Hirota Methods and Integrability

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§1 Hirota's Bilinear Integrability

§2 Bilinearity and Transformations

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§A Lax pair and BT of the KdV equation

§A.1 KdV hierarchy

§A.2 BT of the KdV equation

§1 Hirota's Bilinear Integrability

§1.1 Bilinear operator

Definition

Bilinear derivatives:

$$D_x^m D_y^n f(x,y)\cdot g(x,y) = (\partial_x - \partial_{x'})^m (\partial_y - \partial_{y'})^n f(x,y) g(x',y')|_{x'=x, y'=y}. \tag{1}$$

D: **Hirota's bilinear operator**. Here $\partial_x = \frac{\partial}{\partial x}$.

Another defination:

$$e^{\epsilon D_x + \kappa D_y} f(x,y)\cdot g(x,y) = f(x+\epsilon, y+\kappa) g(x-\epsilon, y-\kappa). \tag{2}$$

In fact, expanding both sides at $(\epsilon,\kappa) = (0,0)$ and comparing coefficients of power $\epsilon^m \kappa^n$ we get definition (1).

Examples:

$$D_x^m D_y^n f(x,y)\cdot g(x,y) = (\partial_x-\partial_{x'})^m(\partial_y-\partial_{y'})^n f(x,y)g(x',y')|_{x'=x,y'=y}.$$

$$\begin{aligned}
D_x f\cdot g &= f_x g - f g_x,\\
D_x^2 f\cdot g &= f_{xx} g - 2 f_x g_x + f g_{xx},\\
D_x^3 f\cdot g &= f_{xxx} g - 3 f_{xx} g_x + 3 f_x g_{xx} - f g_{xxx},\\
D_x D_y f\cdot g &= f_{xy} g - f_x g_y - f_y g_x + f g_{xy},\\
D_x^m f\cdot g &= \sum_{j=0}^{m} (-1)^j \mathrm{C}_m^j f^{(m-j)}(x) g^{(j)}(x),
\end{aligned}$$

where $\mathrm{C}_m^j = \frac{m!}{(m-j)!j!}$, $f^{(j)} = \partial_x^j f(x)$.

Compare: Leibniz's rule for $(fg)^{(m)}$.

Properties:

$$D_x^m f \cdot g = (-1)^m D_x^m g \cdot f,$$
$$(aD_x^m + bD_y^n) f \cdot g = aD_x^m f \cdot g + bD_y^n f \cdot g,$$
$$D_x^m D_y^n (af + bg) \cdot h = aD_x^m D_y^n f \cdot h + bD_x^m D_y^n g \cdot h,$$

where $a, b \in \mathbb{C}$, and particularly,

$$D_x^m f \cdot 1 = \partial_x^m f(x).$$

For linear exponential functions

$$D_x^m D_y^n e^{\eta_1} \cdot e^{\eta_2} = (k_1 - k_2)^m (\omega_1 - \omega_2)^n e^{\eta_1 + \eta_2}, \tag{3}$$

where

$$\eta_i = k_i x + \omega_i y + \eta_i^{(0)}, \quad k_i, \omega_i, \eta_i^{(0)} \in \mathbb{C}. \tag{4}$$

Extend to arbitrary dimensions:

$$\mathbf{t} = (t_1, t_2, \cdots, t_s),\ \mathbf{p} = (p_1, p_2, \cdots, p_s),\ \mathbf{q} = (q_1, q_2, \cdots, q_s)$$

$$\mathbf{p} \cdot \mathbf{t} = \sum_{i=1}^{s} p_i t_i, \quad D_{\mathbf{t}} = (D_{t_1}, D_{t_2}, \cdots, D_{t_s}).$$

Then we can define

$$e^{\mathbf{p} \cdot D_{\mathbf{t}}} f(\mathbf{t}) \cdot g(\mathbf{t}) = f(\mathbf{t} + \mathbf{p}) g(\mathbf{t} - \mathbf{p}). \tag{5}$$

Extend to polynomial $P(D_{\mathbf{t}})$:

$$P(D_{\mathbf{t}}) e^{\mathbf{p} \cdot \mathbf{t}} \cdot e^{\mathbf{q} \cdot \mathbf{t}} = P(\mathbf{p} - \mathbf{q}) e^{(\mathbf{p} + \mathbf{q}) \cdot \mathbf{t}}, \tag{6}$$

$$P(D_{\mathbf{t}}) e^{\mathbf{p} \cdot \mathbf{t}} \cdot 1 = P(\mathbf{p}) e^{\mathbf{p} \cdot \mathbf{t}} = P(\partial_{\mathbf{p}}) e^{\mathbf{p} \cdot \mathbf{t}}. \tag{7}$$

Gauge property of bilinear derivatives:

$$D_x^r D_y^s (e^{\eta_1} f(x,y)) \cdot (e^{\eta_2} f(x,y)) = e^{\eta_1+\eta_2}(D_x + k_1 - k_2)^r (D_y + \omega_1 - \omega_2)^s f(x,y) \cdot g(x,y), \quad (8)$$

when $\eta_1 = \eta_2$, one has gauge property

$$D_x^r D_y^s (e^{\eta_1} f) \cdot (e^{\eta_1} g) = e^{2\eta_1} D_x^r D_y^s f \cdot g, \quad (9)$$

More general case:

$$P(D_{\mathbf{t}})(e^{\mathbf{p}\cdot\mathbf{t}} f(\mathbf{t})) \cdot (e^{\mathbf{q}\cdot\mathbf{t}} g(\mathbf{t})) = e^{(\mathbf{p}+\mathbf{q})\cdot\mathbf{t}} P(D_{\mathbf{t}} + \mathbf{p} - \mathbf{q}) f(\mathbf{t}) \cdot g(\mathbf{t}), \quad (10)$$

$$P(D_{\mathbf{t}})(e^{\mathbf{p}\cdot\mathbf{t}} f) \cdot (e^{\mathbf{p}\cdot\mathbf{t}} g) = e^{2\mathbf{p}\cdot\mathbf{t}} P(D_{\mathbf{t}}) f \cdot g \,. \quad (11)$$

§1.2 N soliton solutions

§1.2.1 NSS of the KdV equation

KdV:

$$u_t + 6uu_x + u_{xxx} = 0. \tag{12}$$

potential form:

$$w_t + 3(w_x)^2 + w_{xxx} = 0, \quad (u = w_x). \tag{13}$$

Transformation

$$u = 2(\ln f)_{xx}, \quad \text{i.e. } w = 2(\ln f)_x, \tag{14}$$

Bilinear form: $f_{xt}f - f_x f_t + f_{xxxx}f - 4f_{xxx}f_x + 3(f_{xx})^2 = 0$, i.e.

$$(D_x D_t + D_x^4)f \cdot f = 0. \tag{15}$$

Expand

$$f = 1 + \sum_{i=1}^{\infty} f^{(i)} \varepsilon^i, \tag{16}$$

$$\varepsilon^1: \quad (\partial_{xt} + \partial_x^4) f^{(1)} = 0, \tag{17a}$$

$$\varepsilon^2: \quad (\partial_{xt} + \partial_x^4) f^{(2)} = -\frac{1}{2}(D_x D_t + D_x^4) f^{(1)} \cdot f^{(1)}, \tag{17b}$$

$$\varepsilon^3: \quad (\partial_{xt} + \partial_x^4) f^{(1)} = -(D_x D_t + D_x^4) f^{(1)} \cdot f^{(2)}, \tag{17c}$$

$$\varepsilon^4: \quad (\partial_{xt} + \partial_x^4) f^{(1)} = -(D_x D_t + D_x^4)(f^{(1)} \cdot f^{(3)} + \frac{1}{2} f^{(2)} \cdot f^{(2)}),$$

Take $f^{(1)} = e^{\eta_i}$, where

$$\eta_i = k_i x - k_i^3 t + \eta_i^{(0)}, \quad k_i, \eta_i^{(0)} \in \mathbb{R}. \tag{18}$$

(17a) is a homogeneous linear equation:

$$f^{(1)} = \sum_{i=1}^{N} e^{\eta_i} \tag{19}$$

1SS: $N = 1$,

$$f = 1 + \varepsilon e^{\eta_1} \tag{20}$$

$$u = 2(\ln f)_{xx} = 2[\ln(1 + e^{\eta_1})]_{xx} \tag{21}$$

2SS: $N = 2$, $u = 2(\ln f)_{xx}$,

$$f = 1 + \varepsilon(e^{\eta_1} + e^{\eta_2}) + \varepsilon^2 A_{12}\, e^{\eta_1+\eta_2}, \quad A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2. \tag{22}$$

3SS: $N = 3$, $u = 2(\ln f)_{xx}$,

$$\begin{aligned} f =& 1 + \varepsilon(e^{\eta_1} + e^{\eta_2} + e^{\eta_3}) \\ & + \varepsilon^2(A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_3+\eta_3}) \\ & + \varepsilon^3 A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3}, \quad A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2. \end{aligned} \tag{23}$$

NSS: For a general number N, Hirota gave the following compact form:

$$f = \sum_{\mu=0,1} \exp\left(\sum_{j=1}^{N} \mu_j \eta_j + \sum_{1\leq i<j}^{N} \mu_i \mu_j a_{ij}\right), \qquad (24)$$

where η_j is defined as in (18), $e^{a_{ij}} = A_{ij}$, and the summation of μ means to take all possible $\mu_j = \{0, 1\}$ $(j = 1, 2, \cdots, N)$.

