Integrable Systems (1834-1984)

Da-jun Zhang, Shanghai University


F
E

Da-jun Zhang

- Integrable Systems
- Discrete Integrable Systems (DIS)
- Collaborators:

    Frank Nijhoff (Leeds),

    Jarmo Hietarinta (Turku)

    R. Quispel, P. van der Kamp (Melbourne)

6 lectures

- A brief review of history of integrable systems and soliton theory (2hrs)
- Lax pairs of Integrable systems (2hrs)
- Integrability: Bilinear Approach (6hrs=2hr $\times$ 3)

  2hrs for 3-soliton condition bilinear integralility

  2hrs for Bäcklund transformations

  and vertex operators

  2hrs for Wronskian technique
- Discrete Integrable Systems: Cauchy matrix approach (2hrs)

Integrable Systems

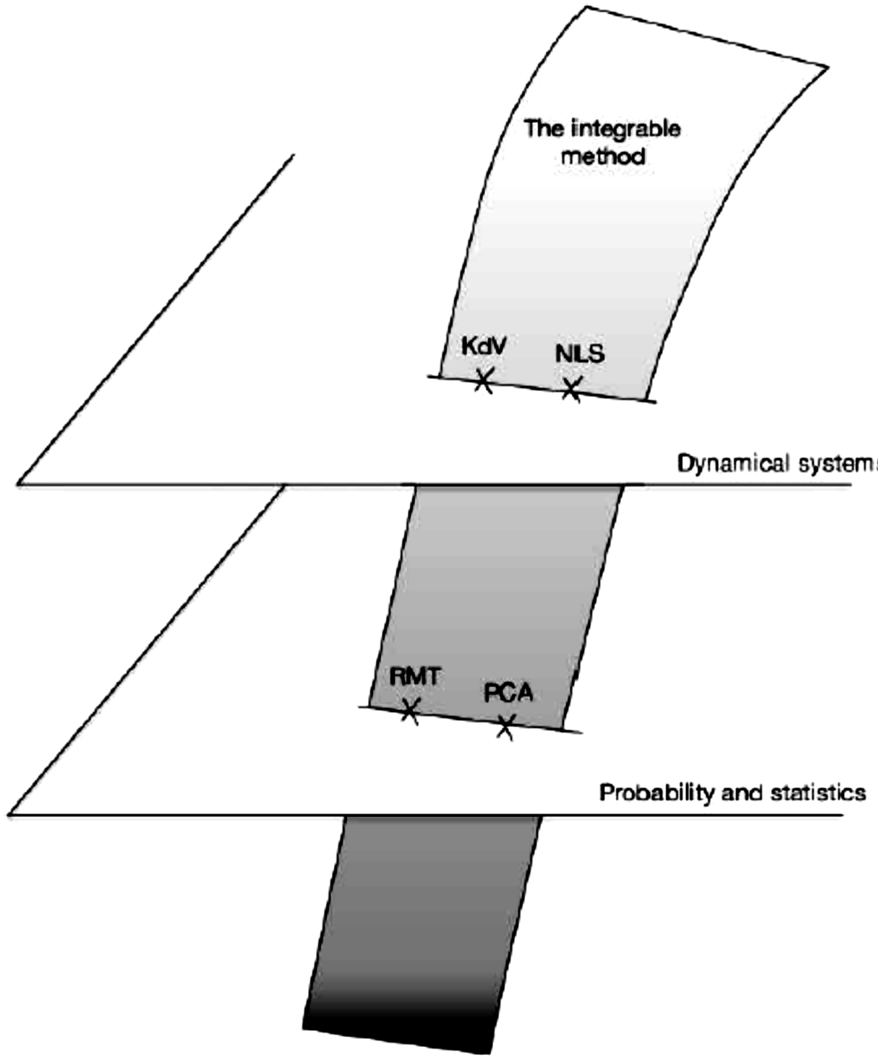
- How does one determine if a system is integrable and how do you integrate it? …. categorically, that I believe there is no systematic answer to this question. **Showing a system is integrable is always a matter of luck and intuition.**
- Viewpoint of Percy Deift on Integrable Systems

  Linearisable-resolve-clear dependence of parameters

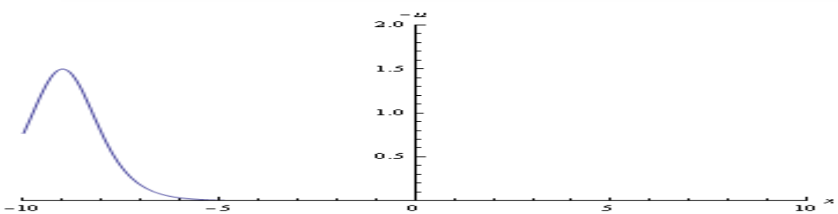
  trigonometric, special, Pinleve functions
- "Fifty Years of KdV: An Integrable System"

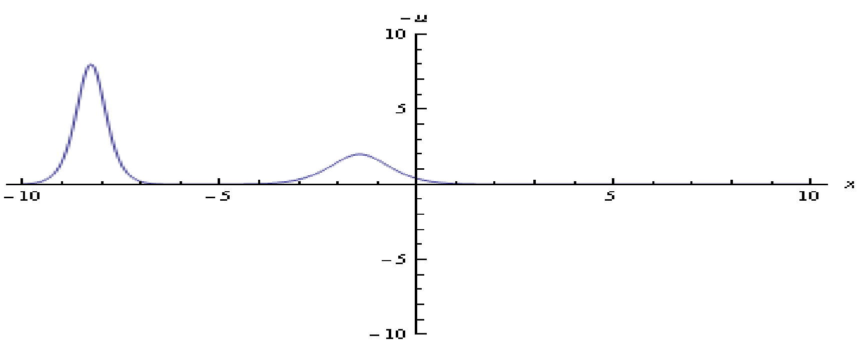
Interaction of Integrable Methods

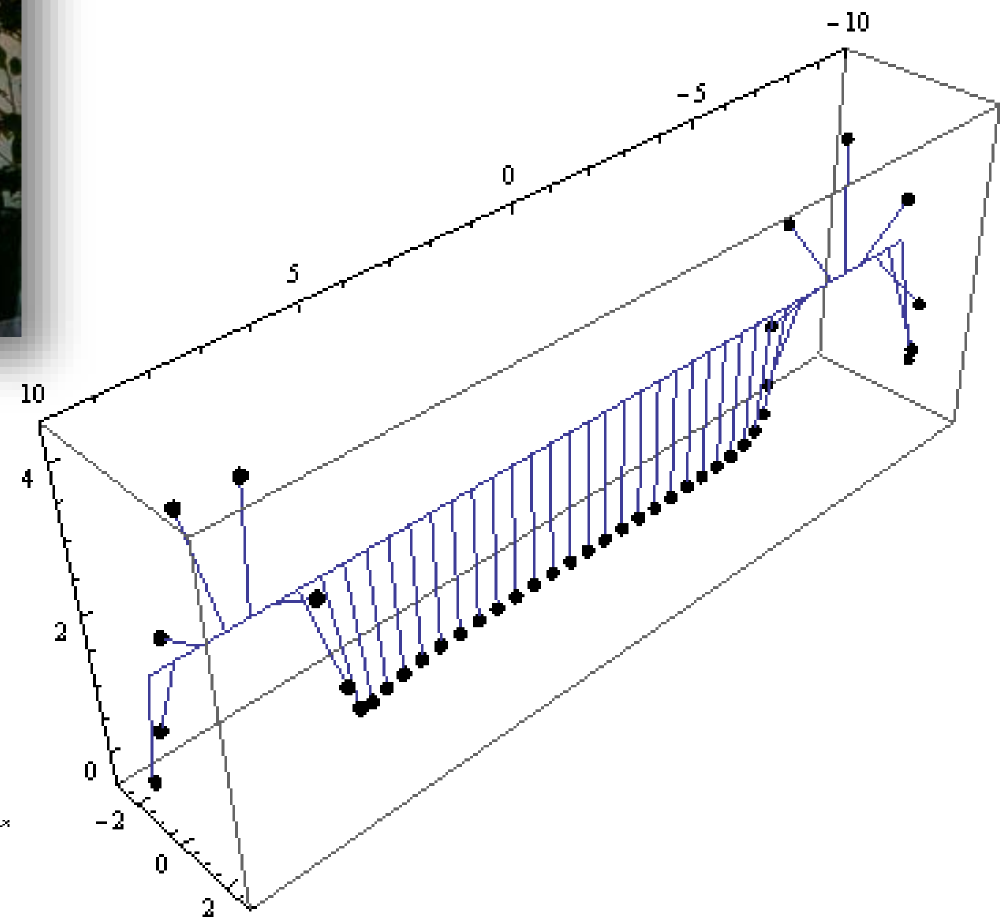
- dynamical systems
- probability theory and statistics
- geometry
- combinatorics
- statistical mechanics
- classical analysis
- numerical analysis
- representation theory
- algebraic geometry
- ……

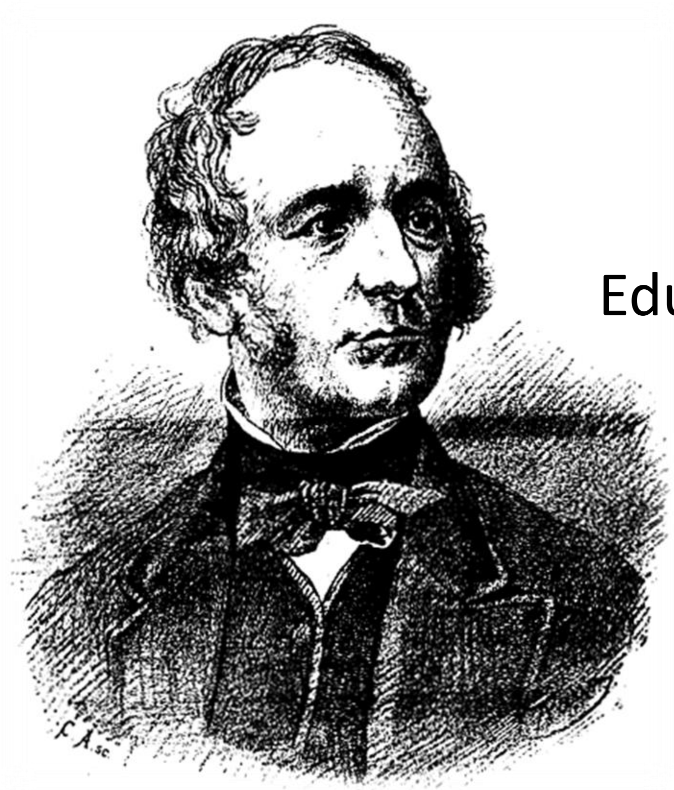


Solitons





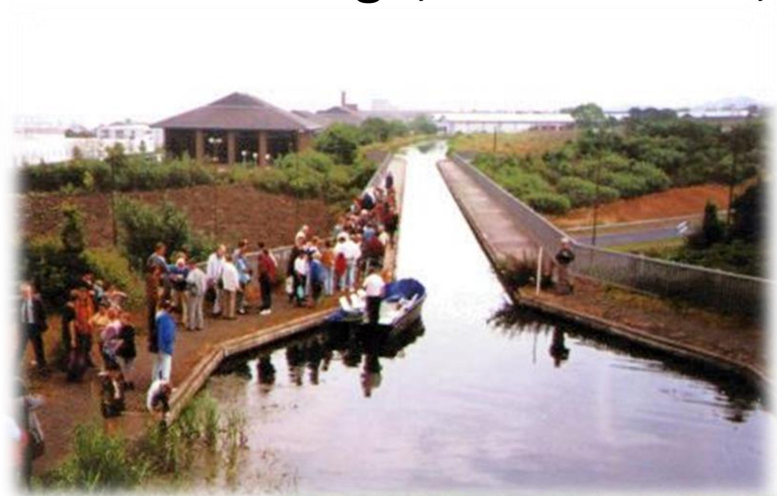


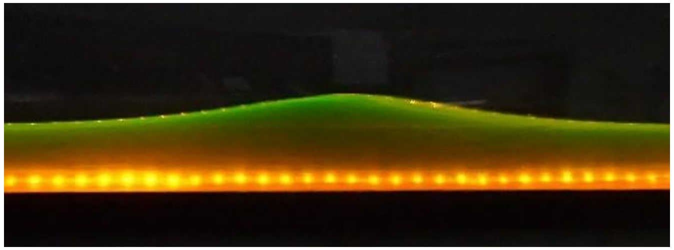


John Scott Russell

(9 May 1808-8 June 1882)

