \le 1, \\ 0, & elsewhere \end{cases}$$ a) Find the probability that at most 50% of the firms make a profit in the first year. b) Find the probability at least 80% of the firms make a profit in the first year.	(10)
Q.4	(a)	Prove that the mean deviation of the normal distribution is approximately $\frac{4}{5}$ of its standard deviation.	(10)
	(b)	A survey of magazine subscribers showed that 45.8% rented a car during the past 12 months for business reasons, 54% rented a car during the past 12 months for personal reasons, and 30% rented a car during the past 12 months for both business and personal reasons. i) What is the probability that a subscriber rented a car during the past 12 months for business or personal reasons? ii) What is the probability that a subscriber did not rent a car during the past 12 months for either business or personal reasons?	(10)

		SECTION-II	
Q.5	**(a)**	Prove that $$R^2_{a.bc} = 2\left(1-\frac{r_{bc}\Delta_{bc}}{\Delta_{aa}}\right)-\left(\frac{\Delta_{bb}+\Delta_{cc}}{\Delta_{aa}}\right).$$	(10)
	(b)	Show that the partial correlation coefficient is the geometric mean between partial regression coefficients.	(10)
Q.6	**(a)**	Given the joint density $$f(x,y)=\begin{cases}\dfrac{2x}{(1+x+xy)^3} & \text{for } x>0 \text{ and } y>0\\ 0 & \text{elsewhere}\end{cases}$$ Find $\mu_{Y/x}$ and show that var(Y/x) does not exist.	(10)
	(b)	Let U and V be two random variables with zero mean, zero correlation and same standard deviations and if and $X = USin\alpha - VCos\alpha,\ Y = UCos\alpha + VSin\alpha$, find the correlation between X and Y.	(10)
Q.7	**(a)**	If the joint probability density of X and Y is given by $$f(x,y)=\begin{cases}16xy, & \text{for } 0<x<2 \text{ and } 0<y<2\\ 0 & \text{elsewhere}\end{cases}$$ Find the probability density function of $Y_1 = XY,\ Y_2 = X+Y$.	(10)
	(b)	Find the moment generating function of Geometric distribution and discuss its coefficients of skewness and kurtosis.	(10)
Q.8	**(a)**	State and prove partitioning property for χ^2-distribution.	(10)
	(b)	Find moments about origin using probability density function of F-distribution.	(10)
Q.9	**(a)**	Show that for a t-distribution, large value of degree of freedom converts the distribution into standardized normal distribution.	(10)
	(b)	Find moment generating function for χ^2-distribution. Use it to evaluate mean and variance of the distribution.	(10)

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Examination – 2020

Roll No. ....

Subject: Mathematics **Paper: IV-VI (Opt. ii) [Computer Applications]** **Time: 3 Hrs. Marks: 50**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

Section – I

Q.1. Write a program to find the inverse of a square matrix. (10)

Q.2. a) Write a program to find a root of a linear equation. (5+5)

b) Write a FORTRAN expression corresponding to the following Mathematical expression:

$$\frac{\sqrt{x^3} + Cosx + e^{x^3+e^x}}{|x - \tan x| + \log x^{2\sqrt{a^3+b^2}}}$$

Q.3. Define: Do loops, While loops, Block IF Statements in FORTRAN Programming. (10)

Q.4. Write a program to find the area of a triangle when two sides and angle between them are given, using Function Subprogram. (10)

Section – II

Q.5. Write a program to find the value of the following integral $\int_4^{10} \frac{1}{x}\,dx$ using Boole's Rule.
Compare the approximate value with the exact value; find the absolute error and relative error. (10)

Q.6. Write a program to solve the following system of equations using Gauss Sidle iterative method: (10)

$$18x - 30y + 2z = 20,\quad 4x + 11y - 13z = 33,\quad 6x + 3y + 12z = 35$$

 P.T.O

Q.7. From the following table write a program to find $f(0.5)$ using Newtons Divided Difference interpolating formula. (10)

Table

x	-1	0	2	3
$f(x)$	-1	0	8	27

Q.8. Write a program to solve the following system of differential equation by Runge Kutta method of order four: (10)

$$\frac{dy}{dx} = y - \frac{2x}{y}, y(0) = 1, \text{ over the interval } [0, 2].$$

Q.9. Write the Mathematicalstatements for the following: (10)

1. Evaluate $\frac{d}{da}\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$.

2. Evaluate $\int_0^1 \pi \, dx$.

3. Plot the graph of $Cot(x)$, $Sec(x)$, $-\frac{\pi}{4} < x < \frac{\pi}{4}$.

4. Solve $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0) = 1$, over the interval $[0, 2]$.

5. Solve $18x - 30y + 2z = 20$, $4x + 11y - 13z = 33$, $6x + 3y + 12z = 35$

UNIVERSITY OF THE PUNJAB

Roll No. ....

M.A./M.Sc. Part – II Annual Examination – 2020

Subject: Mathematics | Paper: IV-VI (Opt. iii) (Group Theory) | Time: 3 Hrs. Marks:

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

		SECTION -1	Marks
Q.1	(a)	Prove that a finite group G whose order is divisible by a prime number p contains a Sylow p - subgroup.	(10)
	(b)	A group G is the direct product of its subgroups A and B if and only if (i) each element of A is permutable with every element of B (ii) each element of G is uniquely expressible as $g = ab, a \in A, b \in B$.	(10)
Q.2	(a)	Define Sylow p - subgroups of a finite group G. Find all Sylow 2-subgroups and Sylow 3-subgroups of the alternating group A_4.	(10)
	(b)	If a subgroup K contains the normalizer of a Sylow p - subgroup of a group G, then prove that K is its own normalizer.	(10)
Q.3	(a)	Prove that centre of a group G is a characteristic subgroup of G.	(10)
	(b)	Define holomorph of a group G, find holomorph of the group $G = \langle a,b : a^3 = b^2 = (ab)^2 = 1\rangle$	(10)
Q.4	(a)	Let O be an orbit of G, then prove that number of elements in O is equal to the index of the stabilizer subgroup G_x in G of any arbitrary element x of O.	(8)
	(b)	Prove that A_4 is not simple.	(4)
	(c)	Define normal product of two subgroups. Determine the normal product of cyclic group of order 4 (C_4) by a cyclic group of order 2 (C_2).	(8)