§1.2.2 NSS of the KP(II) equation

KP(II) ($\sigma = 1$):

$$(4u_t + 6uu_x + u_{xxx})_x + 3\sigma u_{yy} = 0, \quad (\sigma = \pm 1), \tag{25}$$

Transformation

$$u = 2(\ln f)_{xx}, \tag{26}$$

Bilinear KP(II):

$$(4D_x D_t + D_x^4 + 3D_y^2) f \cdot f = 0. \tag{27}$$

With the expension

$$f = 1 + \sum_{i=1}^{\infty} f^{(i)} \varepsilon^i \tag{28}$$

the bilinear KP(II) equation (27) yields

$$\varepsilon^1 : \quad (4\partial_{xt} + \partial_x^4 + 3\partial_y^3) f^{(1)} = 0, \tag{29a}$$

$$\varepsilon^2 : \quad (4\partial_{xt} + \partial_x^4 + 3\partial_y^3) f^{(2)} = -\frac{1}{2}(4D_x D_t + D_x^4 + 3D_y^2) f^{(1)} \cdot f^{(1)}, \tag{29b}$$

$\cdots\cdots$.

For (29a) we can take $f^{(1)} = e^{kx+hy+\omega t}$, and k, h, ω satisfy

$$4k\omega + k^4 + 3h^2 = 0. \tag{30}$$

Parametrisation:

$$k = p - q, \quad h = p^2 - q^2, \quad \omega = -(p^3 - q^3). \tag{31}$$

$$f^{(1)} = \sum_{i=1}^{N} e^{\eta_i}, \tag{32}$$

$$\eta_i = (p_i - q_i)x + (p_i^2 - q_i^2)y - (p_i^3 - q_i^3)t + \eta_i^{(0)}, \quad p_i, q_i, \eta_i^{(0)} \in \mathbb{R}. \tag{33}$$

1SS:

$$f = 1 + f^{(1)} = 1 + e^{\eta_1}, \tag{34}$$

$$u = 2(\ln f)_{xx} = \frac{(p_1 - q_1)^2}{2} \mathrm{sech}^2 \eta_1. \tag{35}$$

2SS:

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}\, e^{\eta_1 + \eta_2}, \tag{36a}$$

$$A_{ij} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)}. \tag{36b}$$

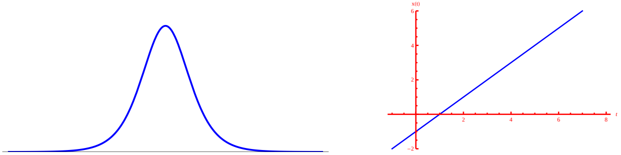
NSS:

$$f = \sum_{\mu=0,1} \exp\left(\sum_{j=1}^{N} \mu_j \eta_j + \sum_{1 \le i < j}^{N} \mu_i \mu_j a_{ij} \right). \tag{37}$$

§1.3 Asymptotic analysis of 2SS

§1.3.1 The KdV equation

$$1SS: \qquad u = \frac{k_1^2}{2}\mathrm{sech}^2\eta_1, \quad \eta_1 = k_1 x - k_1^3 t + \eta^{(0)}, \qquad (38)$$



(a) (b)

Figure: (a) 1SS of the KdV equation. (b) trajectory of the vertex of 1SS.

Amplitude: $\frac{k_1^2}{2}$, speed: $x'(t) = k_1^2$, vertex: $\eta_1 = 0$:

$$x(t) = k_1^2 t + \frac{\eta_1^{(0)}}{k_1}. \qquad (39)$$

2SS:

$$u = 2(\ln f)_{xx}, \tag{40a}$$

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}\, e^{\eta_1+\eta_2}, \tag{40b}$$

$$\eta_i = k_i x - k_i^3 t + \eta_i^{(0)}, \quad A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2. \tag{40c}$$

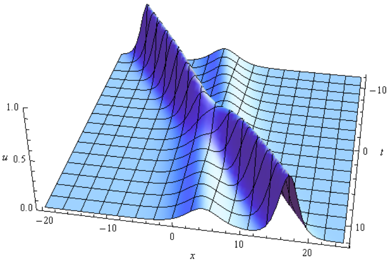


Figure: 2SS (40) of the KdV equation ($k_1 = 0.8,\ k_2 = 1.2,\ \eta_1^{(0)} = \eta_1^{(0)} = 0$).

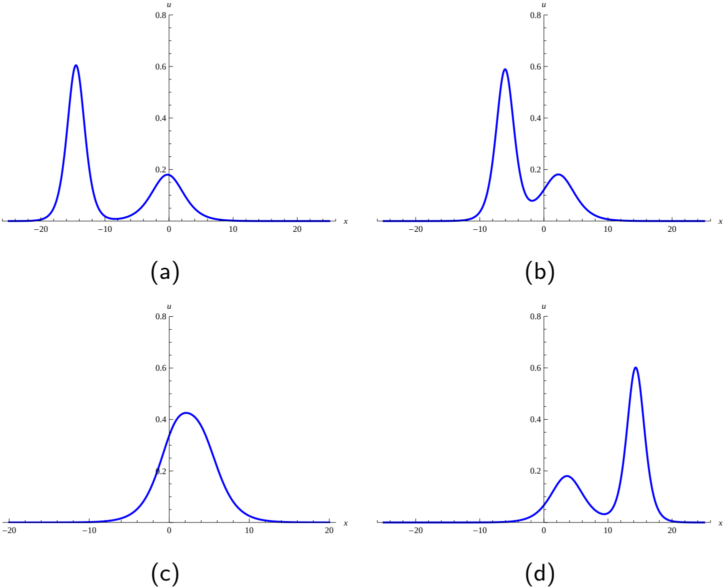


Figure: (a) $t=-12$, (b) $t=-5$, (c) $t=1$, (d) $t=10$.

Assume $k_1 > k_2 > 0$. Suppose $\eta_1 = c$, rewrite (40b) in (η_1, t):

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}\, e^{\eta_1+\eta_2}, \tag{41a}$$

where

$$e^{\eta_2} = \exp\Big[\frac{k_2}{k_1}\eta_1 + k_2(k_1^2 - k_2^2)t + \eta_2^{(0)} - \frac{k_2}{k_1}\eta_1^{(0)}\Big]. \tag{41b}$$

Because of $k_1 > k_2 > 0$, we find

$$e^{\eta_2} \sim \begin{cases} 0, & t \to -\infty, \\ +\infty, & t \to +\infty. \end{cases}$$

Therefore in (η_1, t) we have

$$f \sim \begin{cases} 1 + e^{\eta_1}, & t \to -\infty, \\ e^{\eta_2}(1 + A_{12}e^{\eta_1}), & t \to +\infty. \end{cases} \tag{42}$$

In (η_1, t):

$$f \sim \begin{cases} 1 + e^{\eta_1}, & t \to -\infty, \\ e^{\eta_2}(1 + A_{12}e^{\eta_1}), & t \to +\infty. \end{cases} \quad (43)$$

Gauge property: $f \sim e^{\eta_2}(1 + A_{12}e^{\eta_1}) \sim 1 + A_{12}e^{\eta_1}, \quad t \to +\infty.$

$$D_x^r D_y^s (e^{\eta_1} f) \cdot (e^{\eta_1} g) = e^{2\eta_1} D_x^r D_y^s f \cdot g$$

Thus, if we observe 2SS along the straight line $\eta_1 = c$, when $t \to -\infty$ we only see the 1SS

$$u = 2[\ln(1 + e^{\eta_1})]_{xx}; \quad (44a)$$

and when $t \to +\infty$, we see

$$u = 2[\ln(1 + A_{12}e^{\eta_1})]_{xx}, \quad (44b)$$

which is still the original 1SS (44a) (same amplitude and velocity as before interaction) but obtains a phase shift $-\frac{2}{k_1}\ln\left(\frac{k_1-k_2}{k_1+k_2}\right)$.