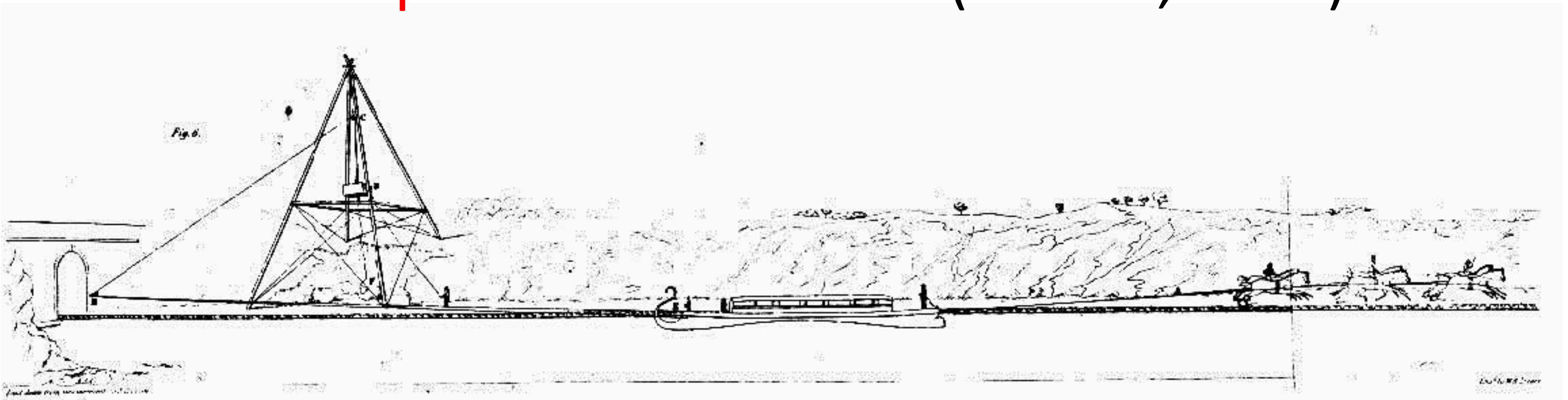
Education: Edinburgh, St. Andrews, Glasgow



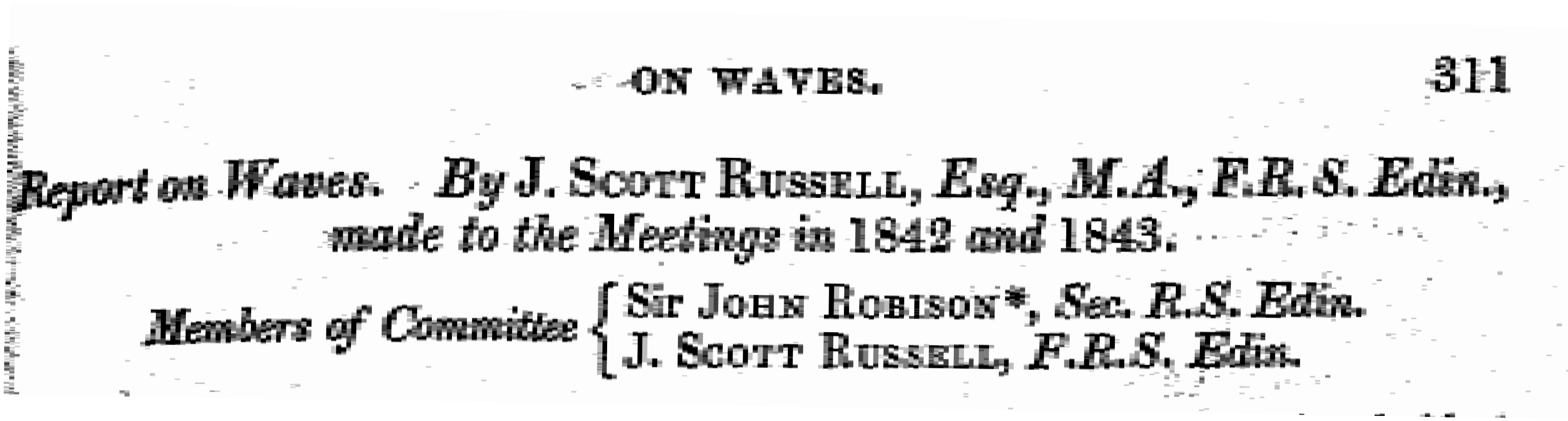
- August, 1834
- 

Russell's observation

- A large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed … Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon. (Russell, 1844)



The Great Wave Translation

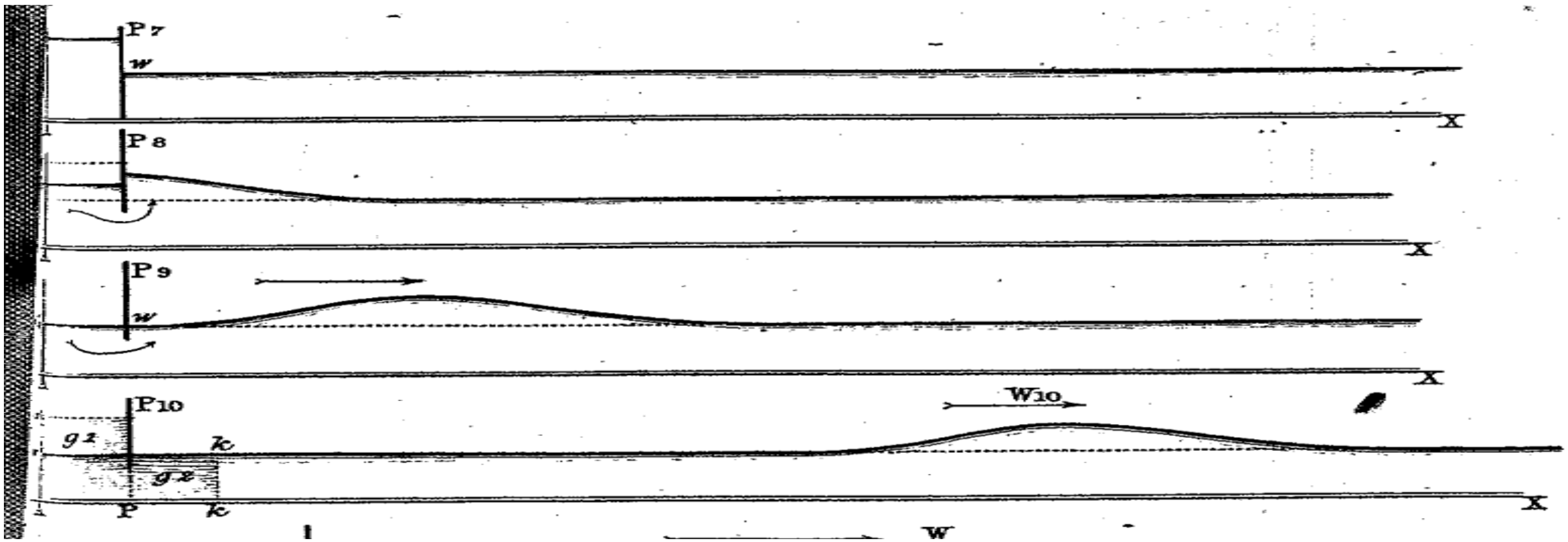

ON WAVES. 311

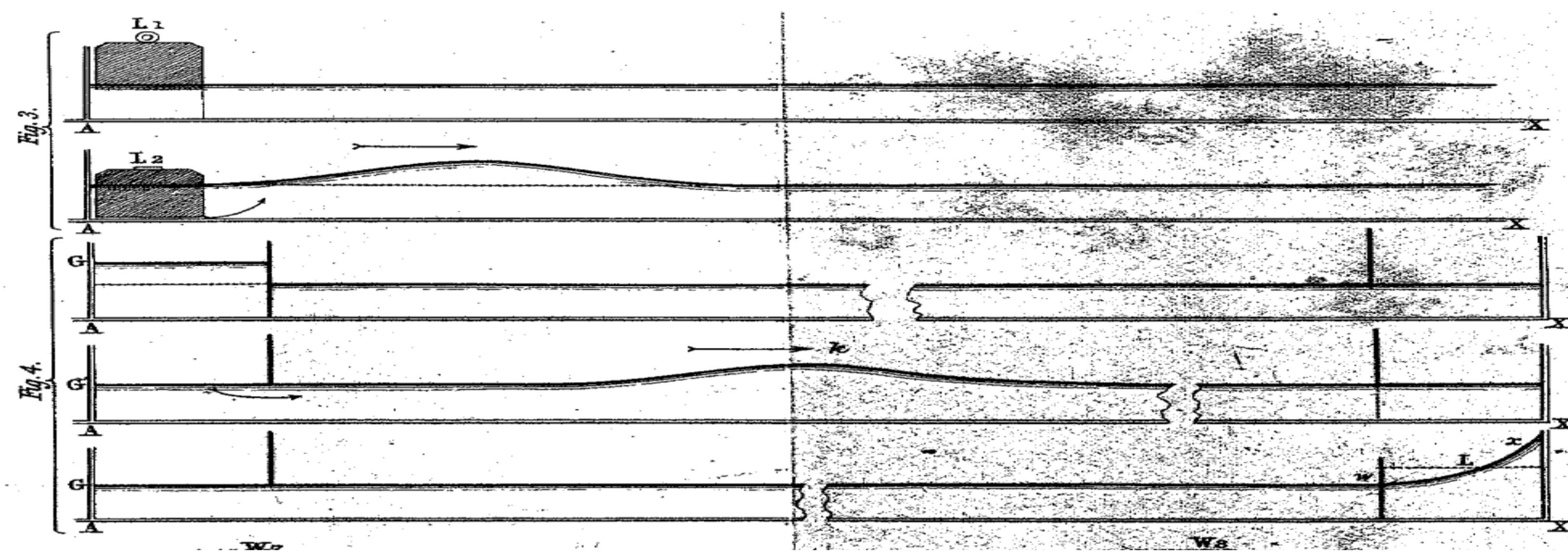
Report on Waves. By J. Scott Russell, *Esq., M.A., F.R.S. Edin.*, *made to the Meetings in* 1842 *and* 1843.