P.T.O

		SECTION -II	
Q.5	(a)	Prove that any two normal series of a group G have isomorphic refinements.	(10)
	(b)	A group G has a composition series if and only if all its ascending and descending normal chains break off.	(10)
Q.6	(a)	Prove that a group G is solvable if and only if it has a normal series $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots\ldots \supseteq G_{k-1} \supseteq G_k = E$ in which G_{i-1}/G_i are abelian for each $1 \le i \le k$.	(10)
	(b)	Let G be a nilpotent group and H a proper subgroup of G. Then prove that H is a proper subgroup of its normalizer $N_G(H)$.	(10)
Q.7	(a)	Construct upper and lower central series of dihedral group D_4 of order 8.	(4)
	(b)	Let $G = G_0 \supset G_1 \supset \ldots\ldots \supset G_k = E$ be a central series for G. Then prove that $G_i \supseteq \gamma_i(G)$, i=0,1,2,.....k and $G_{k-i} \subseteq \xi(G)$ for i=0,1,2,.....k.	(8)
	(c)	Define Frattini subgroup of a group. Write Frattini subgroups of A_4 and the group Q_8 of quaternions.	(8)
Q.8	(a)	Define upper and lower central series of G, let $G = G_0 \supset G_1 \supset \ldots\ldots \supset G_k = E$ be such that G_i is normal in G, $1 \le i \le k$. Then prove that the given series is a central series for G if and only if $[G_{i-1}, G] \subseteq G_i$, $1 \le i \le k$.	(10)
	(b)	If K is a normal subgroup and H a subgroup of G such that $K \subseteq \phi(H)$, then prove that $K \subseteq \phi(G)$.	(10)
Q.9	(a)	Prove that general linear group $GL(n,q)$ is not simple. Also show that $\mid SL(n,q) \mid = (q^n - 1)(q^n - q)\ldots\ldots(q^n - q^{n-1})/(q-1)$	(10)
	(b)	Write a note on (i) General Linear Groups (ii) Representation of a group	(10)

UNIVERSITY OF THE PUNJAB

Roll No. ....

M.A./M.Sc. Part – II Annual Examination – 2020

Subject: Mathematics **Paper: (IV-VI) (Opt. iv) [Rings & Modules]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions selecting at least TWO questions from each section.

Section I

Q 1. a) Prove or disprove that $\mathbb{Z}[\sqrt{-5}\,] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ is a unique factorization domain.

b) By giving an example, justify that an irreducible element in $\mathbb{Z}[\sqrt{-3}\,]$ may not be prime.

Q 2 . a) Prove that a polynomial of degree n over a field has at most n roots, counting multiplicity.

b) Find the remainder when x^{51} is divided by $x + 4$ in $\mathbb{Z}_7[x]$.

Q 3 . a) Prove that, in a principal ideal domain, an element is an irreducible if and only if it is a prime.

b) Determine all the units in the ring of Gaussian integers $\mathbb{Z}[i]$.

Q 4 . a) Let K be an extension field of the field F. Prove that the set of algebraic elements of K over F is a subfield of K.

b) Find the minimal polynomial for $\sqrt[3]{2} + \sqrt[3]{3}$ over $\mathbb{Q}$.

Q 5 . a) Prove that the polynomial $x^5 + x^2 + 1$ is irreducible over $\mathbb{Z}_2$.

b) Define splitting field and find the splitting field for the polynomial $x^4 + 4$ over $\mathbb{Q}$.

Section II

Q 6. a) Let R be a ring and let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ be an ascending chain of submodules of an R-module M. Prove that $\bigcup_{i=1}^{\infty} N_i$ is a submodule M.

b) Let R be a ring and let A, B, and C be the submodules of an R-module M such that $C \supseteq A$. Prove that $A + (B \cap C) = (A + B) \cap C$.

Q 7. a) Let R be a ring, M and N be two R-modules and let $f : M \longrightarrow N$ and $g : N \longrightarrow M$ be the module homomorphisms such that $g \circ f : M \longrightarrow M$ is identity. Show that $N = Imf \oplus Kerg$.

b) Let M be an R-module and let $m \in M$. Define the order ideal $O(m)$. Consider $\mathbb{Z}_{10} = \{\bar{0}, \bar{1}, ..., \bar{9}\}$ as a $\mathbb{Z}$-module. Find $O(\bar{5})$ and $O(\bar{6})$.

Q 8. a) Let R be an integral domain, M be an R-module and let T be the set of all torsion elements of M. Prove that T is a submodule of M and the quotient module M/T is a torsion free R-module.

b) Give an example of a free module in which a linearly independent set cannot be extended to a basis.

Q 9. a) Let R be a ring and let N ba a submodule of an R-module M. Prove that if both N and M/N are finitely generated R-modules then M is also a finitely generated R-module.

b) Show that $\mathbb{Q}$ is not free as a $\mathbb{Z}$-module.

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Examination – 2020

Roll No. ....

Subject: Mathematics | Paper: IV-VI (Opt. v) (Number Theory) | Time: 3 Hrs. | Marks: 100

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

SECTION – I

Q.1 (a) Every integer $n > 1$ can be expressed uniquely as a product of primes in the form **(10)**

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r},$$

where $p_1 < p_2 < \cdots < p_r$ are distinct primes and $\alpha_1, \alpha_2, \cdots, \alpha_r$ and r are positive integers.

(b) Let $n > 1$ such that $(n-1)! \equiv -1 \ (mod\ n)$ then show that n is prime. **(10)**

Q.2 (a) State and prove Lagrange's Theorem for the solution of polynomial congruences modulo a prime number. **(10)**

(b) If p is a prime, then prove that for any integer "a" **(10)**

$$a^p \equiv a(mod\ p)$$

Q.3 (a) State Fermat's last theorem and prove it for $n = 4$. **(10)**

(b) State Mobius Inversion formula. With usual notations, prove that **(2+8)**

$$\frac{\varphi(n)}{n} = \sum_{d|n} \frac{\varphi(d)}{d}$$

Q.4 (a) State and prove Euler's Theorem. **(10)**

(b) Define Mersenne prime and perfect number. Suppose that $m \in \mathbb{N}$. If 2^{m-1} is a prime, then the number $2^{m-1}(2^m - 1)$ is an even perfect number. Show that there exists no other perfect number. **(10)**

Q.5 (a) Prove the existence of primitive roots of any prime number. **(10)**

(b) **(i)** If p is a prime, then there exist exactly $\varphi(p-1)$ primitive roots modulo p. **(5+5)**
(ii) Suppose that p is an odd prime. If g is a primitive root modulo p, then $g + np$ is a primitive root modulo p^2 for exactly $p - 1$ values of n modulo p.