Phase shift:

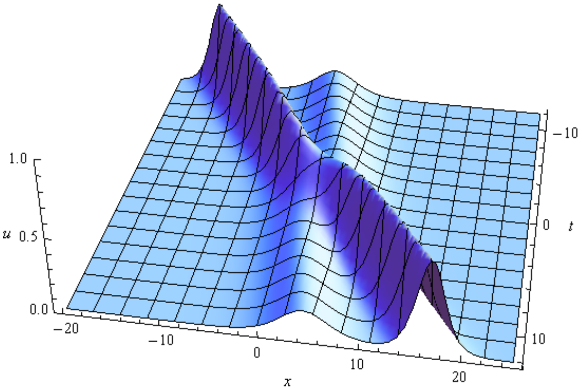


Figure: 2SS (40) of the KdV equation ($k_1 = 0.8,\ k_2 = 1.2,\ \eta_1^{(0)} = \eta_1^{(0)} = 0$).

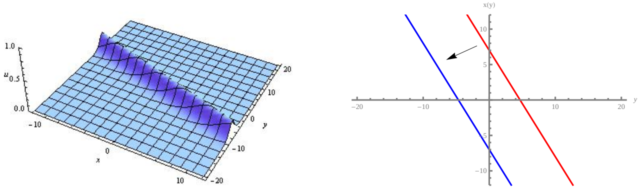
§1.3.2 The KP(II) equation

$$1SS: \qquad u = 2(\ln f)_{xx} = \frac{(p_1 - q_1)^2}{2}\mathrm{sech}^2\eta_1,$$

$$\eta_i = (p_i - q_i)x + (p_i^2 - q_i^2)y + (p_i^3 - q_i^3)t + \eta_i^{(0)}.$$

A straight line on (x, y) plane, amplitude $(p_1 - q_1)^2/2$, velocity:

$$(x'(t), y'(t)) = -(p^2 + pq + q^2)\Big(1, \frac{1}{p+q}\Big).$$



(a) (b)

Figure: (a) 1SS of the KP(II). (b) Trajectory of the line soliton in (a): red line is for $t = -4$ and blue for $t = 4$.

2SS:

$$u = 2(\ln f)_{xx}, \quad f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}\, e^{\eta_1+\eta_2},$$

$$A_{ij} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)}.$$

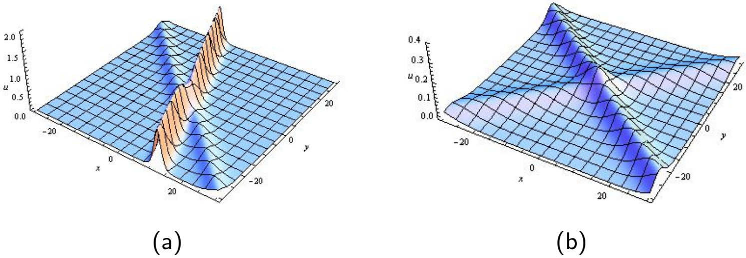


Figure: 2SS of the KP(II) equation.

Resonant Case:

$$2SS: \qquad u = 2(\ln f)_{xx}, \quad f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}\, e^{\eta_1+\eta_2},$$

$$A_{ij} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)}.$$

$p_1 \neq p_2$ but $q_1 = q_2$, we have $A_{12} = 0$ and f degenerates to

$$f = 1 + e^{\eta_1} + e^{\eta_2}, \tag{45}$$

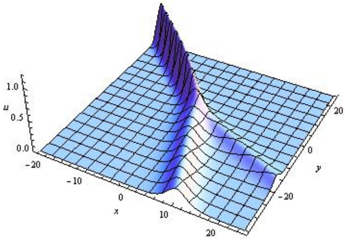


Figure: 2SS resonance of the KP(II) equation.

Asymptotic analysis:

refer to the lecture note

History for “resonance”:

Lamb (1971), Miles, Wadati

Further reading:

Jarmo Hietarinta review

Yuji Kodama, et al

§1.4 2SS of bilinear equations

§1.4.1 Bilinear equations of the KdV-type

Many bilinear equations automatically admit 1SS and 2SS.

KdV-type bilinear equation:

$$P(D_{\mathbf{t}})f\cdot f=0, \tag{46}$$

where P is an even polynomial: $P(\mathbf{t})=P(-\mathbf{t})$, $P(\mathbf{0})=0$.

1SS:

$$f=1+e^{\eta_1},\quad \eta_1=\mathbf{p_1}\cdot\mathbf{t}+\eta_1^{(0)}. \tag{47}$$

$$f\cdot f=1\cdot 1+1\cdot e^{\eta_1}+e^{\eta_1}\cdot 1+e^{\eta_1}\cdot e^{\eta_1}.$$

$$D_x^m D_y^n e^{\eta_1}\cdot e^{\eta_2}=(k_1-k_2)^m(\omega_1-\omega_2)^n e^{\eta_1+\eta_2},$$

$$P(D_{\mathbf{t}})f\cdot f=2P(D_{\mathbf{t}})e^{\eta_1}\cdot 1=2P(\partial_{\mathbf{t}})e^{\eta_1}=2P(\mathbf{p_1})e^{\eta_1}.$$

Dispersion relation (DR): $P(\mathbf{p_1})=0$.

Automatical 2SS:

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2}, \tag{48a}$$

where A_{12} is a constant to be determined,

$$\eta_i = \mathbf{p_i} \cdot \mathbf{t} + \eta_i^{(0)} \tag{48b}$$

satisfying DR

$$P(\mathbf{p_i}) = 0. \tag{48c}$$

Substitute (48a) into the equation (46) and making use of the DR (48c), we find

$$A_{12} = -\frac{P(\mathbf{p_1} - \mathbf{p_2})}{P(\mathbf{p_1} + \mathbf{p_2})}. \tag{48d}$$

It is Hirota who first found this fact [Hirota-1980].

"automatically": no extra condition on $\mathbf{p_i}$ beyond the DR (48c).

§1.4.2 Other cases

Example [Hietarinta-1987c]

$$B(D_{\mathbf{t}})G\cdot F=0, \tag{49a}$$

$$A(D_{\mathbf{t}})(F\cdot F+\epsilon G\cdot G)=0, \tag{49b}$$

where A is an even polynomial and $\epsilon=\pm1$. 1SS;

$$F=1,\quad G=e^{\eta_1},\quad \eta_1=\mathbf{p_1}\cdot\mathbf{t}+\eta_1^{(0)},$$

where η_1 satisfies DR: $B(\mathbf{p_1})=0$. One type of 2SS of (49) is:

$$F=1-A_{12}e^{\eta_1+\eta_2},\quad G=e^{\eta_1}+e^{\eta_2}, \tag{50a}$$

where $\eta_i=\mathbf{p_i}\cdot\mathbf{t}+\eta_i^{(0)}$ satisfies DR $B(\mathbf{p_i})=0$, and

$$A_{12}=-\epsilon\,\frac{A(\mathbf{p_1}-\mathbf{p_2})}{A(\mathbf{p_1}+\mathbf{p_2})}. \tag{50b}$$

Not any bilinear equation (system) automatically admits a 2SS.

Example [Hietarinta-1988]:

$$B(D_{\mathbf{t}})G \cdot F = 0, \tag{51a}$$

$$A(D_{\mathbf{t}})F \cdot F = 2\epsilon|G|^2, \tag{51b}$$

where A is an even polynomial, $F \in \mathbb{R}(\mathbf{t})$ and $G \in \mathbb{C}(\mathbf{t})$. It has 1SS:

$$F = 1 + a\, e^{\eta_1 + \eta_1^*}, \quad G = e^{\eta_1}$$

where $\eta_1 = \mathbf{p_1} \cdot \mathbf{t} + \eta_1^{(0)}$, $\mathbf{p} \in \mathbb{C}^s$, $\eta_1^{(0)} \in \mathbb{C}$, $B(\mathbf{p_1}) = 0$, $a = \frac{-\epsilon}{A(\mathbf{p_1}+\mathbf{p_1}^*)}$, $*$ stands for complex conjugate, $|G|^2 = GG^*$. However, its 2SS does not exist automatically.