Members of Committee { Sir John Robison*, *Sec. R.S. Edin.*
J. Scott Russell, *F.R.S. Edin.* }

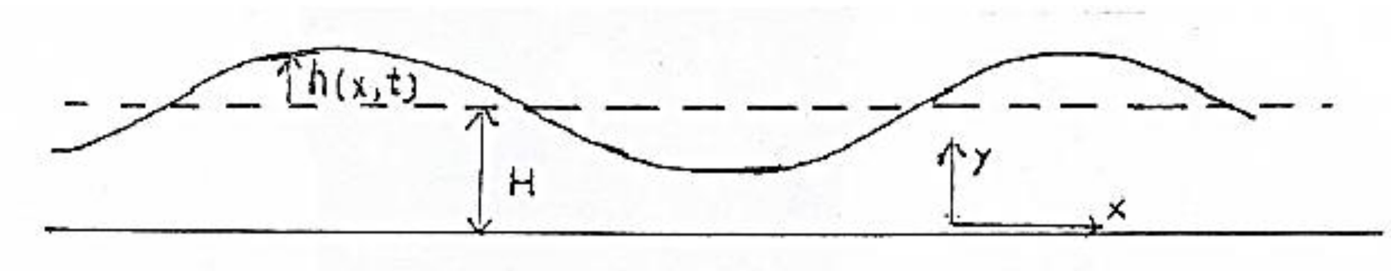
- Solitary waves --- J.S. Russell
- Airy: "even in an uniform-canal of rectangular section, are no longer propagated without change of type." Solitary waves of permanent form do not exist!
- Russell: "completely the opposite of that to which we should be led on the same grounds."

Russell's experiments





Russell-Boussinesq-Korteweg-de Vries



- **Rayleigh**
- **Boussinesq [1872,1877]**

$$\frac{\partial^2 h}{\partial t^2} = gH\frac{\partial^2 h}{\partial x^2} + gH\frac{\partial^2}{\partial x^2}\left[\frac{3h^2}{2H} + \frac{H^2}{3}\frac{\partial^2 h}{\partial x^2}\right]$$

$$\frac{\partial h}{\partial t} + \sqrt{\frac{g}{H}}\frac{3}{2}\frac{\partial}{\partial x}\left(\frac{2}{3}Hh + \frac{1}{2}h^2 + \frac{H^3}{9}\frac{\partial^2 h}{\partial x^2}\right) = 0$$

Russell-Boussinesq-KdV

- **KdV: solitary wave, periodic wave**

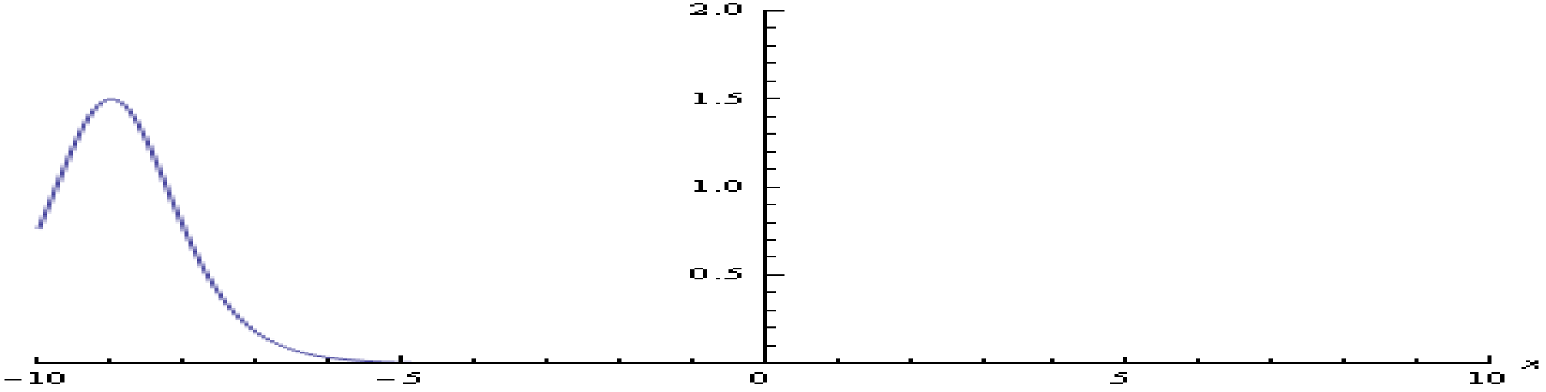
$$h(\xi) = h_2 \operatorname{sech}^2\left(\sqrt{\frac{h_2}{4\sigma}}\xi\right) \quad \xi = x - (\sqrt{gH} - \sqrt{\frac{g}{H}}\alpha)t$$

$$\tilde{h}(\xi) = l\,\mathrm{cn}^2\left(\sqrt{\frac{l+k}{4\sigma}}\xi\right)$$

- **As to the credit of the "a priori demonstration a posteriori" of the stable solitary wave, this credit belongs, of course, to M. Boussinesq.**

E. M. de Jager, On the Origin of the Korteweg-de Vries Equation, arXiv: 0602661

animation

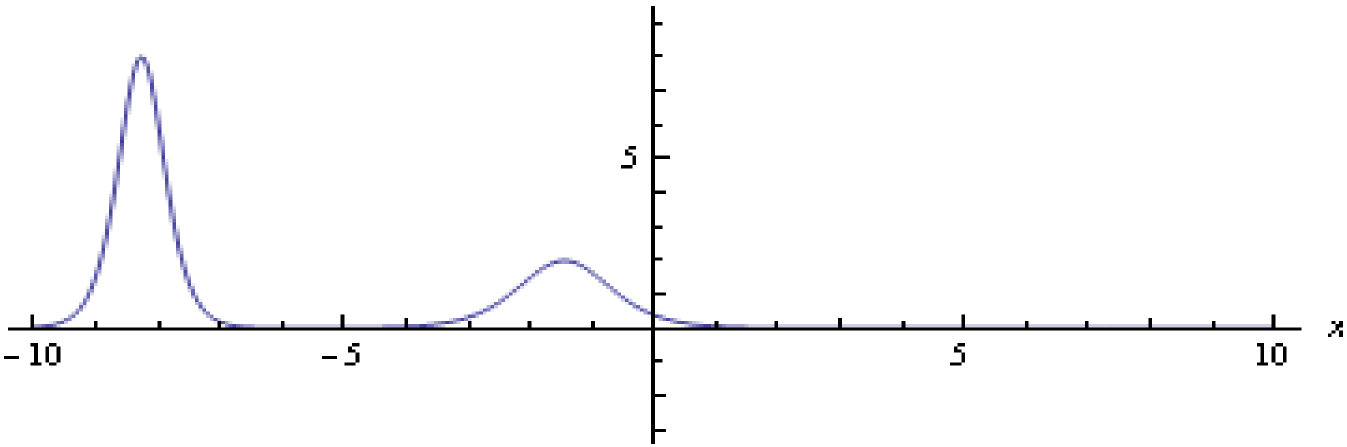


Exact solutions to the KdV

2-soliton solution

$$u = 2[\ln(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1+\xi_2+A_{12}})]_{xx}$$

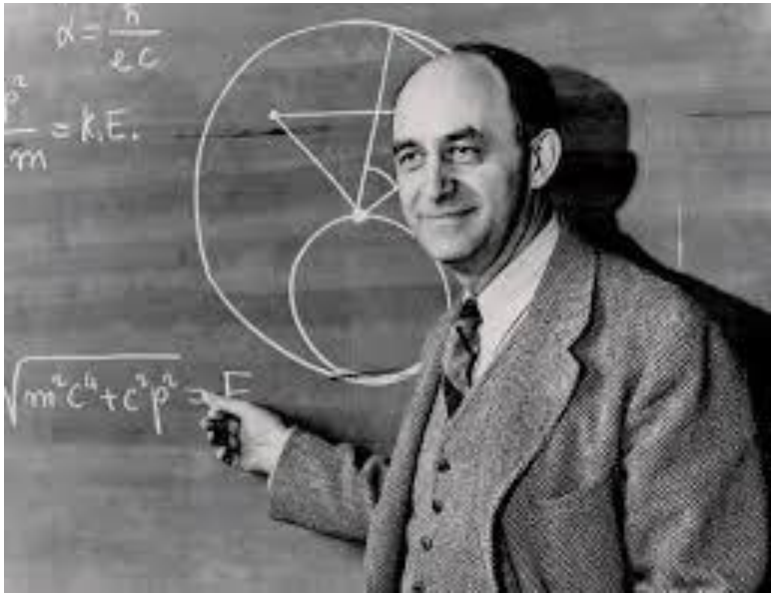
$$\xi_j = k_j x - k_j^3 t + \xi_j^{(0)}, \quad e^{A_{ij}} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2$$

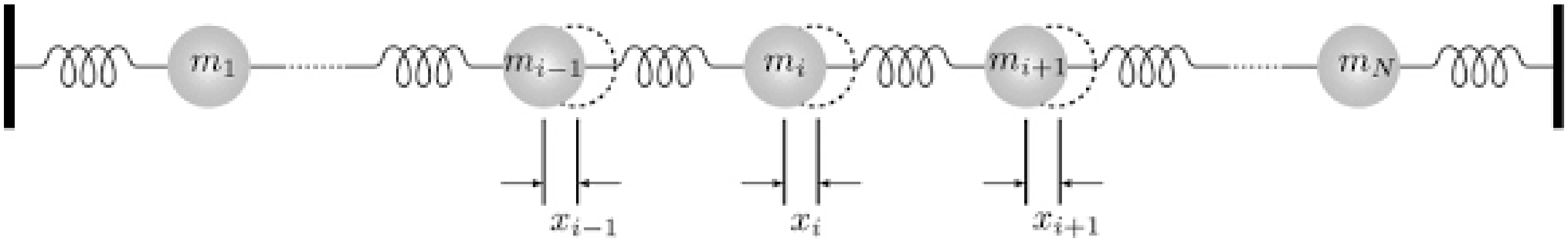


Fermi-Pasta-Ulam (FPU) problem

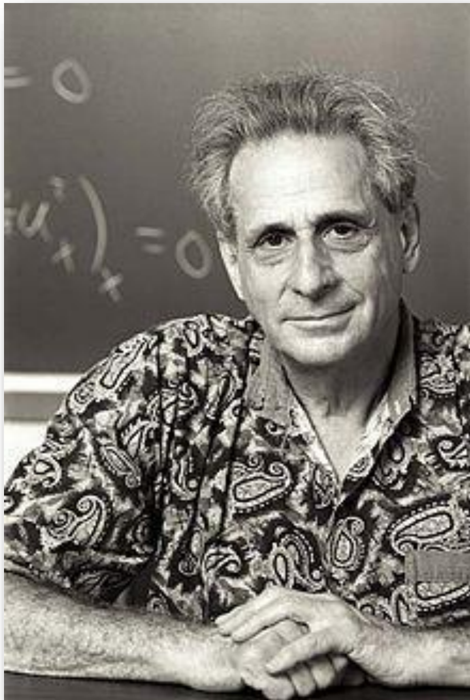
- FPU problem
- Enrico Fermi

  (1901-1954)
- Toda Lattice





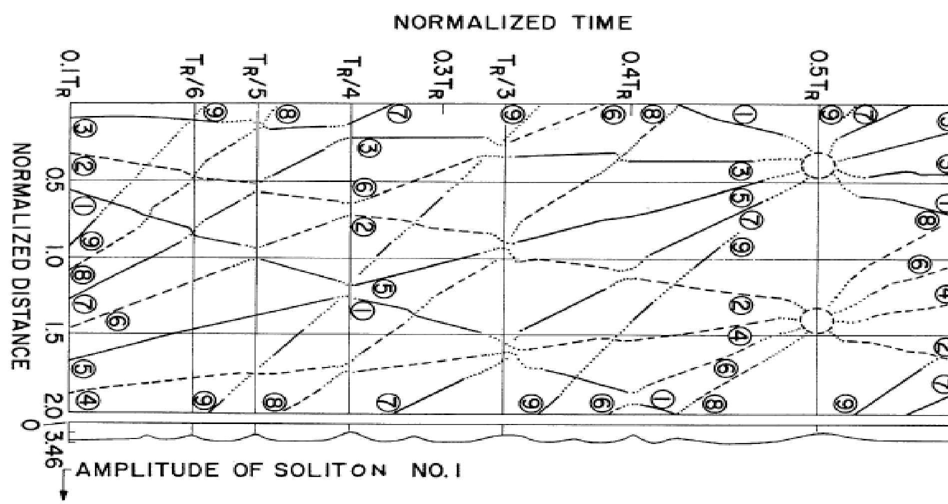
Birth of Solitons



- Martin David Kruskal
- 1925-2006
- Courant
- Father of “Soliton”