SECTION – II

Q.6 (a) Define Legendre and Jacobi symbols. Evaluate **(2+8)**

(i) $\left(\frac{666}{2137}\right) = 1$ (ii) $\left(\frac{402}{991}\right) = -1$

(b) If p and q are distinct odd primes then prove that **(5+5)**

(i) $\left(\frac{p}{q}\right) = -\left(\frac{q}{p}\right), p, q \equiv 3(mood\ 4)$

(ii) $\left(\frac{p}{q}\right) = \left(\frac{p}{q}\right), p\ or\ q \equiv 1(mood\ 4)$

Q.7. (a) Define a primitive polynomial, and show that the product of primitive polynomial is primitive. **(2+8)**

(b) If K is the fine extension over F, and E over K. **(10)**
Then prove that E is finite over F. Hence deduced that if K is of degree n over F, then any element of K is algebraic over F of degree dividing n.

Q.8 (a) Define an algebraic number. Find the polynomial for the algebraic number **(10)**

$$\sqrt{5+\sqrt{7}}.$$

(b) Prove that any two integral bases have same discriminant. **(10)**

Q.9 (a) If ζ is primitive pth root of unity, p an odd prime then show that **(10)**

$$D(\zeta) = (-1)^{\frac{p-1}{2}} p^{p-2}$$

(b) State and prove the existence of transcendental number. **(10)**

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Examination – 2020

Roll No. ....

Subject: Mathematics | Paper: IV-VI (opt.vi) (Fluid Mechanics) | Time: 3 Hrs. | Marks: 100

NOTE: Attempt any FIVE questions by selecting atleast TWO questions from each section. All questions carry equal marks.

Section I

Q.1(a)	A plate having an area of $0.6m^2$ in sliding down the inclined plane at $30°$ to the horizontal with a velocity of $0.36ms^{-1}$. There is a cushion of fluid *1.8 mm* thick between the plane and the plate. Find the viscosity of the fluid if the weight of the plate is *280N*.	(10)
(b)	Derive the Bernoulli's equation for unsteady, irrotational and inviscid flow under conservative forces.	(10)
Q.2(a)	The velocity distribution of flow over a plate is given by $u = 2y - y^2$, where u is the velocity in ms^{-1} at a distance y meters above the plate. Determine the velocity gradient and shear stress at the boundary and 1.5 m from it by taking the fluid viscosity of the fluid as 0.9 Ns/m^2.	(10)
(b)	For the velocity field $\vec{V} = (3x+2)\hat{\imath} + (2y-4)\hat{\jmath} + 5z\hat{k}$, evaluate the volumetric flow rate ($\iint \vec{V} \cdot \hat{n}\, dS$) through the square surface whose vertices are (0, 0, 2), (5, 0, 2), (5, 5, 2) and (0, 5, 2).	(10)
Q.3(a)	Prove that the circulation of an inviscid incompressible fluid around any closed curve moving with the fluid remains constant at all times provided that the external forces are conservative.	(10)
(b)	What are the general methods for describing the fluid motion? Explain those methods.	(10)
Q.4(a)	What is material derivative? What is the local rate of change? Explain convective rate of change in Cartesian coordinates.	(10)
(b)	Define equipotential lines and streamlines. Show that the equipotential lines are orthogonal to the streamlines.	(10)
Q. 5	The complex velocity potential for a certain fluid flow is $w(z) = A\left(z^n + \frac{a^{2n}}{z^n}\right)$, where A and n are real constants. Find the velocity components; speed; stagnation points; velocity potential; streamlines; equipotential lines and streamlines.	(20)

Section II

Q.6	Discuss briefly the flow by superposition of vortex, doublet and uniform stream. Also show that the lift produced is independent of the radius of the cylinder and is equal to the product of the fluid density, the free stream velocity and the circulation.	(20)
Q.7(a)	State and prove the theorem of Kutta and Joukowski.	(10)
(b)	Derive the stress - strain relationship for a Newtonian fluid.	(10)
Q.8(a)	What is the Stokes first problem? Write its mathematical formulation and solve it by Laplace transform for the velocity field.	(10)
(b)	Derive the boundary layer equations for two dimensional viscous incompressible flow.	(10)
Q.9(a)	Define Hagen-Poiseuille flow, Formulate it and calculate the velocity field, average velocity, Maximum velocity and shearing stress.	(10)
(b)	Prove that for a turbulent flow: (i) The mean value of a fluctuating quantity is always zero. (ii) The mean value of the square of a fluctuating quantity is always positive.	(10)

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Examination – 2020 (CLASH)

Roll No. ....

Subject: Mathematics **Paper: IV-VI (opt.vii) [Quantum Mechanics]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

Section-I

1. (a) Discuss the difference between wave and particle phenomena using double slit experiment. (10)

   (b) Describe the quantum mechanical representation of physical observables. Further explain how the momentum operator definition $\hat{p} \equiv -i\hbar\frac{\partial}{\partial x}$ leads to de Broglie wavelength $h = \frac{\lambda}{p}$ (10)

2. (a) Show that $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + \left[\hat{A}, \hat{B}\right] + \frac{1}{2!}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + \frac{1}{3!}\left[\hat{A}, \left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right]\right] + \ldots.$ (10)

   (b) Consider the wavefunction

$$\psi(x,t) = A\exp\left[-\frac{(x-x_0)^2}{4a^2}\right]\exp\left(\frac{ip_0x}{\hbar}\right)\exp\left(-i\omega_0t\right)$$

   Find the real constant A such that $|\psi|^2$ satisfies the Born's postulate. Further show that $<p>= p_0$ and $(\Delta p)^2 = \frac{\hbar^2}{4a^2}$. (10)

3. (a) Let $\hat{\mathbb{P}}_\pm = \frac{1\pm\hat{\mathbb{P}}}{2}$ be the projections of parity operator $\hat{\mathbb{P}}$, defined by $\hat{\mathbb{P}}_\pm f(x) = f_\pm$, where (10)

$$f_\pm = \frac{f(x) \pm f(-x)}{2}.$$

   Show that

   - $(\hat{\mathbb{P}}_\pm)^2 = \hat{\mathbb{P}}_\pm$
   - $[\hat{\mathbb{P}}_+, \hat{\mathbb{P}}_-] = 0$

   (b) Let $\hat{A}$ and $\hat{B}$ be Hermitian operators. (10)

   - Show that $\hat{A}\hat{B}$ is not necessarily Hermitian operator. Further, how can we construct a Hermitian operator using $\hat{A}\hat{B}$?
   - Show that expectation value of $\hat{A}^2$ is non-negative real number i.e., $\langle\hat{A}^2\rangle \geq 0$.