§1.5 Hirota's integrability and 3SS condition

Hirota integrable:

Take the KdV-type bilinear equation (46) as an example:

$$P(D_{\mathbf{t}})f \cdot f = 0, \tag{52}$$

it is said to be Hirota-integrable if for all positive integers N it has NSS of the form

$$f = 1 + \varepsilon \sum_{i=1}^{N} e^{\eta_i} + \{\text{finite number of higher-order terms in } \varepsilon\}, \tag{53}$$

without any further conditions on the parameters p_i beyond DR

$$P(\mathbf{p_i}) = 0. \tag{54}$$

Note: 1SS-condition (e.g. DR).

Hirota's NSS for KdV-type bilinear equation (52) [Hirota-1980]:

$$f = \sum_{\mu=0,1} \exp\left(\sum_{j=1}^{N} \mu_j \eta_j + \sum_{1\le i<j}^{N} \mu_i \mu_j a_{ij}\right), \tag{55a}$$

where $\eta_j = \mathbf{p_j} \cdot \mathbf{t} + \eta_j^{(0)}$,

$$P(\mathbf{p_i}) = 0, \quad e^{a_{ij}} = A_{ij} = -\frac{P(\mathbf{p_i} - \mathbf{p_j})}{P(\mathbf{p_i} + \mathbf{p_j})}, \tag{55b}$$

and further than that, the following condition is needed:

$$\sum_{\sigma=\pm1} \left[P(\sum_{j=1}^{N} \sigma_j \mathbf{p_j}) \times \Big(\prod_{1\le i<j\le N} \sigma_i \sigma_j P(\sigma_i \mathbf{p_i} - \sigma_j \mathbf{p_j}) \Big) \right] = 0. \tag{56}$$

This condition holds automatically for the $N = 2$ case.

Example: Having 3SS but not Hirota integrable: [Hirota-1973]

(2+1)-dimensional sine-Gordon equation

$$\varphi_{xx} + \varphi_{yy} - \varphi_{tt} = \sin\varphi, \tag{57}$$

transformation

$$\varphi = 4 \arctan \frac{g}{f}, \tag{58}$$

bilinear form

$$(D_x^2 + D_y^2 - D_t^2) g \cdot f = gf, \tag{59a}$$

$$(D_x^2 + D_y^2 - D_t^2)(f \cdot f - g \cdot g) = 0. \tag{59b}$$

2SS automatically (see (49) and (50)), and its 3SS reads

$$f = 1 + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3}, \tag{60a}$$

$$g = e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3}, \tag{60b}$$

where

$$\eta_i = a_i x + b_i y - c_i t + \eta_i^{(0)}, \tag{60c}$$

$$\mathrm{DR}: \ a_i^2 + b_i^2 - c_i^2 = 1, \tag{60d}$$

$$A_{ij} = \frac{(a_i - a_j)^2 + (b_i - b_j)^2 - (c_i - c_j)^2}{(a_i + a_j)^2 + (b_i + b_j)^2 - (c_i + c_j)^2}, \tag{60e}$$

and an extra condition is needed:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0. \tag{60f}$$

Hirota's question in [Hirota-1980]:

For the KdV-type bilinear equations: "Under what conditions does P satisfy the identity (56)?"

3SS-condition:

a bilinear equation (system) has a 3SS, and the condition on each η_i is nothing beyond the 1SS-condition.

Conjecture: 3SS-condition equivalent to Hirota integrable.

Elastic scattering property of multi-solitons:

"Removing a soliton from NSS, the left (N-1) solitons keep the elastic scattering structure of (N-1)SS".

Example: KdV-type: elastic scattering behavior of multi-solitons determines structure of NSS.

KdV-type admits 2SS automatically.

Removing a soliton: either $e^{\eta_k} \to 0$ or $e^{\eta_k} \to \infty$.

2SS of the KdV-type bilinear equation (52):

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1+\eta_2}, \tag{61a}$$

$$P(\mathbf{p_i}) = 0, \tag{61b}$$

$$A_{ij} = -\frac{P(\mathbf{p_i} - \mathbf{p_j})}{P(\mathbf{p_i} + \mathbf{p_j})}. \tag{61c}$$

With the requirement of elastic scattering, if no further condition on $\mathbf{p_i}$ beyond (61b), 3SS (if it exists) must be

$$\begin{aligned} f =&1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} \\ &+ A_{12} e^{\eta_1+\eta_2} + A_{13} e^{\eta_1+\eta_3} + A_{23} e^{\eta_3+\eta_3} \\ &+ A_{12}A_{13}A_{23} e^{\eta_1+\eta_2+\eta_3}, \end{aligned} \tag{62}$$

3SS to NSS:

If we start from the 3SS (62) and using the requirement of elastic scattering once again, we can reach a form for 4SS (if existing). Continuing such a procedure one can obtain 5SS, 6SS, $\cdots\cdots$.

Thus, for the KdV-type bilinear equation (52), if we only require the DR (61b) and elastic scattering property, its NSS (if it has) can only be the form (55), i.e.

$$f = \sum_{\mu=0,1} \exp\left(\sum_{j=1}^{N} \mu_j \eta_j + \sum_{1\leq i<j}^{N} \mu_i \mu_j a_{ij}\right). \tag{63}$$

3SS-condition:

Jarmo Hietarinta found that the KdV-type bilinear equations that satisfy 3SS-condition are: [Hietarinta-1987]

$$
\begin{aligned}
&(D_x^4 - 4D_xD_t + 3D_y^2)f \cdot f = 0,\\
&(D_x^3D_t + aD_x^2 + D_tD_y)f \cdot f = 0,\\
&[D_xD_t(D_x^2 + \sqrt{3}D_xD_t + D_t^2) + aD_x^2 + bD_xD_t + cD_t^2]f \cdot f = 0,\\
&(D_x^6 + 5D_x^3D_t - 5D_t^2 + D_xD_y)f \cdot f = 0,
\end{aligned}
$$

etc., where, a, b, c are arbitrary constants.

Further reading: [Hietarinta-1987,1988]

§2 Bilinearity and Transformations

§2 Bilinearity and Transformations

§2.1 Bilinear identities

Property

The following equality holds:

$$\begin{aligned} & e^{D_1}(e^{D_2}a\cdot b)\cdot(e^{D_3}c\cdot d) \\ = & \, e^{\frac{1}{2}(D_2-D_3)}(e^{\frac{1}{2}(D_2+D_3)+D_1}a\cdot d)\cdot(e^{\frac{1}{2}(D_2+D_3)-D_1}c\cdot b), \end{aligned} \quad (64)$$

where $D_i=\varepsilon_i D_x+\delta_i D_t$, $\varepsilon_i,\delta_i\in\mathbb{R}$, *and* a,b,c,d *are sufficiently smooth functions of* (x,t).

Example 2.1.1: Taking $D_2 = D_3$ in (64) yields

$$e^{D_1}(e^{D_2}a \cdot b) \cdot (e^{D_2}c \cdot d) = (e^{D_2+D_1}a \cdot d) \cdot (e^{D_2-D_1}c \cdot b). \quad (65)$$

Next, taking $D_1 = \delta D_x,\ D_2 = \varepsilon D_x$ in (65) and expanding the exponential functions of both sides, we have

$$\begin{aligned}&(1+\delta D_x+\cdots)[(1+\varepsilon D_x+\cdots)a \cdot b)] \cdot [(1+\varepsilon D_x+\cdots)c \cdot d)]\\ =& [(1+(\varepsilon+\delta)D_x+\frac{1}{2}(\varepsilon+\delta)^2 D_x^2+\cdots)a \cdot b)]\\ &\times [(1+(\varepsilon-\delta)D_x+\frac{1}{2}(\varepsilon-\delta)^2 D_x^2+\cdots)c \cdot d)].\end{aligned}$$

The coefficient of the term $\varepsilon\delta$ leads to a bilinear identity

$$D_x[(D_x a \cdot b) \cdot (cd) - (D_x c \cdot d) \cdot (ab)] = (D_x^2 a \cdot b)cd - (D_x^2 c \cdot d)ab. \quad (66)$$

Example 2.1.2: Taking $D_2 = D_3, b = c, d = a$ in (64) yields

$$e^{D_1}(e^{D_2} a \cdot c) \cdot (e^{D_2} c \cdot a) = (e^{D_2+D_1} a \cdot a) \cdot (e^{D_2-D_1} c \cdot c). \quad (67)$$

Then we take $D_1 = \varepsilon D_x,\ D_2 = \delta D_t$, and from the coefficient of $\varepsilon\delta$ term in the expansion we find

$$2D_x(D_t a \cdot c) \cdot (ac) = (D_x D_t a \cdot a)c^2 - (D_x D_t c \cdot c)a^2; \quad (68)$$

from ε^4 term we find

$$2D_x[(D_x^3 a \cdot c) \cdot (ac) - 3(D_x^2 a \cdot c) \cdot (D_x a \cdot c)] = (D_x^4 a \cdot a)c^2 - (D_x^4 c \cdot c)a^2. \quad (69)$$