- FPU-Toda Lattice
- KdV

Solitons(partical property, 1965)



Inverse Scattering Transform

VOLUME 19, NUMBER 19 PHYSICAL REVIEW LETTERS 6 NOVEMBER 1967

METHOD FOR SOLVING THE KORTEWEG-deVRIES EQUATION*

Clifford S. Gardner, John M. Greene, Martin D. Kruskal, and Robert M. Miura
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey
(Received 15 September 1967)

A method for solving the initial-value problem of the Korteweg-deVries equation is presented which is applicable to initial data that approach a constant sufficiently rapidly as $|x| \to \infty$. The method can be used to predict exactly the "solitons," or solitary waves, which emerge from arbitrary initial conditions. Solutions that describe any finite number of solitons in interaction can be expressed in closed form.

Inverse Scattering Transform

（starting point of modern integrable theory）

CLs：Miura, Gardner

Miura Trans： $u = v^2 + v_x$

modified KdV: $v_t - 6v^2 v_x + v_{xxx} = 0$

$$u_t - 6uu_x + u_{xxx} = 0$$

$$u = v^2 + v_x \xrightarrow{v = \frac{\psi_x}{\psi}} \psi_{xx} - u\psi = 0$$

Galilean: $u \to u - \lambda,\ t \to t,\ x \to x + 6\lambda t$

modified KdV: $\xrightarrow{v = \frac{\psi_x}{\psi}}$

$$\psi_{xx} - (u - \lambda)\psi = 0$$

$$\psi_t + \psi_{xxx} - 3(u + \lambda)\psi_x = 0$$

Lax Pair

Lax pair:

$$\psi_{xx} - (u - \lambda)\psi = 0$$

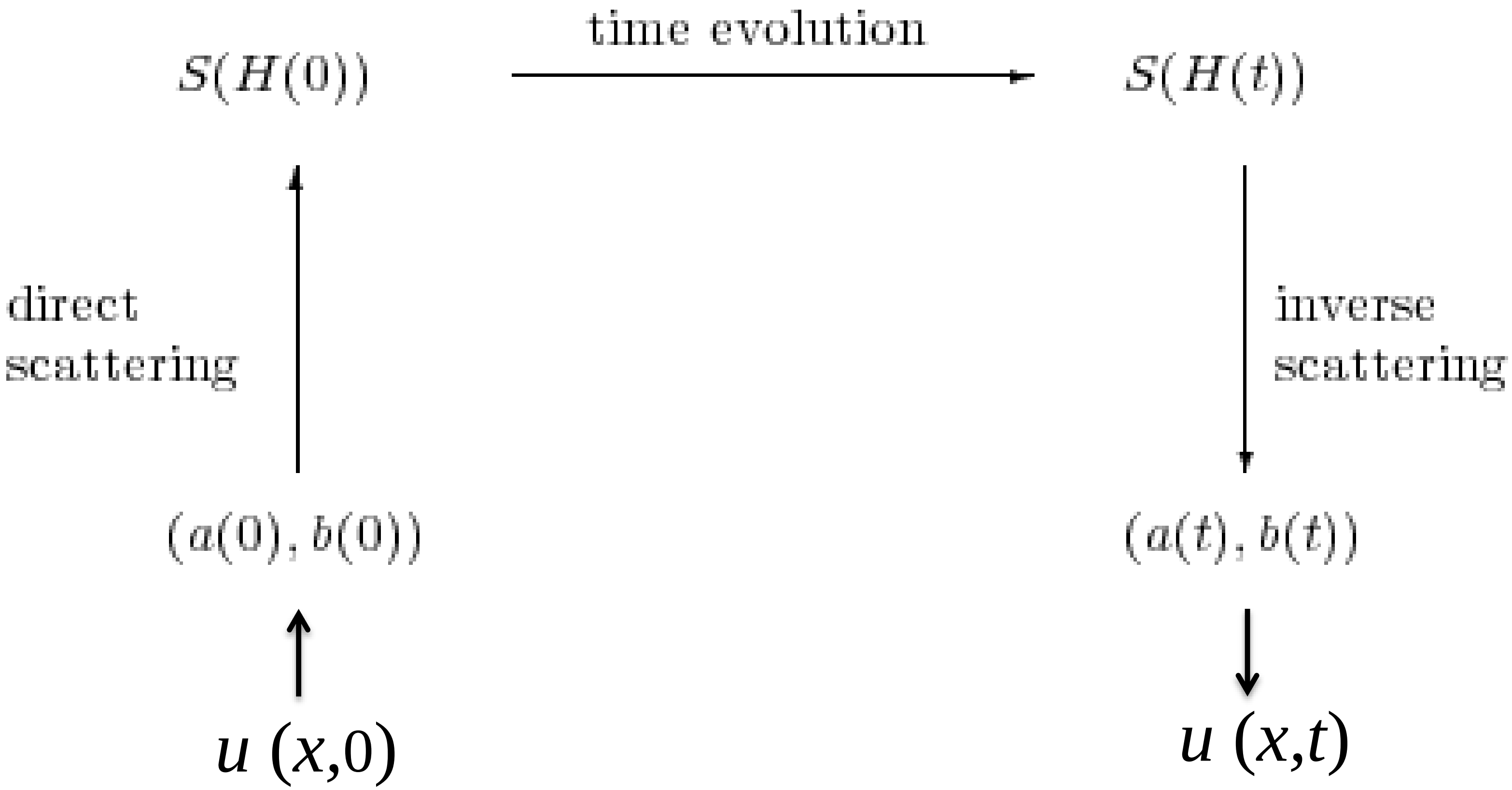
$$\psi_t + \psi_{xxx} - 3(u + \lambda)\psi_x = 0$$

compatibility: $\psi_{xx,t} = \psi_{t,xx}$

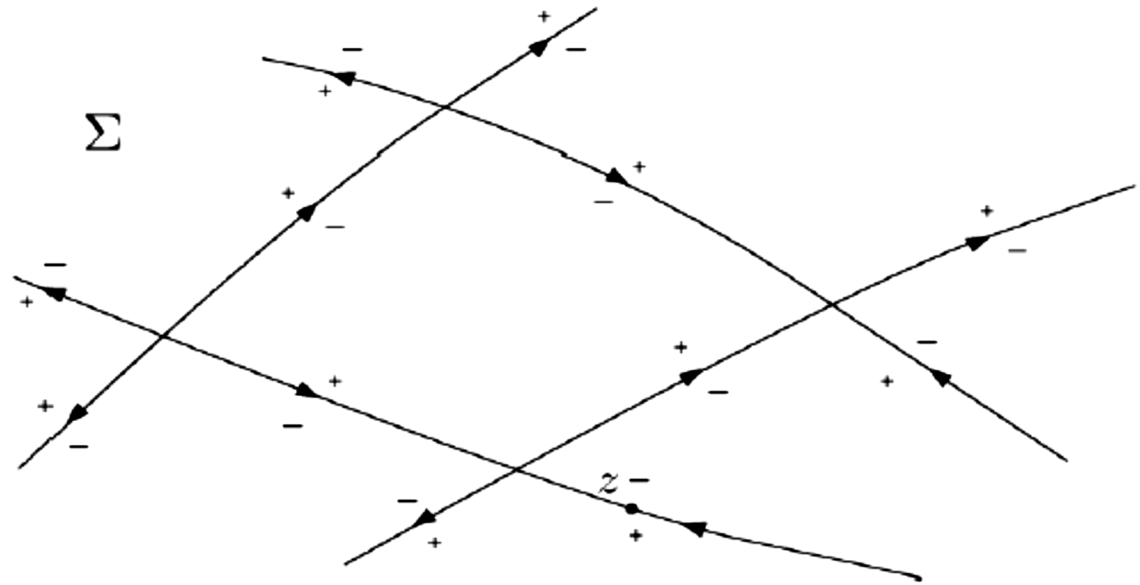
KdV:

$$u_t - 6uu_x + u_{xxx} = 0$$

Inverse Scattering Transform



RHP

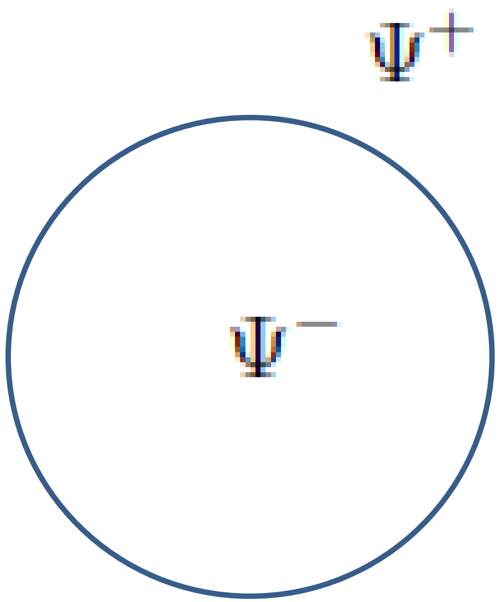


- $m(z)$ is analytic in $\mathbb{C}/\Sigma$
- $m_+(z) = m_-(z)\, v(z) \quad , \quad z \in \Sigma$

  where $\quad m_\pm(z) = \lim\limits_{z' \to z_\pm} m(z')$
- $m(z) \to I_n \quad$ as $\quad z \to \infty$

IST and RHP

- RH Problem



- IST

- New RHP ??