4. (a) Derive reflection and transmission coefficient for rectangular barrier scattering where the energy E of the particles in the incident beam is greater than the height V of potential barrier. (10)

(b) Given that $\phi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$ are the normalized eigenstates of particle in 1-D box problem with eigenenergies $E_n = n^2 E_1$, $n = 1, 2, 3, \ldots$.
If initial state of the particle is given by the superposition $\psi(x,0) = \frac{2}{\sqrt{5}}\phi_1 + \frac{1}{\sqrt{5}}\phi_2$
Find the evolved wave function $\psi(x,t)$. Further show that expectation value of the energy operator $\hat{H}$ remains constant. (10)

Section-II

5. (a) Let $|n\rangle$ be the n^{th} eigenfunction (Dirac notation) of quantum mechanical harmonic oscillator. If $\hat{a}$ and $\hat{a}^{\dagger}$ denote annihilation and creation operators, respectively, then evaluate (10)

- $\langle n|\hat{x}|n\rangle$,
- $\langle n|\hat{x}^2|n\rangle$; where $\hat{x} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^{\dagger}\right)$.

(b) Find the first order corrections, $E_n^{(1)}$, to the eigen energies of Anharmonic oscillator whose perturbation Hamiltonian is given by $\hat{H}' = K'x^4$, where $K' \ll 1$ and $\hat{x} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^{\dagger}\right)$. (10)

6. (a) How does the twofold-degenerate energy $E = 2\hbar\omega_0$ of the two-dimensional harmonic oscillator separate due to the perturbation $H' = K'xy$, where $\hat{x} = \frac{1}{\sqrt{2\beta}}\left(\hat{a} + \hat{a}^{\dagger}\right)$ and $\hat{y} = \frac{1}{\sqrt{2\beta}}\left(\hat{b} + \hat{b}^{\dagger}\right)$? (10)

(b) Find expressions for first order corrections to the eigenvalues and eigenfunctions for very small perturbation to the time independent, non-degenerate Hamiltonian $\hat{H}_0$. (10)

7. (a) Suppose that a rigid rotator is in eigenstate of $\hat{L}^2$ and $\hat{L}_z$ corresponding to $l = 1$ and $m = +1$. What is the probability that measurement of L_x finds the respective values $m = 0, \pm 1$? (10)

(b) Prove the following statements. (10)

- $[\hat{L}_+, \hat{L}_-] = 2\hbar\hat{L}_z$, where $\hat{L}_\pm$ are ladder operators.
- If ϕ_m is an eigenfunction of $\hat{L}_z$ for eigenvalue $\hbar m$, then show that $\hat{L}_+\phi_m$ is also an eigenfunction of $\hat{L}_z$ corresponding to eigenvalue $\hbar(m+1)$.

8. (a) Explain scattering cross section for 3-D scattering and obtain general expression. (10)

(b) Using Born's approximation, compute the phase shift δ_1 for scattering in a centrally symmetric field. (10)

9. (a) Derive an expression for $\hat{L}_z$ in spherical polar co-ordinates. (10)

(b) Compute the expression for ionization potential of hydrogen, helium and lithium atoms. (10)

UNIVERSITY OF THE PUNJAB

Roll No. ....

M.A./M.Sc. Part – II Annual Examination – 2020

Subject: Mathematics **Paper: IV-VI (opt.vii) [Quantum Mechanics]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

Section-I

1. (a) Discuss how did Bohr atomic model explain the discrete emission spectra of Hydrogen atom. (10)

   (b) Describe the quantum mechanical representation of physical observables. Further explain how the momentum operator definition $\hat{p} \equiv -i\hbar\frac{\partial}{\partial x}$ leads to de Broglie wavelength $h = \frac{\lambda}{p}$ (10)

2. (a) The displacement operator $\hat{D}$ is defined by the equation

$$\hat{D}f(x) = f(x+\xi).$$

   Show that the eigenfunctions of $\hat{D}$ are of the form

$$\phi_\beta = e^{\beta x}g(x),$$

   where $g(x+\xi) = g(x)$. (10)

   (b) Consider the wavefunction

$$\psi(x,t) = A\exp\left[-\frac{(x-x_0)^2}{4a^2}\right]\exp\left(\frac{ip_0x}{\hbar}\right)\exp\left(-i\omega_0 t\right)$$

   Find the real constant A such that $|\psi|^2$ satisfies the Born's postulate. Further show that $< x >= x_0$ and $(\Delta x)^2 = a^2$ (10)

3. (a) Define the parity operator $\hat{\mathbb{P}}$ and find its eigenvalues and eigenfunctions. Also determine the parities of the following wavefunctions (10)

   - $\psi(x,y,z) = (x+y+z)e^{-(x^2+y^2+z^2)}$
   - $\psi(r,\theta) = re^{-r^2}\sin\theta$

   (b) Let $\hat{A}$ and $\hat{B}$ be Hermitian operators. (10)

   - Show that $\hat{A}\hat{B}$ is not necessarily Hermitian operator. Further, how can we construct a Hermitian operator using $\hat{A}\hat{B}$?
   - Show that expectation value of $\hat{A}^2$ is non-negative real number i.e., $\langle \hat{A}^2 \rangle \geq 0$.

4. (a) Let $\psi(x,t)$ represents the wavefunction of an unbound state and $\mathbb{J}_x$ be the corresponding current density in 1-D dimension barrier problem. Then show that (10)

$$\mathbb{J}_x = \frac{\hbar}{2mi}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right).$$

(b) Let $\psi(\mathbf{r},0)$ represents the wave or state function of a quantum mechanical system at time $t=0$ with time independent Hamiltonian $\hat{H}$. i.e., $\frac{\partial}{\partial t}\hat{H}=0$. Show that the evolved state function, $\psi(\mathbf{r},t)$ at some later time $t>0$, is given by

$$\psi(\mathbf{r},t)=\exp\left(-i\frac{t\hat{H}}{\hbar}\right)\psi(\mathbf{r},0).$$

Further discuss the case when $\psi(\mathbf{r},0)$ is an eigenfunction, $\phi_n(\mathbf{r})$ of $\hat{H}$. That is, $\hat{H}\phi_n(\mathbf{r})=E_n\phi_n(\mathbf{r})$. (10)

Section-II

5. (a) Let $|n\rangle$ be the n^{th} eigenfunction (Dirac notation) of quantum mechanical harmonic oscillator. If $\hat{a}$ and $\hat{a}^\dagger$ denote annihilation and creation operators, respectively, then show that (10)

   - $\left[\hat{a},\hat{a}^\dagger\right]=\mathbb{I}$.
   - $\langle n|\hat{x}|n\rangle=0$; where $\hat{x}=\frac{1}{\sqrt{2}}\left(\hat{a}+\hat{a}^\dagger\right)$.