§2.2 Bilinear Bäcklund transformations

Bilinear KdV equation (15):

$$(D_x D_t + D_x^4) f \cdot f = 0 \tag{70}$$

Suppose that g is also a solution of (70), i.e.

$$(D_x D_t + D_x^4) g \cdot g = 0. \tag{71}$$

Then

$$g^2 (D_x D_t + D_x^4) f \cdot f - f^2 (D_x D_t + D_x^4) g \cdot g = 0, \tag{72}$$

i.e.

$$[(D_x D_t f \cdot f) g^2 - (D_x D_t g \cdot g) f^2] + [(D_x^4 f \cdot f) g^2 - (D_x^4 g \cdot g) f^2] = 0. \tag{73}$$

Employing the identities (68) and (69), (taking $a = f, c = g$):

$$2D_x(D_t a \cdot c) \cdot (ac) = (D_x D_t a \cdot a)c^2 - (D_x D_t c \cdot c)a^2; \tag{68}$$

$$2D_x[(D_x^3 a \cdot c) \cdot (ac) - 3(D_x^2 a \cdot c) \cdot (D_x a \cdot c)] = (D_x^4 a \cdot a)c^2 - (D_x^4 c \cdot c)a^2. \tag{69}$$

rewrite (73) as

$$2D_x[(D_x^3 + D_t)f \cdot g] \cdot (fg) + 6D_x[(D_x f \cdot g) \cdot (D_x^2 f \cdot g)] = 0. \tag{74}$$

Next, introduce

$$D_x^2 f \cdot g = \lambda fg, \tag{75a}$$

where λ is a constant, by which (74) yields

$$2D_x[(D_x^3 + D_t + \lambda D_x)f \cdot g] \cdot (fg) = 0.$$

Then we can take

$$(D_x^3 + D_t + 3\lambda D_x)f \cdot g = 0. \tag{75b}$$

(75a,b) compose a bilinear Bäcklund transformation of KdV.

Get a solution using BT:

First, taking $\lambda = \frac{k_1^2}{4}$ and $f = 1$ (noticing that $f = 1$ is a solution to (70)), BT (75) $\Longrightarrow$

$$g_{xx} = \frac{k_1^2}{4} g,$$
$$g_t + g_{xxx} + \frac{3}{4} k_1^2 g_x = 0.$$

Solutoion

$$g = g_1 = e^{\frac{\eta_1}{2}} + e^{-\frac{\eta_1}{2}},$$
$$\eta_i = k_i x - k_i^3 t + \eta_i^{(0)}, \quad k_i, \eta_i \in \mathbb{R}. \tag{76}$$

1SS of the KdV: $u = 2(\ln g_1)_{xx}$.

Get a solution using BT:

Next, taking $f = g_1,\ \lambda = \frac{k_2^2}{4}$, BT (75) $\Longrightarrow$

$$(D_x^2 - \frac{k_2^2}{4}) g \cdot (e^{\frac{\eta_1}{2}} + e^{-\frac{\eta_1}{2}}) = 0, \tag{77a}$$

$$(D_x^3 + D_t + \frac{3}{4} k_2^2 D_x) g \cdot (e^{\frac{\eta_1}{2}} + e^{-\frac{\eta_1}{2}}) = 0. \tag{77b}$$

Assume

$$g = g_2 = \alpha(e^{\frac{\eta_1+\eta_2}{2}} + e^{-\frac{\eta_1+\eta_2}{2}}) + \beta(e^{\frac{\eta_1-\eta_2}{2}} + e^{-\frac{\eta_1-\eta_2}{2}}),$$

where η_i is defined as (76) and α, β are undetermined constants.

(77) $\Longrightarrow \alpha = k_1 - k_2$ and $\beta = -(k_1 + k_2)$.

For N:

$$g_N = \sum_{\varepsilon=\pm 1} \left[\prod_{1 \leq j < l}^{N} \varepsilon_l (\varepsilon_j k_j - \varepsilon_l k_l) e^{\frac{1}{2} \sum_{j=1}^{N} \varepsilon_j \eta_j} \right], \tag{78}$$

where η_j is defined as (76).

§2.3 Deformations of bilinear BT

Why deformation? Simplify calculation, $f=1,\ g=1$ not solution.

$$D_x^2 f\cdot g=\lambda fg, \tag{75a}$$

$$(D_x^3+D_t+3\lambda D_x)f\cdot g=0. \tag{75b}$$

How? (Keep solutions for KdV.)

$$f\to e^{\xi_1}f,\quad g\to e^{\xi_2}g,\ \ \xi_i=p_ix+q_it+\xi_i^{(0)},\ p_i,q_i,\xi_i^{(0)}\in\mathbb{R}. \tag{79}$$

Tools: Identity (8),

$$D_x(e^{\xi_1}f)\cdot(e^{\xi_2}g)=e^{\xi_1+\xi_2}[(p_1-p_2)fg+D_xf\cdot g],$$

$$D_x^2(e^{\xi_1}f)\cdot(e^{\xi_2}g)=e^{\xi_1+\xi_2}[(p_1-p_2)^2fg+2(p_1-p_2)D_xf\cdot g+D_x^2f\cdot g],$$

$$D_x^3(e^{\xi_1}f)\cdot(e^{\xi_2}g)=e^{\xi_1+\xi_2}[(p_1-p_2)^3fg+3(p_1-p_2)^2D_xf\cdot g$$
$$+3(p_1-p_2)D_x^2f\cdot g+D_x^3f\cdot g],$$

$$D_t(e^{\xi_1}f)\cdot(e^{\xi_2}g)=e^{\xi_1+\xi_2}[(q_1-q_2)fg+D_xf\cdot g].$$

Rewrite BT (75) into

$$[D_x^2 + 2(p_1 - p_2)D_x]f \cdot g = [\lambda - (p_1 - p_2)^2]fg, \tag{80a}$$

$$\{D_t + D_x^3 + 3(p_1 - p_2)D_x^2 + [3\lambda - (p_1 - p_2)^2 D_x]\}f \cdot g$$
$$= -[(q_1 - q_2) + (p_1 - p_2)^3 + 3\lambda(p_1 - p_2)^2]fg. \tag{80b}$$

Introducing

$$2(p_1 - q_1) = \lambda', \quad \lambda = (p_1 - p_2)^2, \quad (q_1 - q_2) + 4(p_1 - p_2)^3 = 0,$$

to simplify (80a) to

$$(D_x^2 + \lambda' D_x)f \cdot g = 0, \tag{81a}$$

and eliminating D_x^2 term in (80b), we reach

$$(D_t + D_x^3)f \cdot g = 0. \tag{81b}$$

(81a,b) compose a deformed bilinear BT of KdV.

Advantage: more freedom

$$f=1+\sum_{i=1}^{\infty} f^{(i)}\varepsilon^{i}, \quad g=1+\sum_{i=1}^{\infty} g^{(i)}\varepsilon^{i} \tag{82}$$

$$(\partial_x^2+\lambda'\partial_x)(f^{(1)}-g^{(1)})=0, \tag{83a}$$

$$(\partial_x^2+\lambda'\partial_x)(f^{(2)}-g^{(2)})=-(D_x^2+\lambda' D_x)f^{(1)}\cdot g^{(1)}, \tag{83b}$$

$$(\partial_x^2+\lambda'\partial_x)(f^{(3)}-g^{(3)})=-(D_x^2+\lambda' D_x)(f^{(1)}\cdot g^{(2)}-f^{(2)}\cdot g^{(1)}),$$

$$\cdots\cdots;$$

$$(\partial_t+\partial_x^3)(f^{(1)}-g^{(1)})=0, \tag{83c}$$

$$(\partial_t+\partial_x^3)(f^{(2)}-g^{(2)})=-(D_t+D_x^3)f^{(1)}\cdot g^{(1)}, \tag{83d}$$

$$(\partial_t+\partial_x^3)(f^{(3)}-g^{(3)})=-(D_t+D_x^3)(f^{(1)}\cdot g^{(2)}-f^{(2)}\cdot g^{(1)}),$$

$$\cdots\cdots.$$

§2.4 BT and Lax pairs

A bilinear Bäcklund transformation appears as an equation system and actually requires compatibility among these equations.

Lax pair is a pair of linear equations of which the compatibility yields integrable equations.