Peter D Lax

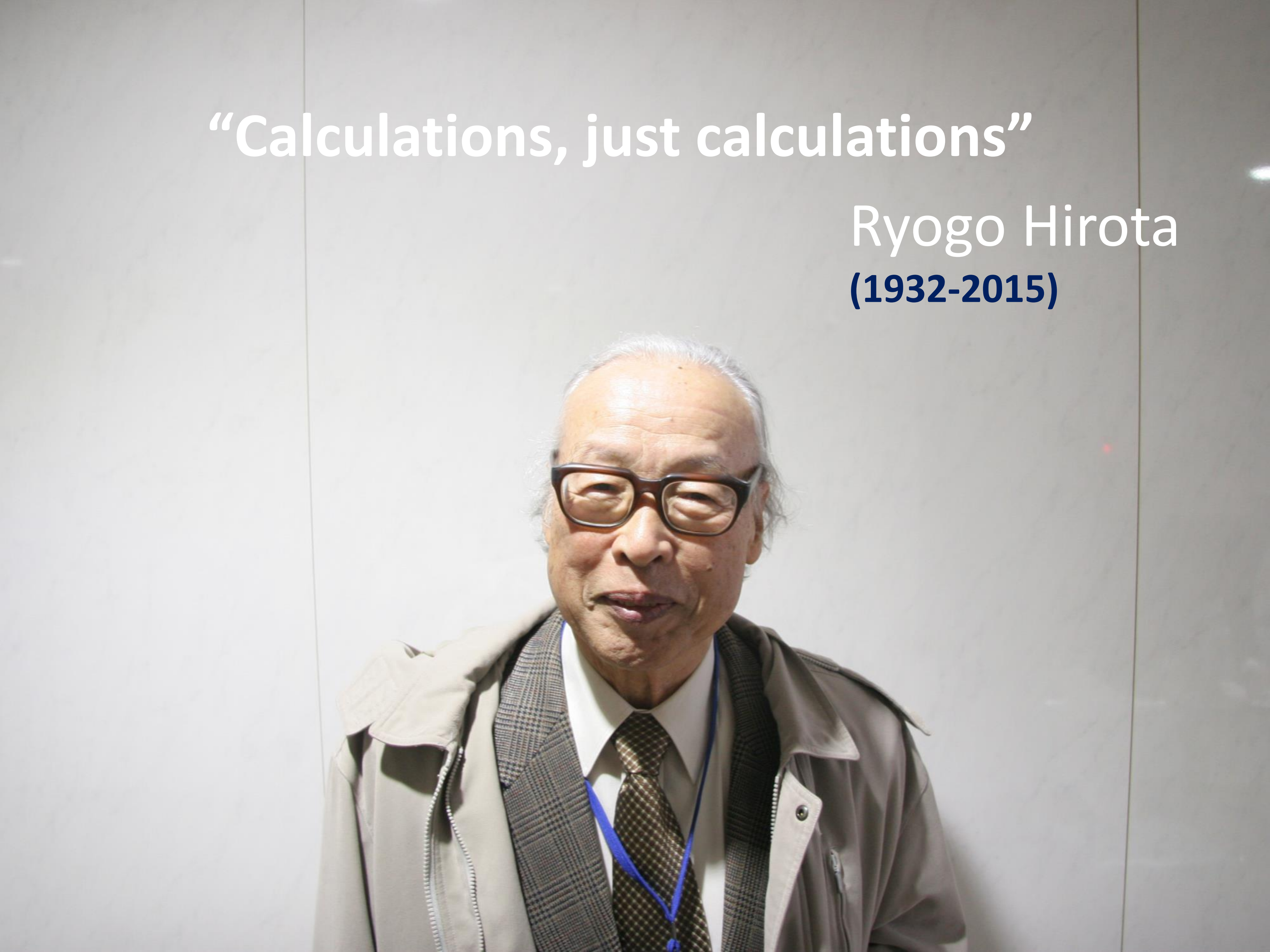
- Peter David Lax
- 1 May 1926-
- Courant Institute
- Hungarian
- **1968-CPAM**： symmetry, conserved covarient, gradient of eigenvalue, asymototic analysis

Solitons in 1970’s

- **Hirota method**
- **Action-angle variables**
- **Zakharov+Shabat:NLS/IST**
- **AKNS (ZS-AKNS)**
- **Recursion op： Lenard, AKNS, Olver**
- **Symmetry, biHamiltonian**
- **Bäcklund trans（1973-76）**
- **Darboux (Moutard) trans（1979）**

Solitons in 1970’s

- **Continuous spectruum：long time behavior analysis**
- **Finite-Gap solutions**
- **Quantum IST**
- **Integrable systems from geometry**
- **Optical solitons**


“Calculations, just calculations”
Ryogo Hirota
(1932-2015)

Autumn meeting of JPS in 1971

Multiple collision of K-dV solitons --- Exact solution

1p—F—9　　K-dV ソリトンの多重衝突 ―厳密解

RCA基礎研　　広田 良吾

Ryogo Hirota

次の形の Korteweg-de Vries 方程式を考える.

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

この式の解 $u(x,t)$ を, $u(x,t) = -2\frac{\partial^2}{\partial x^2}\log[f(x,t)]$ とおくと, $f(x,t)$ は次式を満足する,

$$f_{xt}f - f_t f_x + f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2 = 0.$$

$f(x,t)$ として次の $n\times n$ マトリックス M の行列式を考えると, これは上式の解であり, $u(x,t)$ は n 個の高さの違ったソリトンの多重衝突を表わしている事が示される.

$$f(x,t) = \det|M|$$

$$M_{ij}(x,t) = \delta_{ij} + \frac{2\sqrt{P_i}\sqrt{P_j}}{P_i+P_j}\exp\left[\tfrac{1}{2}(\xi_i+\xi_j)\right], \quad i,j = 1,2,\cdots,n$$

$$\xi_i = P_i x - \Omega_i t - \xi_i^0, \qquad \Omega_i = P_i^3,$$

ここで, P_i と ξ_i^0 は任意常数で i 番目のソリトンの高さと位相に関係している量である。P_i はお互に違った値であると仮定している。ここで得られた $u(x,t)$ の形は t をパラメータと考えると, Kay と Moses (J. Appl. Phys. 27('56) 1503) の一次元Schrödinger 方程式の無反射ポテンシャルに一致する。

The date of discovery of Hirota's direct method

- Autumn meeting of JPS (submitted in Jul, 1971)
  "Multiple collision of K-dV solitons --- Exact solution"

- Submission of paper to PRL (received Sep 17, 1971)
  Ryogo Hirota: "Exact Solution of the Korteweg-de Vries Equation for Multiple Collisions of Solitons",
  Phys. Rev. Letters 27 pp.1192–1194 (1971)

- Annual (spring) meeting of JPS (submitted in Dec, 1971)
  "N-soliton solution to modified K-dV equation"

Action-angle variables

- Finite-D Hamiltonian system： Liouville
- Infinite-D
- V. Zakharov and L. Faddeev, Korteweg-de Vries equation: A completely integrable Hamiltonian system, Funktsional. Anal. i Prilozhen, 5:4, 1971, 18–27.

Solitons in 1970's

- **Hirota method**
- **Action-angle variables**
- **Zakharov+Shabat: NLS/IST (1972)**
- **AKNS (ZS-AKNS) (1973)**

$$\Phi_x = \begin{pmatrix} \eta & q \\ r & -\eta \end{pmatrix} \Phi, \quad \Phi = (\phi_1, \phi_2)^T$$

Solitons in 1970's

- **Hirota method**
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Andrew Lenard's 15 mins

$$G_0 = \frac{\delta}{\delta u} \int \frac{1}{2} u dx$$

P_1 ↘ $u_t = u_x$ ↙ P_0

$$G_1 = \frac{\delta}{\delta u} \int \frac{1}{2} u^2 dx$$

P_1 ↘ $u_t = 6uu_x + u_{xxx}$ ↙ P_0

$$G_2 = \frac{\delta}{\delta u} \int \left(u^3 - \frac{1}{2} u_x^2 \right) dx$$

P_1 ↘ $u_t = 30u^2 u_x + 20u_x u_{xx} + 10u_x u_{xxx} + u^{(5)}$ ↙ P_0

$$G_3 = \frac{\delta}{\delta u} \int \left(\frac{5}{2} u^4 - 5uu_x^2 + \frac{1}{2} u_{xx}^2 \right) dx$$

P_1 ↘ ...

$$P_0 = \frac{\partial}{\partial x} \qquad P_1 = \frac{\partial^3}{\partial x^3} + 4u \frac{\partial}{\partial x} + 2u_x$$

Figure 1. The Lenard recursion formula.

Symmetries

$$u_t = K(u)$$

$$\text{symmetry}: \ \sigma_t = K'[\sigma]$$

$$g(\epsilon): \ u \to \bar{u}(\epsilon), \quad \frac{d}{d\epsilon}\bar{u} = \sigma(\bar{u}), \ \bar{u}|_{\epsilon=0} = u$$

$$\bar{u}_t = K(\bar{u})$$

$$u_{t_j} = K_j = L^j K_0$$

$$\text{Strong symmetry}: \ L_t = [K', L]$$

$$\text{Hereditary}: \ L'(L[f]g - L[g]f) = L(L'[f]g - L'[g]f)$$

$$[\![K_i, K_j]\!] = 0$$

- **Fokas, Fuchssteiner**

Bi-Hamiltonian Structure

- Magri (1978), Gel'fand-Dorfman (1979)

$$u_t = K(u) = \theta_1 \frac{\delta H_1}{\delta u} = \theta_2 \frac{\delta H_2}{\delta u}$$

$$\text{compatibility}: \ \theta = a_1\theta_1 + a_2\theta_2$$

$$\text{recursion operator}: L = \theta_2\theta_1^{-1}: \ u_{t_j} = L^j K(u), \ \{H_i, H_j\} = 0$$

- **Fokas, Fuchssteiner (1981)**

$$u_{t_j} = K_j = L^j K_0$$

$$\text{implectic} - \text{symplectic factorization}: \ L = \theta J, \ L^* = J\theta$$

$$\text{Hereditary} \iff \text{compatibility}$$

$$u_{t_j} = K_j = \theta \frac{\delta H_1}{\delta u} = \theta J \theta \frac{\delta H_2}{\delta u} = \theta (L^*)^2 \frac{\delta H_3}{\delta u} = \cdots$$

Solitons in 1970’s

- **Hirota method**
- **Action-angle variables**
- **Zakharov+Shabat:NLS/IST**
- **AKNS (ZS-AKNS)**
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- **Darboux (Moutard) trans（1979）**

Notes:

- Bäcklund Transformations:[C. Rogers, W.K. Schief-2002] [Robert Prus and Antoni Sym-1998]
  - Bianchi (1892): Demonstrated that the Bäcklund Transformation $\mathbb{B}_\sigma$ admits a commutativity property $\mathbb{B}_{\sigma_1}\mathbb{B}_{\sigma_2} = \mathbb{B}_{\sigma_2}\mathbb{B}_{\sigma_1}$, a consequence of which is a nonlinear superposition principle embodied in what is termed a Permutability Theorem.
  - It was neither Bianchi nor Bäcklund who was the first to write down the special Bäcklund transformation for the sine-Gordon equation. It was the great French geometer Gaston Darboux (in 1883).

Darboux found (1883) that if θ is a solution of sG equation

$$\theta_{uu} - \theta_{vv} = \sin\theta\cos\theta,$$

and ϕ satisfies

$$\phi_u + \theta_v = \sin\phi\cos\theta, \quad \phi_v + \theta_u = -\cos\phi\sin\theta, \tag{117}$$

then ϕ is a solution of sG equation $\phi_{uu} - \phi_{vv} = \sin\phi\cos\phi$.