   (b) Find the first order corrections, $E_n^{(1)}$, to the eigen energies of Anharmonic oscillator whose perturbation Hamiltonian is given by $\hat{H}'=K'x^4$, where $K'\ll 1$ and $\hat{x}=\frac{1}{\sqrt{2}}\left(\hat{a}+\hat{a}^\dagger\right)$. (10)

6. (a) Assume that a particle has an orbital angular momentum with z component $\hbar m$ and square magnitude $\hbar^2 l(l+1)$. Show that (10)

   - $\langle L_x\rangle=\langle L_y\rangle=0$
   - $\langle L_x^2\rangle=\langle L_y^2\rangle=\frac{\hbar^2 l(l+1)-m^2\hbar^2}{2}$

   (b) Find expressions for first order corrections to the eigenvalues and eigenfunctions for very small perturbation to the time independent, non-degenerate Hamiltonian $\hat{H}_0$. (10)

7. (a) Suppose that a rigid rotator is in eigenstate of $\hat{L}^2$ and $\hat{L}_z$ corresponding to $l=1$ and $m=-1$. What is the probability that measurement of L_x finds the respective values $m=0,\pm1$? (10)

   (b) Prove the following statements. (10)

   - $[\hat{L}_+,\hat{L}_-]=2\hbar\hat{L}_z$, where $\hat{L}_\pm$ are ladder operators.
   - If ϕ_m is an eigenfunction of $\hat{L}_z$ for eigenvalue $\hbar m$, then show that $\hat{L}_-\phi_m$ is also an eigenfunction of $\hat{L}_z$ corresponding to eigenvalue $\hbar(m-1)$.

8. (a) Explain scattering cross section for 3-D scattering and obtain general expression. (10)

   (b) Explain scattering amplitude for 3-D scattering and obtain general expression relating it to scattering cross section. (10)

9. (a) Derive an expression for $\hat{L}^2$ in spherical polar co-ordinates. (10)

   (b) Obtain the radial part of the Scrhödinger's wave equation. Also find the allowed eigen energies for infinite spherical well given by (10)

$$V(r)=\begin{cases}0, & r\le a;\\ \infty, & r>a.\end{cases}$$

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Examination – 2020

Roll No. ....

Subject: Mathematics

Paper: IV-VI (Opt.viii) (Special Theory of Relativity and Analytical Dynamics) **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

SECTION – I

Q.1 (a) If there are two boats A and B in a river of width D flowing with speed V, find the difference of time taken by the two boats, when one of the boats A crosses the river to the point directly opposite to the stationary point and returns back to its initial point whereas the other boat B heads down the stream and covers the same distance D as that of the first boat and then returns to its initial point. **(10)**

(b) Explain Michelson Morley experiment using reasoning developed in the analogy in Q. 1 (A). **(10)**

Q.2 (a) Use Lorentz transformations to explain: **(10)**
(a) Non-synchronism of clocks
(b) Relativity of Simultaneity
(c) Relativistic composition of accelerations.

(b) Formulate the energy momentum transformations and show that $p^2 - E^2/c^2$ is Lorentz invariant. **(10)**

Q.3 (a) Show that the relativistic resultant of three co-linear speeds u_1, u_2, u_3 is given by **(10)**

$$\frac{u_1+u_2+u_3+(u_1u_2u_3)/c^2}{1+(u_1u_2+u_2u_3+u_3u_1)/c^2}.$$

(b) What are time like, space like and light like vectors? A null cone? For the Minkowski Metric $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$, identify the vectors as time like, space like or light like:
(a) $T^\mu = (1, 0, 0, 0)$, (b) $X^\mu = (0, 1, 0, 0)$, (c) $Y^\mu = (0, 0, 1, 0)$, (d) $Z^\mu = (1, 0, 1, 0)$.
Show that the only non-vanishing inner products between vectors are $T^2 = -X^2 = -Y^2 = 1$. **(10)**

Q.4 (a) If the angle between the direction of motion of a light source of frequency ν_0 and the direction from it to an observer is ϑ, then show that expression for the frequency ν the observer finds is given by $\nu = \nu_0 \frac{\sqrt{1-v^2/c^2}}{1-(v/c)\cos\vartheta}$ where v is the relative speed of the source.
Hence find frequency when the observer is:
(a) receding away from the light source, (b) approaching the light source,
(c) moving perpendicular to a line between him and the light source. **(10)**

(b) What is Comptons Effect. Derive the expression for Compton shift and hence deduce the expression for the recoiling electrons energy and momentum. **(10)**

Q.5 (a) Derive the expression for $F^{\mu\nu}$, the Maxwell field tensor and ${}^*F^{\mu\nu}$, the dual field tensor. **(10)**

(b) Using matrix form of Maxwell field tensor show that: **(10)**
(a) $F^{\mu\nu} = -F^{\nu\mu}$, (b) $F^{\mu\nu}F_{\mu\nu} = -2\mu_0(\varepsilon_0 \mathbf{E}\bullet\mathbf{E} - \mathbf{B}\bullet\mathbf{B}/\mu_0)$, (c) ${}^*F^{\mu\nu}F_{\mu\nu} = 4(\mathbf{E}\bullet\mathbf{B}/c)$.

SECTION – II

Q.6 (a) What is the difference between usual derivative d/dt and δ–variation of position vector $r_v = r_v\ (t, q_s)$, $v = 1, 2, \ldots N$, $s = 1, 2, \ldots n$ of a particle? Prove that $d\ \delta = \delta\ d$ with the usual notation, where d denotes the usual derivative. **(10)**

(b) What are generalized coordinates? Generalized velocities? Give example in each case. State and prove D'Alembert Principle for a dynamical system. **(10)**

Q.7 (a) Derive Lagrange's EOM for a holonomic system. Define a conservative system and show that Lagrange's EOM reduce to: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = \frac{\partial L}{\partial q_\alpha}$ Where $L = T - V$. **(10)**

(b) Use Lagrange's EOM to describe the motion of a projectile launched with speed V_0 at angle α with the horizontal. **(10)**

Q.8 (a) What is a simple harmonic oscillator (SHO)? Prove that the force $F = -k\,x\,i$ acting on the SHO is conservative. Set up the Lagrangian and discuss the motion. **(10)**

(b) State and prove Hamilton's principle of least action. Derive Hamilton's equations of motion from this principle. **(10)**

Q.9 (a) What do you understand by Energy Integral? Hence derive Whittaker's equation of motion using Energy Integral. Illustrate with the help of an example. **(10)**

(b) Let (F, G) be the Poisson bracket then find the expression for the Poisson brackets **(a)** (F_1F_2, G), **(b)** $\partial(F, G)/\partial t$, **(c)** $(F_1, (F_2, F_3))$ and **(d)** (F, H) where H is the Hamiltonian and F, F_1, F_2, F_3, G are arbitrary functions depending on q_i, p_i and t. **(10)**

UNIVERSITY OF THE PUNJAB

Roll No. ....