For the KdV equation (12), i.e.

$$u_t + 6uu_x + u_{xxx} = 0, \tag{84}$$

its Lax pair reads

$$\phi_{xx} + u\phi = \lambda\phi, \tag{85a}$$

$$\phi_t = \phi_{xxx} + (3\lambda + u)\phi_x. \tag{85b}$$

Derive KdV hierarchy

$$\phi_{xx} + u\phi = \lambda\phi, \tag{86}$$

$$\phi_t = A\phi + B\phi_x, \tag{87}$$

A and B undetermined functions of u and λ, $(\lambda_t = 0)$. From the compatibility condition $(\phi_{xx})_t = (\phi_t)_{xx}$ we have

$$(2A_x + B_{xx})\phi_x + [u_t + A_{xx} + 2(\lambda - u)B_x - u_x B]\phi = 0,$$

and then

$$2A_x + B_{xx} = 0, \tag{88a}$$

$$u_t = -A_{xx} - 2(\lambda - u)B_x + u_x B. \tag{88b}$$

Eliminating A $(A = -B_x/2)$ yields

$$u_t = TB - 2\lambda B_x, \quad T = \frac{1}{2}\partial_x^3 + 2u\partial_x + u_x. \tag{89}$$

$$u_t = TB - 2\lambda B_x, \quad T = \frac{1}{2}\partial_x^3 + 2u\partial_x + u_x. \tag{89}$$

Assume B:
$$B = \sum_{j=0}^{n} b_j (2\lambda)^{n-j}. \tag{90}$$

(89) $\Longrightarrow$
$$u_t = Tb_n, \tag{91a}$$
$$b_{j+1,x} = Tb_j \quad (j = 0, 1, \cdots, n-1), \tag{91b}$$
$$b_{0,x} = 0. \tag{91c}$$

Take integration constants: $B|_{u=0} = (4\lambda)^n$. (91) $\Longrightarrow$

$b_0 = 2^n, \; b_1 = 2^n u, \; b_2 = 2^{n-1}(u_{xx} + 3u^2)$,

$$b_{j+1} = 2^{n-j}\partial_x^{-1} L^j u_x, \quad j = 0, 1, \cdots, n-1,$$

$$L = 2T\partial_x^{-1} = \partial_x^2 + 4u + 2u_x\partial_x^{-1} \tag{92}$$

L is called the recursion operator of the KdV hierarchy.

The KdV hierarchy:

$$u_{t_n} = L^n K_0, \quad K_0 = u_x, \ \ (n = 0, 1, 2, \cdots). \tag{93}$$

When $n = 1, 2$, there are

$$u_t = K_1 = u_{xxx} + 6uu_x, \quad \text{(KdV)} \tag{94a}$$

$$u_t = K_2 = u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x, \tag{94b}$$

When $n = 1$: $b_0 = 2$, $b_1 = 2u$, i.e. $B = 4\lambda + 2u$; $A = -u_x$. (87)

$\Longrightarrow$

$$\phi_t = -u_x\phi + (4\lambda + 2u)\phi_x. \tag{95}$$

or

$$\phi_t = u_x\phi - (4\lambda + 2u)\phi_x, \ \ (t \to -t), \tag{96}$$

$$\phi_t = \phi_{xxx} + (3\lambda + u)\phi_x, \tag{97}$$

$$\phi_t = 4\phi_{xxx} + 6u\phi_x + 3u_x\phi. \tag{98}$$

Miura's Transformation

$$u = -v_x - v^2, \tag{99}$$

modified KdV (mKdV) equation:

$$v_t - 6v^2 v_x + v_{xxx} = 0. \tag{100}$$

Relation:

$$u_t + 6uu_x + u_{xxx} = -(2v + \partial_x)(v_t - 6v^2 v_x + v_{xxx}).$$

More:

- Miura's transformation $\Longleftrightarrow$ Schrödinger spectral problem:
  λ: Galilean transformation $(u(x,t) \to \lambda + u(x - 6\lambda t, t))$,
  Cole-Hopf transformation $v = \phi_x/\phi$.
- mKdV $\Longleftrightarrow$ the time evolution part in Lax pair of KdV:
  Under the Miura transformation (99) and $v = \phi_x/\phi$.

Miura's transformation to BT [Wahlquist,Estabrook-1973]

Two Miura's transformations: $(v \to -v)$:

$$\widetilde{u} = \lambda + v_x - v^2, \quad u = \lambda - v_x - v^2. \tag{101}$$

$$\widetilde{u} + u = 2(\lambda - v^2), \quad \widetilde{u} - u = 2v_x. \tag{102}$$

Introducing $u = w_x,\ \widetilde{u} = \widetilde{w}_x \implies \widetilde{w} - w = 2v \implies$

$$(\widetilde{w} + w)_x = 2\lambda - \frac{1}{2}(\widetilde{w} - w)^2. \tag{103a}$$

Next, $\quad w_t + 3(w^2)_x + w_{xxx} = 0, \quad \widetilde{w}_t + 3(\widetilde{w}^2)_x + \widetilde{w}_{xxx} = 0,$

subtracting them each other yields

$$(\widetilde{w} - w)_t = -3(\widetilde{w} - w)_x(\widetilde{w} + w)_x - (\widetilde{w} - w)_{xxx},$$

$$(\widetilde{w} - w)_t = \frac{1}{2}[(\widetilde{w} - w)^3]_x - 6\lambda(\widetilde{w} - w)_x - (\widetilde{w} - w)_{xxx}. \tag{103b}$$

(103a) and (103b) compose a BT for KdV.

Connection: Bilinear BT, nonlinear BT, Lax pair

$$D_x^2 f \cdot g = \lambda f g, \tag{75a}$$

$$(D_x^3 + D_t + 3\lambda D_x) f \cdot g = 0. \tag{75b}$$

$$\phi_{xx} + u\phi = \lambda\phi, \tag{85a}$$

$$\phi_t = \phi_{xxx} + (3\lambda + u)\phi_x. \tag{85b}$$

$$(\widetilde{w} + w)_x = 2\lambda - \frac{1}{2}(\widetilde{w} - w)^2, \tag{103a}$$

$$(\widetilde{w} - w)_t = \frac{1}{2}[(\widetilde{w} - w)^3]_x - 6\lambda(\widetilde{w} - w)_x - (\widetilde{w} - w)_{xxx}. \tag{103b}$$

$$u = 2(\ln f)_{xx}, \quad \widetilde{u} = 2(\ln g)_{xx},$$

$$\text{Darboux transformation}: \quad \widetilde{u} = u + 2(\ln\phi)_{xx}, \quad \phi = \frac{g}{f}, \tag{104}$$

$$w = 2(\ln f)_x, \quad \widetilde{w} = 2(\ln g)_x, \tag{105}$$

§2.5 BTs and superposition formulas

§2.5.1 Nonlinear BTs and superposition formulas

Find a solution using BT:

$$(\widetilde{w}+w)_x = 2\lambda - \frac{1}{2}(\widetilde{w}-w)^2, \tag{103a}$$

$$(\widetilde{w}-w)_t = \frac{1}{2}[(\widetilde{w}-w)^3]_x - 6\lambda(\widetilde{w}-w)_x - (\widetilde{w}-w)_{xxx}. \tag{103b}$$

Taking $w=0$ and $\lambda=k_1^2$, we have

$$\widetilde{w}_x = 2k_1^2 - \frac{1}{2}\widetilde{w}^2, \tag{106a}$$

$$\widetilde{w}_t = \frac{1}{2}(\widetilde{w}^3)_x - 6k_1^2\widetilde{w}_x - \widetilde{w}_{xxx}. \tag{106b}$$

From (106a) we assume

$$\widetilde{w} = 2k_1 \tanh(kx + c(t)),$$

where $c(t)$ is undetermined. (106b) $\Longrightarrow c(t) = 4k_1^3 t + \eta_1^{(0)}$.