Bäcklund constructed the transformation:

$$\phi_u + \theta_v = (\sin\phi\cos\theta + \sin\sigma\cos\phi\sin\theta)/\cos\sigma,$$
$$\phi_v + \theta_u = (-\cos\phi\sin\theta + \sin\sigma\sin\phi\cos\theta)/\cos\sigma,$$

where σ is an arbitrary parameter. When $\sigma = 0$ it is (117).

Solitons in 1970's

- **Continuous spectruum：long time behavior analysis**
- **Finite-Gap solutions**
- **Quantum IST**
- **Integrable systems from geometry**
- **Optical solitons**

Solitons in 1970's

- **Continuous spectruum： long time behavior analysis**

  Zakharov, Manakov (IST,1976), Deift, Zhou (RHP,1993) ,

  Its, Bobenko (DIS, 2017)

- **Finite-Gap solutions (Novikov, 1974-76)**

  30 years of finite-gap integration theory (Matveev, 2008)

- **Quantum IST (Faddeev, Sklyanin)**

- **Integrable systems from geometry**

  Hasimoto (1972): Vortex filament and NLS

  Lamb (1976): space curve and NLS, mKdV

Solitons in 1970's

- **Soliton (optical) (1973)**

Akira Hasegawa, Fred Tappert (AT&T Bell Labs)

Robin Bullough made the first mathematical

Solitons in early 1980's

- **Sato's KP Theory (Sato's talk,1981)**

  Bilinear identity

  Vertex operator, tau function, affine Lie algebra

  Discretisation (Miwa, Jimbo, Date, Takasaki)
- **Painleve (Kyoto School)**
- **Integrable systems and Lie algebra**

  Drinfel'd, Sokolov (1981,84)
- **Direct linearisation approach**

  Fokas, Ablowitz (1981)
- **Dutch group (DIS)**

Direct linearisation approach(DLA)

- **DLA[PRL-1981-46-1096-110]**

VOLUME 47, NUMBER 16 PHYSICAL REVIEW LETTERS 19 OCTOBER 1981

Linearization of the Korteweg–de Vries and Painlevé II Equations

A. S. Fokas and M. J. Ablowitz

$$\varphi(k;x,t)+i\exp[i(kx+k^3t)]\int_L \frac{\varphi(l;x,t)}{l+k}\,d\lambda(l)=\exp[i(kx+k^3t)]$$

$$u_t+6uu_x+u_{xxx}=0 \qquad u=-\frac{\partial}{\partial x}\int_L \varphi(k;x,t)\,d\lambda(k)$$

Solitons in early 1980's

- **BT and DIS (1980)**

  Decio Levi and R Benguria

- **Wronskian (1983)**
- **…………..**

$$\Phi_x = M\Phi, \;\; M = \begin{pmatrix} \eta & u \\ v & -\eta \end{pmatrix} \Phi$$

$$\widetilde{\Phi} = T\Phi, \; T = T(\gamma, U, \widetilde{U}) = \begin{pmatrix} 2(\eta-\gamma) + u\widetilde{v} & u \\ \widetilde{v} & 1 \end{pmatrix} \Phi$$

$$T_x - \widetilde{M}T + TM = 0,$$

Thank You

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Introduction to Discrete Integrable Systems

Da-jun Zhang
Shanghai University

(2019-11-13)

SIDE series

(Symmetries and Integrability of difference Equations)

- 13th International Conference on Symmetries and Integrability of Difference Equations (SIDE 13)
- Venue: Jr Hakata City Conference Rooms, Fukuoka, Japan
- SIDE-14: 2020, Poland

First Look

$$\Phi_x = \begin{pmatrix} \eta & q \\ r & -\eta \end{pmatrix} \Phi, \quad \Phi = (\phi_1, \phi_2)^T$$

$$\Phi_{n+1} = \begin{pmatrix} \lambda & Q_n \\ R_n & 1/\lambda \end{pmatrix} \Phi_n, \quad \Phi_n = (\phi_{1,n}, \phi_{2,n})^T$$

$$\Phi(n+j) = \Phi(x + j\epsilon), \quad (Q_n, R_n) = \epsilon(q, r), \quad \lambda = e^{\epsilon\eta}$$

- **Ablowitz-Ladik (JMP, SAM 1975/76)**
- **Hirota (JPSJ-I,II,III 1977)**

- **M.J. Ablowitz, J.F. Ladik, Nonlinear dierential-dierence equations, J. Math. Phys., 16 (1975) 598-603.**
- **M.J. Ablowitz, J.F. Ladik, Nonlinear dierential-dierence equations and Fourier analysis, J. Math. Phys., 17 (1976) 1011-8.**
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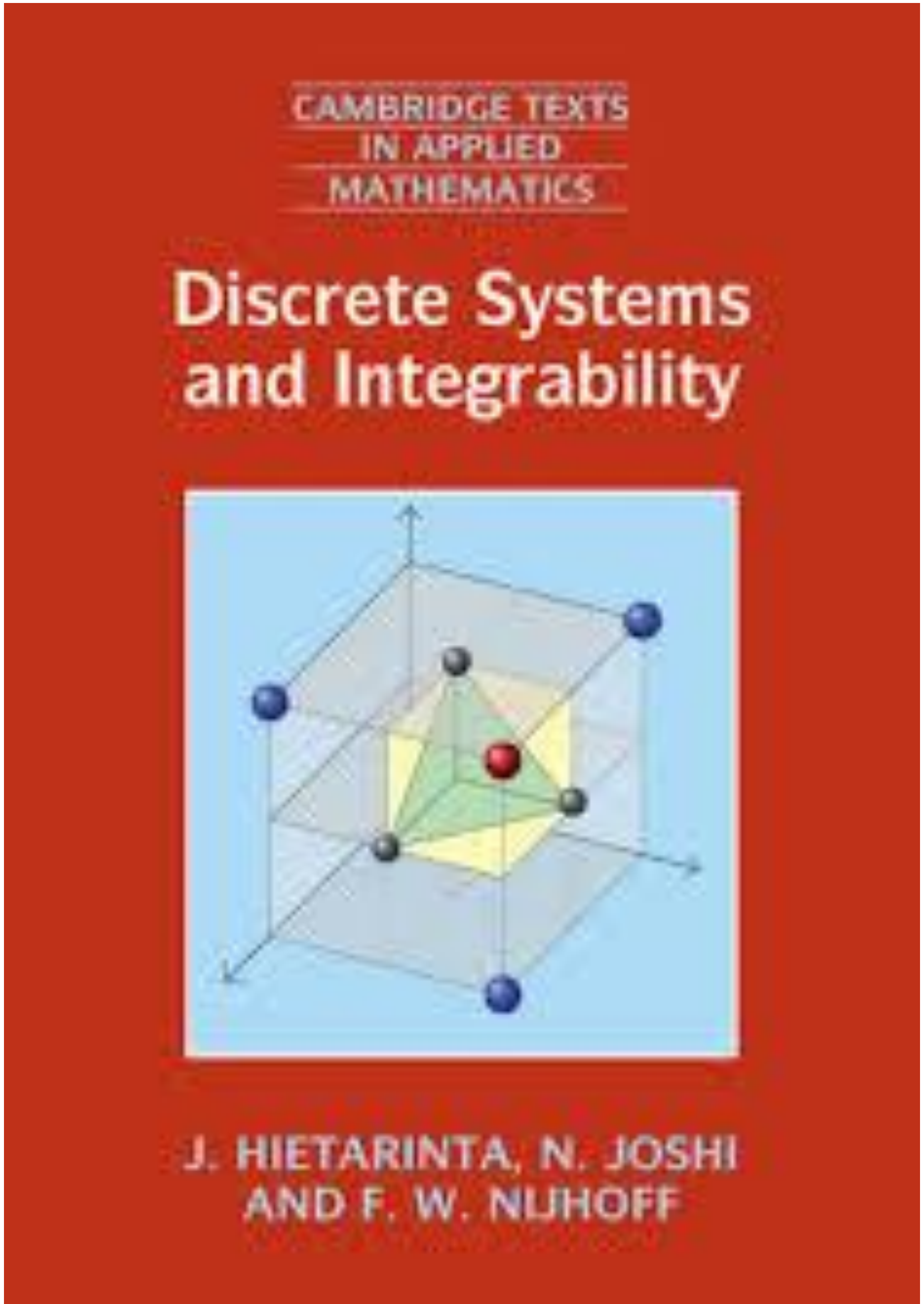
Two Books

- A.I. Bobenko, Yu.B. Suris, Discrete Differential Geometry, 2008 AMS
- J. Hietarinta, N. Joshi, F.W. Nijhoff, Discrete Systems and Integrability, 2016 Cam Univ Press.

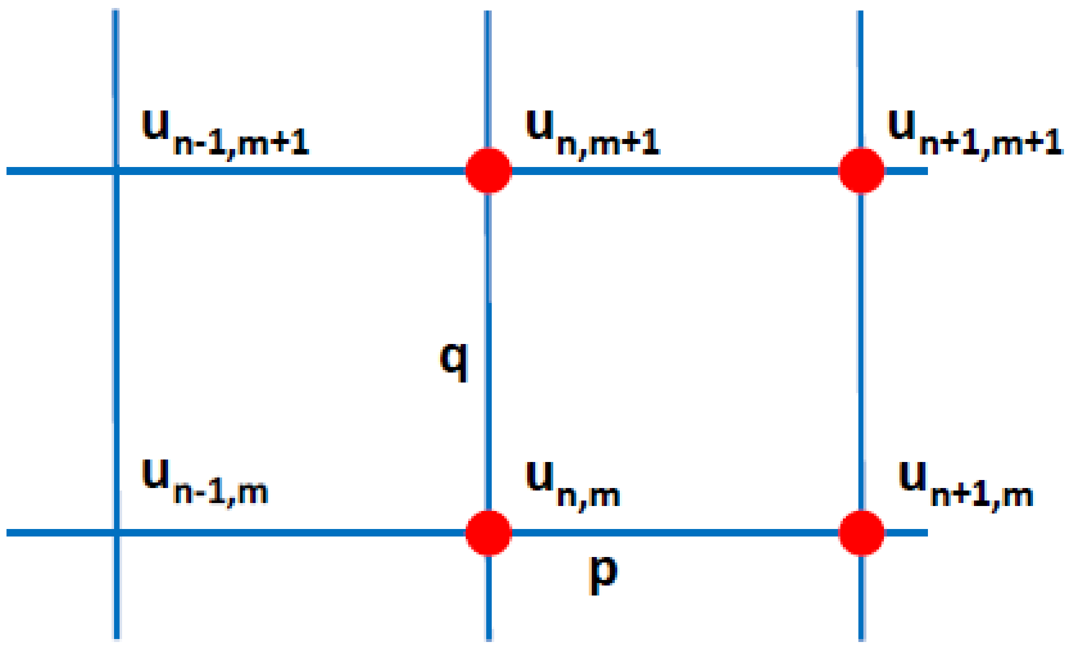

CAMBRIDGE TEXTS
IN APPLIED
MATHEMATICS
Discrete Systems
and Integrability
J. HIETARINTA, N. JOSHI
AND F. W. NIJHOFF

- Discrete Integrable Systems (DIS):

$$(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) = p^2 - q^2 \qquad \text{(lpKdV)}$$