M.A./M.Sc. Part – II Annual Examination – 2020

Subject: Mathematics **Paper: IV-VI (opt. ix) (Electromagnetic Theory)** **Time: 3 Hrs. Marks: 100**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

SECTION-I

1. (a) Find the relation between incident, reflected and transmitted waves.
   (b) There are two small identical conducting spheres having charges $3 \times 10^{-8}C$ and $-0.5 \times 10^{-9}C$, respectively, when they are placed 5cm apart. How much force they exert on each other?
2. (a) Find the potential at any point of the electric field in the presence of conducting plates.
   (b) Two-point charges are at a distance 'a' apart along the z axis. Find the electric field at any point in the $z=0$ plane when the charges are of opposite polarity but equal magnitude?
3. (a) Discuss the relation between potential and electric field.
   (b) Describes the effect of time on free charge density.
4. (a) Deduce the Lorentz condition and discuss it for electrostatic field.
   (b) Calculate the electromotance induced in a loop by a pair of long parallel wires carrying a variable current.
5. (a) Find the potential difference between two points "a" and "b" ($b > a$) lying on spherical radial line ($\theta, \emptyset = constant$) from the origin.
   (b) What would be the magnitude of uniform electric field if it has to carry the same energy density as that possessed by a $0.4\ T$ magnetic field.

SECTION-II

6. (a) Drive the electromagnetic wave equations for field vectors.
   (b) Use the Ampere-Maxwell law to find the magnetic field between the circular plates of a parallel-plate capacitor that is charging. The radius of the plates is R. Ignore the fringing field.
7. (a) Show that Poynting vector points everywhere radially into the cylindrical volume.
   (b) A radio station transmits a *10-kW* signal at a frequency of *100 MHz.* For simplicity, assume that it radiates as a point source. At a distance of *1 km* from the antenna, find: (i) the amplitude of the electric and magnetic field strengths, and (ii) the energy incident normally on a square plate of side *10 cm* in *5 min.*
8. (a) Work out the coefficients of reflection and transmission at an interface using Fresnel's equations for case when $\overline{E}$ is polarized normal to the plane of incidence.
   (b) Discuss the transverse electric waves in a hollow cylindrical waveguide.
9. (a) State and prove Snell's law.
   (b) Light travels from air into an optical fiber with an index of refraction of 1.44. (i) In which direction does the light bend? (ii) If the angle of incidence on the end of the fiber is 22°, what is the angle of refraction inside the fiber? (iii) Sketch the path of light as it changes media

UNIVERSITY OF THE PUNJAB

Roll No. ....

M.A./M.Sc. Part – II Annual Examination – 2020

Subject: Mathematics **Paper: IV-VI (opt. x) [Operations Research]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions selecting at least TWO questions from each section.

SECTION-I

Q.1. Garima Enterprises manufactures three types of dolls. The boy-doll requires ½ meter of red cloth, 1½ meters of green and 1½ meters of black cloth and 5 kg of fibre. The girl-doll requires ½ meter of red cloth, 2 meters of green cloth and 1 meter of black and 6 kg of fibre. The dog-doll requires ½ meter of red, 1 meter of green, ¼ meter of black, and 2 kg of fibre. The profits on the three are Rs. 3.00, Rs. 5.00 and Rs. 2.00, respectively. The firm has 1000 meters of red, 1500 meters of green, 2000 meter of black and 6000 kg of fibre. Set up a linear programming model and use graphical method to find the number of dolls of each type to be manufactured for maximum profit. (20)

Q.2. Solve the following LP model by using two phase method (20)

$$Max\ z = 5x_1 - 4x_2 + 3x_3$$

Subject to

$$2x_1 + x_2 - 6x_3 = 20$$
$$6x_1 + 5x_2 + 10x_3 \le 76$$
$$8x_1 - 3x_2 + 6x_3 \le 50$$
$$x_1, x_2, x_3 \ge 0$$

Q.3. a) Prove that the dual of dual of a primal is the primal LPP. (6)

b) Solve the dual of the following LP model. Also determine the optimal values of primal variables using optimal dual solution. (14)

$$Min\ z = 20x_1 + 10x_2$$

Subject to

$$x_1 + x_2 \ge 10$$
$$3x_1 + 2x_2 \ge 24$$
$$x_1, x_2 \ge 0$$

Q.4. Given the following LP model (20)

$$Max\ z = 10x_1 + 15x_2 + 20x_3$$

Subject to

$$2x_1 + 4x_2 + 6x_3 \le 24$$
$$3x_1 + 9x_2 + 6x_3 \le 30$$
$$x_1, x_2, x_3 \ge 0$$

Find the optimal solution. Also find the optimality ranges. Moreover, check whether optimality is affected, if profit coefficients are changed from (10, 15, 20) to (7, 14, 15). If so, find the revised optimum solution.

Q.5. a) Find the non-degenerate basic feasible solution for the following Transportation problem using: (8)

(i) North-west corner method

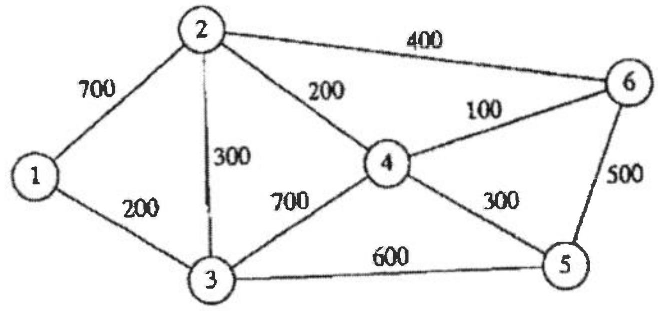
(ii) Vogel's Approximation method.