Nonlinear superposition formula:

$$(\widetilde{w}+w)_x = 2\lambda - \frac{1}{2}(\widetilde{w}-w)^2, \tag{103a}$$

$$(\widetilde{w}-w)_t = \frac{1}{2}[(\widetilde{w}-w)^3]_x - 6\lambda(\widetilde{w}-w)_x - (\widetilde{w}-w)_{xxx}. \tag{103b}$$

$$(w_1+w)_x = 2\lambda_1 - \frac{1}{2}(w_1-w)^2, \tag{107a}$$

$$(w_2+w)_x = 2\lambda_2 - \frac{1}{2}(w_2-w)^2. \tag{107b}$$

$$(w_{12}+w_1)_x = 2\lambda_2 - \frac{1}{2}(w_{12}-w_1)^2; \tag{108a}$$

$$(w_{21}+w_2)_x = 2\lambda_1 - \frac{1}{2}(w_{21}-w_2)^2. \tag{108b}$$

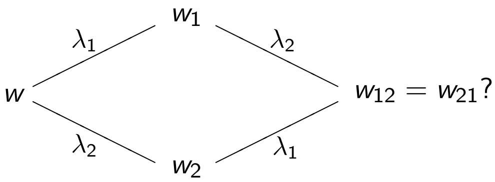


Figure: Permutability property of BT

Eliminate $w_{1,x}$ from (107a) and (108a) and we reach

$$w_1 = \frac{1}{2}(w_{12} + w) + \frac{2(\lambda_1 - \lambda_2)}{w_{12} - w} + [\ln(w_{12} - w]_x.$$

Substituting it into (108a) we find

$$\begin{aligned}\lambda_1 + \lambda_2 =& (w_{12} + w)_x + [\ln(w_{12} - w)]_{xx} + \frac{1}{2}[\ln(w_{12} - w)]_x^2 \\ &+ \frac{1}{8}(w_{12} - w)^2 + \frac{2(\lambda_1 - \lambda_2)^2}{(w_{12} - w)^2}.\end{aligned} \tag{109}$$

From (107a) and (107b) we have

$$(w_1 - w_2)_x = 2(\lambda_1 - \lambda_2) + \frac{1}{2}(w_1 - w_2)(w_1 + w_2 - 2w);$$

and from (108a) and (108b) we have (noticing that $w_{12} = w_{21}$)

$$(w_1 - w_2)_x = -2(\lambda_1 - \lambda_2) + \frac{1}{2}(w_1 - w_2)(w_1 + w_2 - 2w_{12}).$$

Eliminating derivative terms from them yields

$$4(\lambda_1 - \lambda_2) = (w_1 - w_2)(w_{12} - w), \qquad (110)$$

$$w_{12} = w + \frac{4(\lambda_1 - \lambda_2)}{w_1 - w_2}.$$

Viewed as a discrete equation: (with 3D consistence)

$$(w_{n+1,m} - w_{n,m+1})(w_{n,m} - w_{n+1,m+1}) = q^2 - p^2, \qquad (111)$$

p and q are spacing parameters of n- and m- direction.

§2.5.2 Bilinear BT and superposition formula

$$D_x^2 f \cdot \widetilde{f} = \lambda f \widetilde{f}, \tag{112}$$

$$(D_x^2 - \lambda_1) f \cdot f_1 = 0, \tag{113a}$$

$$(D_x^2 - \lambda_2) f \cdot f_2 = 0, \tag{113b}$$

$$(D_x^2 - \lambda_2) f_1 \cdot f_{12} = 0, \tag{113c}$$

$$(D_x^2 - \lambda_1) f_2 \cdot f_{21} = 0. \tag{113d}$$

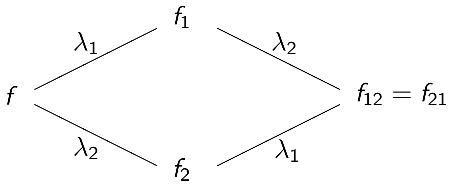


Figure: Permutability property of bilinear BT. ($w = 2(\ln f)_x$)

Now, $f_2 f_{12} \times$ (113a) $- ff_1 \times$ (113d) yields

$$(D_x^2 f \cdot f_1) f_2 f_{12} - (D_x^2 f_2 \cdot f_{12}) ff_1 = 0; \qquad (114)$$

meanwhile, using bilinear identity (66) we have

$$(D_x^2 f \cdot f_1) f_2 f_{12} - (D_x^2 f_2 \cdot f_{12}) ff_1 = D_x[(D_x f \cdot f_{12}) \cdot (f_1 f_2) - D_x(f_2 \cdot f_1) \cdot (ff_{12})].$$

$\Longrightarrow$

$$D_x[(D_x f \cdot f_{12}) \cdot (f_1 f_2) - (D_x f_2 \cdot f_1) \cdot (ff_{12})] = 0;$$

Switching indices: $1 \leftrightarrow 2$, yields

$$D_x[(D_x f \cdot f_{12}) \cdot (f_1 f_2) - (D_x f_1 \cdot f_2) \cdot (ff_{12})] = 0.$$

Adding and subtracting each other yield, respectively

$$D_x(D_x f \cdot f_{12}) \cdot (f_1 f_2) = 0,$$

$$D_x(D_x f_1 \cdot f_2) \cdot (ff_{12}) = 0.$$

$$D_x(D_x f \cdot f_{12}) \cdot (f_1 f_2) = 0,$$
$$D_x(D_x f_1 \cdot f_2) \cdot (f f_{12}) = 0.$$

Then, noticing the property $D_x g \cdot g = 0$ we can take

$$D_x f \cdot f_{12} = \alpha f_1 f_2, \tag{115a}$$
$$D_x f_2 \cdot f_1 = \beta f f_{12}, \tag{115b}$$

where $\alpha, \beta \in \mathbb{R}$. These two equations together compose a nonlinear superposition formula for the bilinear KdV equation (15).

To nonlinear superposition formula (110):

$$(D_x f \cdot f_{12}) \times (D_x f_2 \cdot f_1) = \alpha\beta f f_1 f_2 f_{12}.$$

Then introducing $w = 2(\ln f)_x$, we reach

$$(w - w_{12})(w_1 - w_2) = -\alpha\beta. \tag{116}$$

Notes:

- Bäcklund Transformations:[C. Rogers, W.K. Schief-2002] [Robert Prus and Antoni Sym-1998]
  - Bianchi (habilitation thesis-1879): A purely geometric construction for pseudospherical surfaces was reformulated in mathematical terms as a transformation $\mathbb{B}$ (parameter-independent).
  - Bäcklund (1883): Published details of his celebrated transformation $\mathbb{B}_\sigma$ which extends Bianchi's construction and allows the iterative construction of pseudospherical surfaces.
  - Lie (1883): Connect Bianchi's transformation and Bäcklund Transformation. $\mathbb{B}_\sigma = \mathbb{L}_\sigma^{-1}\mathbb{B}_0\mathbb{L}_\sigma$.
  - Bianchi (1885): Showed BT to be associated with an elegant invariance of the sine-Gordon equation.

Notes:

- Bäcklund Transformations:[C. Rogers, W.K. Schief-2002] [Robert Prus and Antoni Sym-1998]
  - Bianchi (1892): Demonstrated that the Bäcklund Transformation $\mathbb{B}_\sigma$ admits a commutativity property $\mathbb{B}_{\sigma_1}\mathbb{B}_{\sigma_2} = \mathbb{B}_{\sigma_2}\mathbb{B}_{\sigma_1}$, a consequence of which is a nonlinear superposition principle embodied in what is termed a Permutability Theorem.
  - It was neither Bianchi nor Bäcklund who was the first to write down the special Bäcklund transformation for the sine-Gordon equation. It was the great French geometer Gaston Darboux (in 1883).

Darboux found (1883) that if θ is a solution of sG equation

$$\theta_{uu} - \theta_{vv} = \sin\theta\cos\theta,$$

and ϕ satisfies

$$\phi_u + \theta_v = \sin\phi\cos\theta, \quad \phi_v + \theta_u = -\cos\phi\sin\theta, \tag{117}$$

then ϕ is a solution of sG equation $\phi_{uu} - \phi_{vv} = \sin\phi\cos\phi$.

Bäcklund constructed the transformation:

$$\phi_u + \theta_v = (\sin\phi\cos\theta + \sin\sigma\cos\phi\sin\theta)/\cos\sigma,$$
$$\phi_v + \theta_u = (-\cos\phi\sin\theta + \sin\sigma\sin\phi\cos\theta)/\cos\sigma,$$

where σ is an arbitrary parameter. When $\sigma = 0$ it is (117).

Notes:

- Mathematicians:
  - Jean-Gaston Darboux: French, (14.08.1842-23.02.1917)
  - Marius Sophus Lie: Norwegian, (17.12.1842-18.02.1899)
  - Albert Victor Bäcklund: Swedish, (11.01.1845-23.02.1922)
  - Luigi Bianchi: Italian, (18.01.1856-01.06.1928)
- Story from R.M. Miura
- Superposition formulas and quadrilateral equations

§2.6 Vertex operators

§2.6.1 Vertex operator of the KdV equation

$\tau_N = f$: NSS of the bilinear KdV.