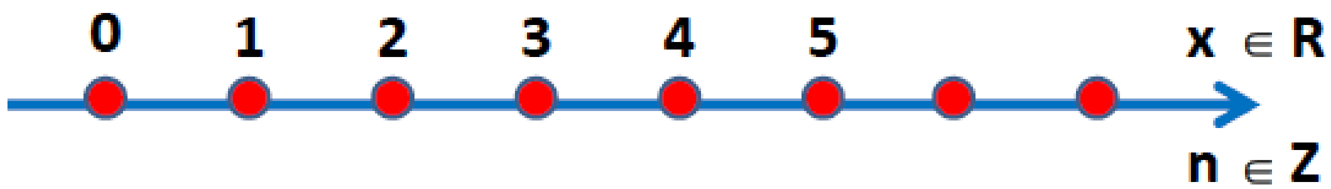
Korteweg-de Vries equation (KdV):

$$u_t + 6uu_x + u_{xxx} = 0 \qquad \text{(KdV)}$$



Discretisation

- **x ∈ R → n ∈ Z** $\psi(x) + V(x)\psi(x) = \lambda\psi(x)$



Discrete Schrödinger spectral problem on (half) line:

$$-\psi(n+1) + 2\psi(n) - \psi(n-1) + V(n)\psi(n) = \lambda\psi(n), \quad n \in \mathbb{Z}^+$$

Boundary cond. e.g. $\psi(0) = 0, \psi(1) = 1$

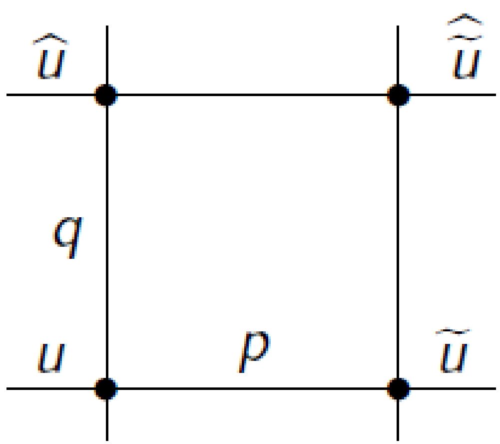
Spectral theory and OP

Hermite polynomials $H_{n+1}(x) = 2x\,H_n(x) - 2n\,H_{n-1}(x)$

Discretisation

- **Numerical**

$$u(x,t):\ x = x_0 + nh,\ t = t_0 + mk, \quad u(x,t) \Rightarrow u_{n,m}$$

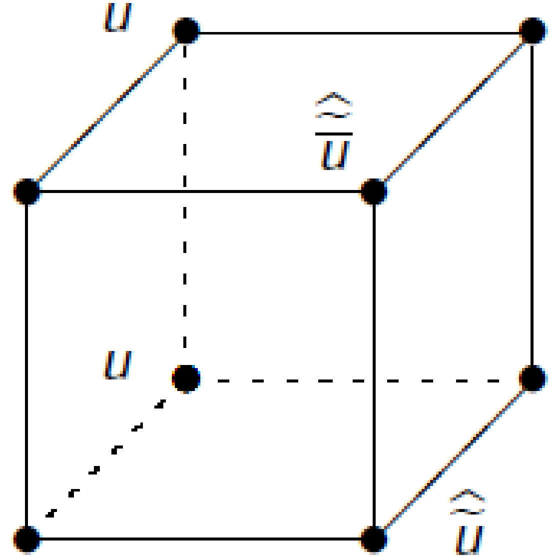


$$u \equiv u_{n,m}, \quad \widetilde{u} \equiv u_{n+1,m},$$

$$\widehat{u} \equiv u_{n,m+1}, \quad \widehat{\widetilde{u}} \equiv u_{n+1,m+1}$$

$$(u - \widehat{\widetilde{u}})(\widetilde{u} - \widehat{u}) = p - q$$

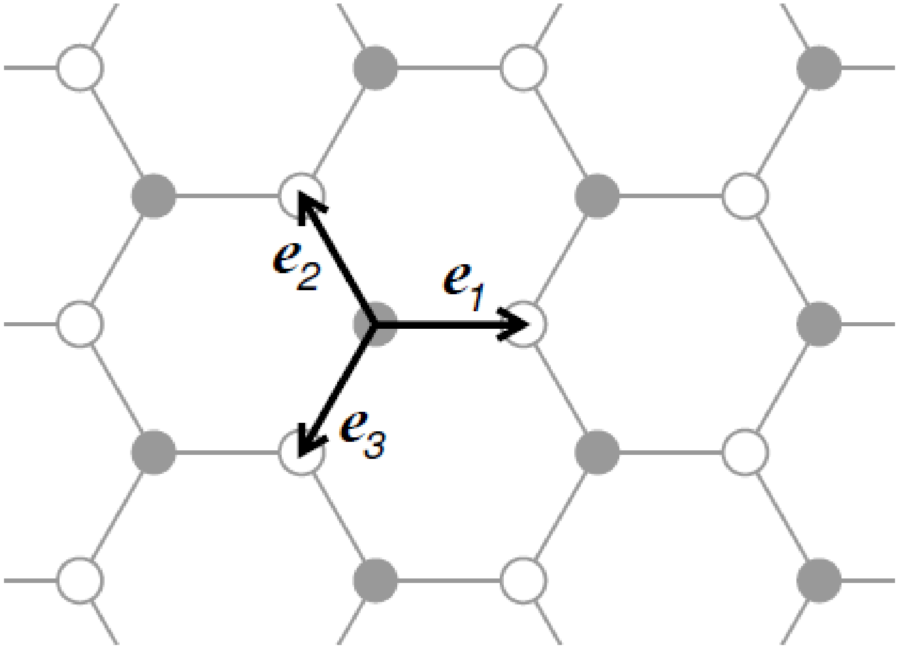
3D example



$$\frac{(p-r+\overline{u}-\widetilde{u})\widehat{\ }}{p-r+\overline{u}-\widetilde{u}} = \frac{(q-r+\overline{u}-\widehat{u})\widetilde{\ }}{q-r+\overline{u}-\widehat{u}} = \frac{(p-q+\widehat{u}-\widetilde{u})^{-}}{p-q+\widehat{u}-\widetilde{u}}$$

Example

- **Honeycomb lattice**



Example

- **Recursion relation**

Recurrence relation: Eg. Bessel function

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{k=0}^{\infty} \frac{(-x^2/4)^k}{k!\,\Gamma(\alpha+k+1)}$$

Satisfies: $x^2 w'' + x w' + (x^2 - \alpha^2) w = 0$

Recurrence relation: $x(w_{\alpha+1} + w_{\alpha-1}) - 2\alpha w_\alpha = 0$

[J. Hietarinta, N. Joshi, F.W. Nijhoff, Discrete Systems and Integrability, page32-34]

Example

- **Recursion relation**

Painlevé II (P_{II})

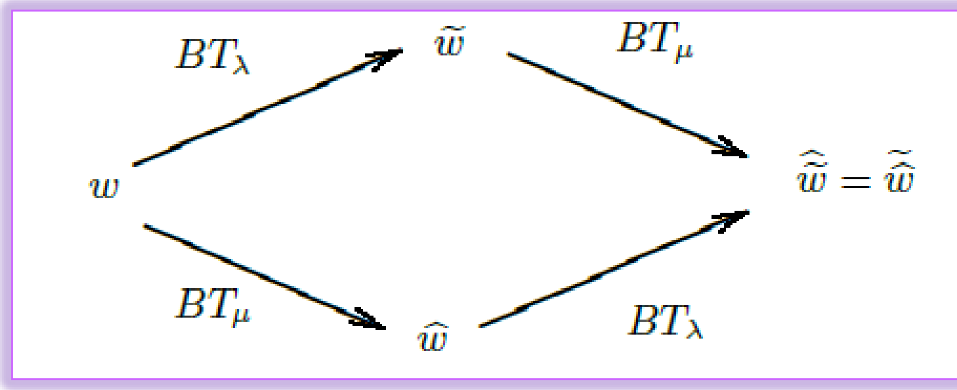
$$\frac{d^2 f}{dt^2} = 2f^3 + tf - \alpha$$

$$f_{\alpha+1}(t) = -f_\alpha(t) - \frac{(\alpha + 1/2)}{f'_\alpha(t) - f^2_\alpha(t) - t/2}$$

$f_\alpha(t)$ **satisfy P_{II}**

[H.Flaschka, A.C. Newell, Monodromy- and spectrum-preserving deformations. I. Commun. Math. Phys., 76 (1980) 65–116.]

Example



- **NSF**

$$u_t = 6uu_x + u_{xxx}$$

$$\begin{aligned} (w_1 + w)_x &= 2\lambda - \tfrac{1}{2}(w_1 - w)^2 \\ (w_1 + w)_t &= (w_1 - w)(w - w_1)_{xx} + 2(w_x^2 + w_x w_{1,x} + w_{1,x}^2) \end{aligned}$$

$$\begin{array}{l} w \xrightarrow{\lambda} w_1 \xrightarrow{\mu} w_{12} \\ w \xrightarrow{\mu} w_2 \xrightarrow{\lambda} w_{21} \end{array} \Bigg\downarrow \quad w_{12} = w_{21} \ \text{(permutability)}$$

$$(w - w_{12})(w_1 - w_2) = 4(\mu - \lambda)$$

$$(w - \widehat{\widetilde{w}})(\widetilde{w} - \widehat{w}) = p - q$$

Exampla

- NSF
  - sG，mKdV： $$p\left(v\hat{v}-\widetilde{v}\widehat{\widetilde{v}}\right)=q\left(v\widetilde{v}-\hat{v}\widehat{\widetilde{v}}\right)$$
  - AKNS： $$(\tilde{u}-\bar{u})(u\bar{\tilde{v}}+1)+(\gamma_1-\gamma_2)u=0,$$
    $$(\tilde{v}-\bar{v})(u\bar{\tilde{v}}+1)-(\gamma_1-\gamma_2)\bar{\tilde{v}}=0.$$

Example

- **NSF**
  - **Krichever-Novikov :** $$u_t = \frac{1}{4}u_{xxx} + \frac{3}{8}\frac{r(u) - u_{xx}^2}{u_x}$$

$$r(u)=4u^3 - g_2 u - g_3$$

$$p(u\tilde{u} + \hat{u}\widehat{\tilde{u}}) - q(u\hat{u} + \tilde{u}\widehat{\tilde{u}}) - r(u\widehat{\tilde{u}} + \tilde{u}\hat{u}) + pqr(1 + u\tilde{u}\hat{u}\widehat{\tilde{u}}) = 0$$

$$(p, P) = (\sqrt{k}\,\mathrm{sn}(\alpha; k), \mathrm{sn}'(\alpha; k)), \quad (q, R) = (\sqrt{k}\,\mathrm{sn}(\beta; k), \mathrm{sn}'(\beta; k)$$

$$(r, R) = (\sqrt{k}\,\mathrm{sn}(\gamma; k), \mathrm{sn}'(\gamma; k)), \quad \gamma = \alpha - \beta$$

points on the elliptic curve:

$$\Gamma = \{(x, X) : X^2 = x^4 + 1 - (k + 1/k)x^2\}$$

Example

- **Maps (eg. QRT)**

Integrable mappings of the plane (Quispel, Roberts, Thompson – 1988,89:

$$x \mapsto \widetilde{x} = \frac{f_1(y) - x\, f_2(y)}{f_2(y) - x\, f_3(y)}, \quad y \mapsto \widetilde{y} = \frac{g_1(\widetilde{x}) - y\, g_2(\widetilde{x})}{g_2(\widetilde{x}) - y\, g_3(\widetilde{x})}$$

f_i, g_i are fourth-order polynomials

Integrability: conserved quantities, symmetries, Lax pair, the behavior around singularities ……