	1	2	3	4	Supply
1	10	20	5	7	10
2	13	9	12	8	20
3	4	5	7	9	30
4	14	7	1	9	40
5	3	12	5	19	50
Demand	60	60	20	10	

b) Write a note on UV multiplier method for solving transportation model. (12)

P.T.O

SECTION-II

Q.6.	The Warid mobile phone company services six geographical areas. The satellite distances (in miles) among the six areas are given in the following network. Warid needs to determine the most efficient message routes that should be established between each two areas in the network. Use Floyd's algorithm to solve the problem. 	(20)
Q.7.	Solve the following by mixed fractional cut method $Max\ z = 3x_1 + x_2 + 3x_3$ Subject to $-x_1 + 2x_2 + x_3 \leq 4$ $4x_1 - 3x_3 \leq 2$ $x_1 - 3x_2 + 2x_3 \leq 3,$ where x_1 and x_3 are non-negative integers and $x_2 \geq 0$.	(20)
Q.8.	Use branch and bound method to solve the following integer programming problem $Max\ z = 7x_1 + 9x_2$ Subject to $-x_1 + 3x_2 \leq 6$ $7x_1 + x_2 \leq 35$ $x_2 \leq 7,$ where $x_1, x_2 \geq 0$ are non- negative integers.	(20)
Q.9.	Use bounded variable algorithm to solve the following LP model: $Max\ z = 3x_1 + 5x_2 + 2x_3$ Subject to $x_1 + 2x_2 + 2x_3 \leq 14$ $2x_1 + 4x_2 + 3x_3 \leq 23$ $0 \leq x_1 \leq 4, \quad 2 \leq x_2 \leq 5, \quad 0 \leq x_3 \leq 3.$	(20)

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Annual Examination – 2020

Roll No. ....

Subject: Mathematics **Paper: IV-VI (opt.xi) [Theory of Approximation & Splines]** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions selecting at least TWO questions from each section.

SECTION I

Q1. (a) Find the least-squares line $y = f(x) = Ax + B$ for the data and calculate $E_2(f)$ (10)

x_k	y_k	$f(x_k)$
-2	1	1.2
-1	2	1.9
0	3	2.6
1	3	3.3
2	4	4.0

(b) Show that the composition of reflection and rotation is reflection. (10)

Q2. (a) Derive the normal equations for finding the least-squares parabola $y = Ax^2 + B$. (10)

(b) Establish the Pade approximation (10)

$$e^x \approx R_{3,3}(x) = \frac{120 + 60x + 12x^2 + x^3}{120 - 60x + 12x^2 - x^3}$$

Q3. (a) Determine the image of the circle $x^2 + y^2 = 4$, under the transformation of reflection through the angle $\frac{\pi}{2}$. (10)

(b) Discuss in detail the properties of Chebyshev polynomials. (10)

Q4. (a) Prove that the every ellipse is affine-congruent to the unit circle with equation $x^2 + y^2 = 1$. (10)

(b) Find the maximum and minimum values of Chebyshev polynomial $T_3(x)$ over the interval $[-1, 1]$. (10)

SECTION II

Q5. (a) Let $t(x) = \sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{k} c_i (x - x_i)_+^n$

Determine whether $t^{(n)}(x_i^+) = t^{(n)}(x_i^-),\ i = 0,1,2,\ldots,k$ (10)

(b) Find the cubic spline $S(x)$ that passes (1,4), (2,0.7), (3,6) and (4,3.75) with the boundary conditions $S'(1) = 4$, $S'(4) = -3.5$. (10)

P.T.O

Q6. **(a)** Define uniform B-Spline. Show that $N_0^{(4)}(t)$ is a spline of degree 3. (10)

(b) Show that $\theta = \sum_{i=0}^{n} B_i^n(\theta)\frac{i}{n}$ (10)

Q7. **(a)** For the control point form

$$\underline{P(\theta)} = (1-\theta)^2(2\theta - k\theta + 1)\underline{b_0} + k(1-\theta)^2\theta\underline{b_1} + k(1-\theta)\theta^2\underline{b_2} + \theta^2(-2\theta + k\theta + 3 - k)\underline{b_3}.$$

Show that $k=3$ is the Bernstein Bezier cubic form. Also show that $\underline{P(\theta)}$ satisfy the convex hull property. (10)

(b) Calculate new control points for Bernstein Bezier cubic form for the interval $[\frac{2}{3},1]$ using subdivision algorithm. (10)

Q8. **(a)** Derive the degree raising algorithm to express $\underline{P}(\theta) = \sum_{i=0}^{n} B_i^n(\theta)\underline{b}_i,\ \theta \in [0,1]$ as

$$\underline{P}(\theta) = \sum_{i=0}^{n+1} B_i^{n+1}(\theta)\underline{b}_i^*,\ \theta \in [0,1]$$

(10)

(b) Determine a function $P:[0,2] \to \Re$

such that P is cubic over $[0,1[$ and quadratic over $[1,2]$ Moreover P, P', P'' are continuous and $P(0)=1$, $P(1)=2$, $P(2)=5$, $P'(1)=\frac{3}{2}$. (10)

Q9. **(a)** Show that the Bernstein Bezier rational quadratic form satisfies the convex hull property. (10)

(b) Using De Casteljau's algorithm, find the value of Bernstein Bezier curve

$$P(\theta) = \sum_{i=0}^{3} B_i^3(\theta)\underline{b}_i \quad \text{at } \theta = \frac{1}{3}.$$

(10)

UNIVERSITY OF THE PUNJAB

Roll No. ....

M.A./M.Sc. Part – II Annual Examination – 2020

Subject: Mathematics **Paper: (IV-VI) (Opt. xii) (Advanced Functional Analysis)** **Time: 3 Hrs.** **Marks: 100**

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.

SECTION – I

Q.1 **(a)** For any metric space (X,d)), there exists a complete metric space $\hat{X} = (\hat{X}, \hat{d})$ which has a subspace W that is isometric with X and is dense in $\hat{X}$. This space $\hat{X}$ is unique except for isometries, that is, if $\tilde{X}$ is any complete metric space having a dense subspace $\tilde{W}$ isometric with X, then $\tilde{X}$ and $\hat{X}$ are isometric. (15)

(b) Show that $C[a,b]$ is complete. (5)

Q.2 **(a)** (i) Prove that the set Q of rational numbers is of the first category. (10)

(ii) Prove that the set Z of integers is nowhere dense in real line R.

(b) Let S be a closed subspace of a Banach space $(N, \|.\|)$. Show that the quotient space N/S is also a Banach space with the norm defined by $\|x + S\|_1 = \inf_{x \in S} \|x + s\|$. (10)

Q.3 **(a)** Let $\{e_1, e_2, \ldots, e_n, \ldots\}$ be an orthonormal set in a Hilbert space H, for any choice of scalars $\{c_k\}$, prove that the following statements are equivalent: (10)

(i) $\{c_k\} \in l^2$ (ii) $\sum_{k=1}^{\infty} c_k e_k$ converges in H

(iii) there is an element $x \in H$ such that $\langle x, e_k \rangle = c_k$, $k = 1, 2, \ldots$

(b) If Y is a closed subspace of a Hilbert space H, then show that H can be written as a direct sum of Y and its orthogonal complement. (10)

Q.4 **(a)** Let H be a Hilbert space and f be any linear functional on H, then there exists a unique $y \in H$ such that $f(x) = \langle x, y \rangle$ for all $x \in H$. (10)

(b) Prove that every bounded linear functional f on a Hilbert space H can be uniquely represented as $f(x) = \langle x, z \rangle$ where z depends on f and $\|z\| = \|f\|$. (10)

SECTION – II

Q.5 **(a)** State and prove Hahn-Banach theorem for complex spaces. (10)

(b) Let X be the vector space of all polynomials of degree ≤ 2. Let ϕ_1, ϕ_2, ϕ_3 be linear functional defined by $\phi_1(f(t)) = \int_0^1 f(t)dt$, $\phi_2(f(t)) = f'(1)$, $\phi_3(f(t)) = f(0)$. Find the basis $\{f_1, f_2, f_3\}$ of X that is dual to $\{\phi_1, \phi_2, \phi_3\}$. (10)

Q.6 **(a)** Verify that the space $\mathrm{NBV}[a,b]$ Is the conjugate space of $C[a,b]$. (10)

(b) Find the norm of the linear functional f on $C[-1,1]$ defined by: (10)

$$f(x) = \int_{-1}^{0} x(t)dt - \int_{0}^{1} x(t)dt.$$

Q.7 **(a)** For any $a = (a_1, a_2, \ldots a_n) \in R^n$ define $f_a : R^n \to R$ by $f_a(x) = \sum_{i=1}^{n} a_i x_i, x \in R^n$. Prove that (i) f_a is linear functional (ii) f_a is bounded (iii) $\|f_a\| = \|a\|$. (10)

(b) State and prove Riesz representation theorem for bounded linear functionals on $C[a,b]$. (10)

Q.8 **(a)** If X is a normed space and $x_n \xrightarrow{w} x$, then there exists some positive constant M such that $\|x_n\| < M$ for all n. (10)

(b) If $\dim(X) < \infty$ then show that weak convergence implies strong convergence. (10)

Q.9 **(a)** Show that the dual space of l^1 is l^∞. (10)

(b) Show that a finite dimensional vector space is algebraically reflexive. (10)

UNIVERSITY OF THE PUNJAB

Roll No. ....

M.A./M.Sc. Part – II Annual Examination – 2020

Subject: Mathematics Paper: (IV-VI) (Opt. xiv) (Theory of Optimization) Time: 3 Hrs. Marks: 100

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.

SECTION – A

Q.1. Find the primal problem of

$$\text{Minimize } 1500y_1+1575y_2,$$

Subject to
$$4y_1+5y_2+y_3 \geq 13$$
$$5y_1+3y_2+2y_3 \geq 11, \qquad y_1, y_2, y_3 \geq 0$$

Apply the slack variables and check the feasible solution in dual case. **[20]**

Q.2. a) State and Prove the Weierstrass Theorem. **[10]**

b) Check by using The Weierstrass Theorem.

Let $\mathbf{D}=[-1,1]$, and let f be given by: $f(x)=\begin{cases} 0, & \text{if } x=\pm 1 \\ x, & \text{if } x=-1<x<1 \end{cases}$ **[10]**

Q.3. a) State Kuhn & Tucker Theorem. **[10]**

b) Maximize $\ln x+\ln y$

subject to $x^2+y^2 \quad x,y\geq 0$ by Using Kuhn & Tucker Conditions. **[10]**

Q.4. a) State and prove Strong duality theorem. **[10]**

b) Use dual simplex method to solve following LP problem **[10]**

$$\max z = 2x_1 + x_2$$
$$\text{s.t } x_1+x_2-x_3 \geq 5$$
$$x_1-2x_2+4x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0$$

Q.5. a) Identify the property of function under which the Kuhn Tucker conditions are both necessary and sufficient. Also minimize f(**x**) subject to $g_j(\mathbf{x}) \leq 0$ for j= 1, 2, 3....p where **x** is in $\mathbf{R}^n$ by using Kuhn-Tucker condition. **[10]**

b) Minimize $\mathbf{f}=x_1^2+x_2^2+60$ subject to constraints

$g_1 = x_1 - 80 \geq 0,$

$g_2 = x_1 + x_2 - 120 \geq 0$ using Kuhn Tucker condition. **[10]**

SECTION – B

Q.6. a) State and briefly explain the condition used for finding the weak extrema of a function. **[10]**

b) For F(x, y, y`) = 2x + xyy` +y`^{2}+y, in order to find the weak extrema identify and find the condition using given function. **[10]**

Q.7. Gout disease is characterized by excess of uric acid in blood,

- define level of uric acid to be x(t)
- in absence of any control, tends to 1 according to.

x` = 1−x

drugs are available to control disease (control u), x` = 1−x−u aim to reduce x to zero as quickly as possible, drug is expensive, and unsafe (side effects). Formulate the problem with constraints and find Hamiltonian. **[20]**

Q.8. a) Define Principle of optimality and the optimality equations in different cases. **[10]**

b) John has $20,000 to invest in three funds F1, F2 and F3. Fund F1 is offers a return of 2% and has a low risk. Fund F2 offers a return of 4% and has a medium risk. Fund F3 offers a return of 5% but has a high risk. To be on the safe side, John invests no more than $3000 in F3 and at least twice as much as in F1 than in F2. Assuming that the rates hold till the end of the year, what amounts should he invest in each fund in order to maximize the year end return? **[10]**

Q.9. a) State and derive Bellman's equation? **[10]**

b) A farmer plans to mix two types of food to make a mix of low cost feed for the animals in his farm. A bag of food A costs $10 and contains 40 units of proteins, 20 units of minerals and 10 units of vitamins. A bag of food B costs $12 and contains 30 units of proteins, 20 units of minerals and 30 units of vitamins. How many bags of food A and B should the consumed by the animals each day in order to meet the minimum daily requirements of 150 units of proteins, 90 units of minerals and 60 units of vitamins at a minimum cost? **[10]**