Example

- Painlevé

  Painlevé I ~ VI

$$f'' = 6f^2 + t$$

$$f'' = 2f^3 + tf + \alpha$$

$$tff'' = t(f')^2 - ff' + \delta t + \beta t + \alpha f^3 + \gamma t f^4$$

Example

- Japan
- Australia
- France

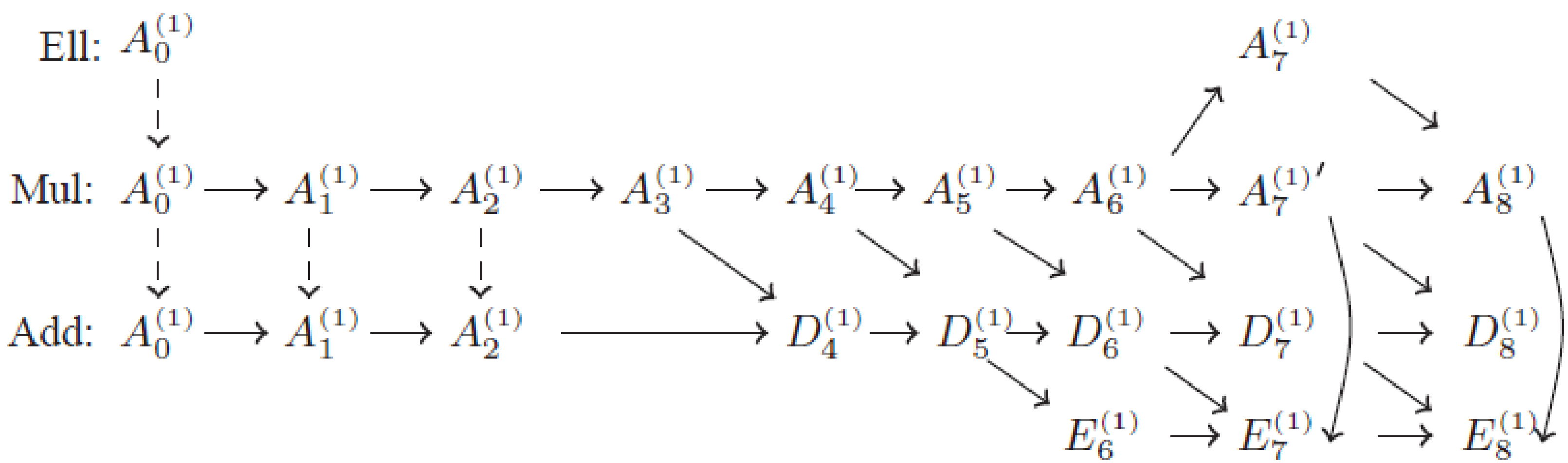
- Discrete Painlevé

d-Painlevé

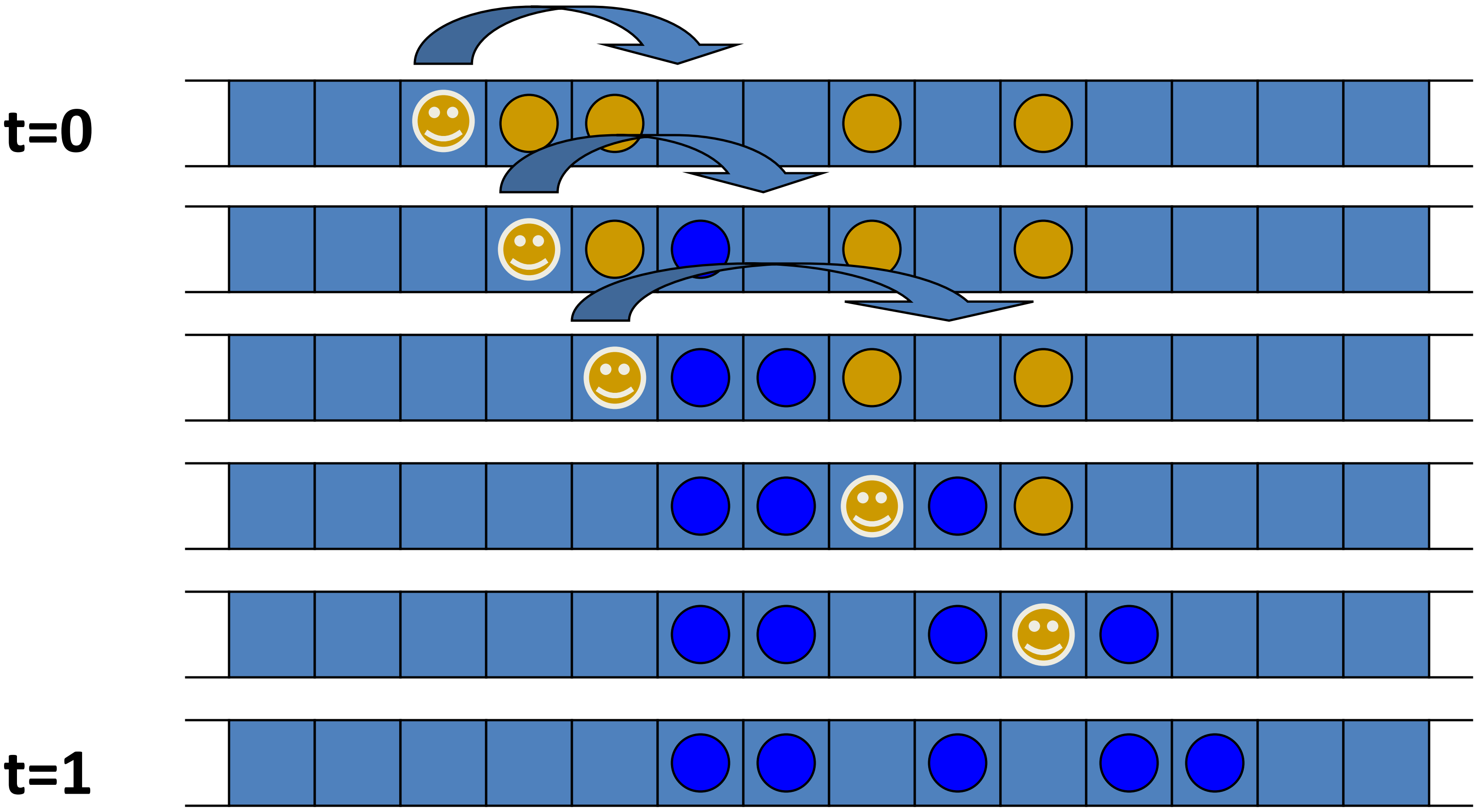
$$\mathrm{dP_I} \;:\quad w\,(\overline{w} + w + \underline{w}) = a\,n + b + c\,w$$

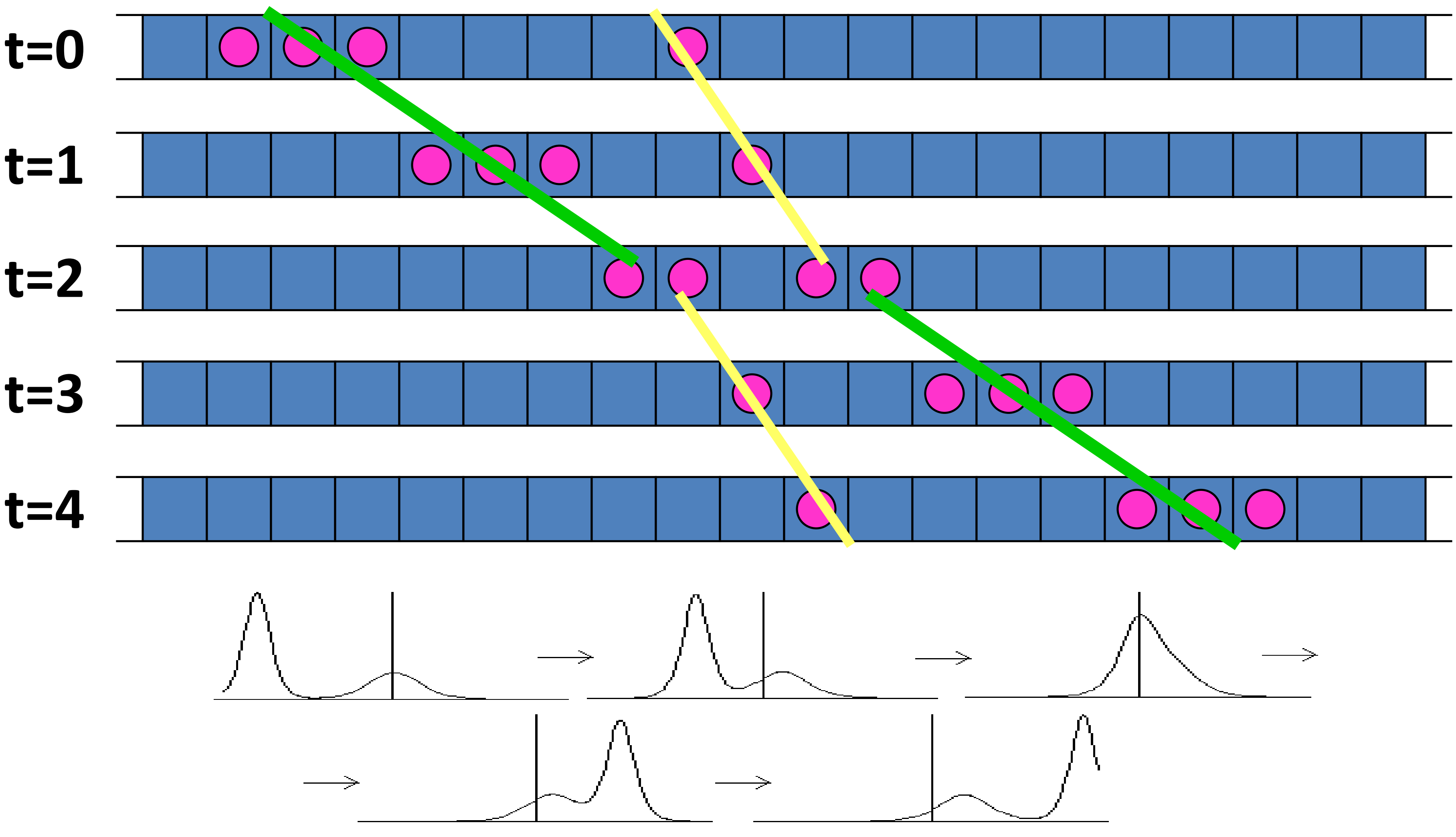
$$\mathrm{dP_{II}} \;:\quad \overline{w} + \underline{w} = \frac{(a\,n + b)\,w + c}{1 - w^2}$$

Sakai's classification (2001):



Takahashi-Satsuma' Rule
--- Box-Ball System (BBS)





t=0
t=1
t=2
t=3
t=4

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Thank You

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For Nov 14

- Lax pair for the KdV
- Lax pair for the AKNS
- Discretisation
- Conservation laws
- KP hierarchy