Bilinear BT: $\tau_N \to \tau_{N+1}$

Vertex operator: $\tau_{N+1} = e^{c_{N+1}X(k_{N+1})}\tau_N$.

Rewrite the KdV equation (by $t \to -4t$) as

$$4u_t - 6uu_x - u_{xxx} = 0. \tag{118}$$

Transformation

$$u = 2(\ln \tau)_{xx}, \tag{119}$$

bilinear KdV equation is

$$(4D_x D_t - D_x^4)\tau \cdot \tau = 0. \tag{120}$$

$t_1 = x,\ t_3 = t$:

$$(4D_1 D_3 - D_1^4)\tau \cdot \tau = 0. \tag{121}$$

NSS:

$$\tau_N=\sum_{\mu=0,1}\exp\left(2\sum_{j=1}^{N}\mu_j(\zeta_j+\zeta_j^{(0)})+\sum_{1\leq i<j}^{N}\mu_i\mu_j a_{ij}\right), \tag{122a}$$

$$\zeta_j=\sum_{i=0}^{\infty}k_j^{2i+1}t_{2i+1},\quad e^{a_{ij}}=A_{ij}=\left(\frac{k_i-k_j}{k_i+k_j}\right)^2, \tag{122b}$$

$\zeta_j^{(0)}\in\mathbb{R}$. Infinite coordinates $(t_1,t_3,t_5,\cdots)$.

The above τ_N can be written as

$$\tau_N=\sum_{J\subset I}\left[\left(\prod_{i\in J}c_i\right)\left(\prod_{\substack{i,j\in J\\ i<j}}A_{ij}\right)\exp\left(2\sum_{i\in J}\zeta_i\right)\right], \tag{123}$$

where $c_i\in\mathbb{R}$, $I=\{1,2,\cdots,N\}$, J is a subset of I.

$J=\varnothing\rightarrow$ "1", $J=\{i\}\rightarrow c_ie^{2\zeta_i}$,

$J=\{1,2\}\rightarrow c_1c_2A_{12}e^{2(\zeta_1+\zeta_2)}$, $\cdots\cdots$. When $N=2$,

$$\tau_2=1+c_1e^{2\zeta_1}+c_2e^{2\zeta_2}+c_1c_2A_{12}e^{2(\zeta_1+\zeta_2)}.$$

Theorem

For the τ function defined by (123), *there is*

$$\tau_{N+1} = e^{c_{N+1}X(k_{N+1})}\tau_N, \tag{124}$$

$$X(k) = e^{2\zeta(\mathbf{t},k)}\, e^{-2\zeta(\widetilde{\partial},\, k^{-1})}, \quad \text{(vertex operator)} \tag{125a}$$

$$\zeta(\mathbf{t},k) = \sum_{j=0}^{\infty} k^{2j+1} t_{2j+1}, \quad \mathbf{t} = (t_1, t_3, t_5, \cdots), \tag{125b}$$

$$\widetilde{\partial} = \left(\partial_1, \frac{\partial_3}{3}, \frac{\partial_5}{5}, \cdots\right), \quad \partial_j = \partial_{t_j}. \tag{125c}$$

Notes: (125a) was constructed by [Lepowsky, Wilson-1978], who also found the operator is isomorphic to affine Lie algebra $A_1^{(1)}$. [Date, Kashiwara, Miwa-1981] found the connection (124), which led to the work on transformation groups and integrable systems.

We prove this theorem through some lemmas.

Lemma

$\forall a, k \in \mathbb{R}$, there is

$$e^{a\zeta(\widetilde{\partial}, k^{-1})} f(\mathbf{t}) = f(\mathbf{t} + a\varepsilon(k)), \tag{126}$$

where

$$\varepsilon(k) = \Big(\frac{1}{k}, \frac{1}{3k^3}, \frac{1}{5k^5}, \cdots\Big).$$

The proof is obvious.

Lemma

$$e^{-4\zeta(\varepsilon(p),q)} = \left(\frac{p-q}{p+q}\right)^2. \tag{127}$$

Proof:

$$\begin{aligned}
\ln\left(\frac{p-q}{p+q}\right) &= \ln(1-q/p) - \ln(1+q/p) \\
&= -\sum_{j=1}^{\infty}\frac{q^j}{j\,p^j} - \sum_{j=1}^{\infty}(-1)^{j+1}\frac{q^j}{j\,p^j} \\
&= -2\sum_{j=0}^{\infty}\frac{q^{2j+1}}{(2j+1)\,p^{2j+1}} \\
&= -2\zeta(\varepsilon(p),q).
\end{aligned}$$

Lemma

$$e^{-2\zeta(\widetilde{\partial},\, k_i^{-1})} e^{2\zeta(\mathbf{t},k_j)} = A_{ij} e^{2\zeta(\mathbf{t},k_j)}\, e^{-2\zeta(\widetilde{\partial},\, k_i^{-1})}. \tag{128}$$

Proof:

$$\begin{aligned}
& e^{-2\zeta(\widetilde{\partial},\, k_i^{-1})} e^{2\zeta(\mathbf{t},k_j)} f(\mathbf{t}) \\
=& e^{2\zeta(\mathbf{t}-2\varepsilon(k_i),k_j)} f(\mathbf{t}-2\varepsilon(k_i)) \\
=& e^{2\zeta(\mathbf{t},k_j)} e^{-4\zeta(\varepsilon(k_i),k_j)} e^{-2\zeta(\widetilde{\partial},\, k_i^{-1})} f(\mathbf{t}) \\
=& A_{ij} e^{2\zeta(\mathbf{t},k_j)}\, e^{-2\zeta(\widetilde{\partial},\, k_i^{-1})} f(\mathbf{t}).
\end{aligned}$$

Another expression for (128): $A = -2\zeta(\widetilde{\partial},\ k_i^{-1}),\ B = 2\zeta(\mathbf{t}, k_j)$,

$$[A, B] = -4\zeta(\varepsilon(k_i), k_j) = \ln A_{ij}, \qquad (129)$$

Noticing that $[\partial_{2r+1},\ t_{2s+1}] = \delta_{r,s}$, we have

$$\begin{aligned}
[A, B] &= -4\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{1}{2s+1}\frac{k_j^{2r+1}}{k_i^{2s+1}}[\partial_{2s+1},\ t_{2r+1}] \\
&= -4\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{1}{2s+1}\frac{k_j^{2r+1}}{k_i^{2s+1}}\delta_{r,s} \\
&= -4\sum_{r=0}^{\infty}\frac{1}{2r+1}\frac{k_j^{2r+1}}{k_i^{2r+1}} = -4\zeta(\varepsilon(k_i), k_j).
\end{aligned}$$

Then, using (129), (128) can be written as

$$e^A e^B = e^{[A,B]} e^B e^A. \qquad (130)$$

Lemma

For the vertex operator $X(k)$ *defined by* (125a), *there is*

$$X(k_i)X(k_j) = A_{ij}e^{2(\zeta(\mathbf{t},k_i)+\zeta(\mathbf{t},k_j))}e^{-2\zeta(\widetilde{\partial},\, k_i^{-1})}e^{-2\zeta(\widetilde{\partial},\, k_j^{-1})}. \quad (131)$$

Proof:

Using formula (128), we find

$$\begin{aligned} X(k_i)X(k_j) &= e^{2\zeta(\mathbf{t},k_i)}e^{-2\zeta(\widetilde{\partial},\, k_i^{-1})}e^{2\zeta(\mathbf{t},k_j))}e^{-2\zeta(\widetilde{\partial},\, k_j^{-1})} \\ &= e^{2\zeta(\mathbf{t},k_i)}A_{ij}\, e^{2\zeta(\mathbf{t},k_j))}e^{-2\zeta(\widetilde{\partial},\, k_i^{-1})}e^{-2\zeta(\widetilde{\partial},\, k_j^{-1})}, \end{aligned}$$

which is (131).

Lemma

$$(X(k))^2 = 0, \tag{132}$$

$$e^{cX(k)} = 1 + cX(k), \tag{133}$$

$$X(k_s)\cdots X(k_2)X(k_1)$$
$$= \left(\prod_{1\le i<j}^{s} A_{ij}\right)\exp\left(2\sum_{j=1}^{s}\zeta(\mathbf{t}, k_j)\right)\exp\left(2\sum_{j=1}^{s}\zeta(\widetilde{\partial}, k_j^{-1})\right); \tag{134}$$

$$X(k)\circ 1 = e^{2\zeta(\mathbf{t},k)}, \tag{135}$$

$$X(k_s)\cdots X(k_2)X(k_1)\circ 1 = \left(\prod_{1\le i<j}^{s} A_{ij}\right)\exp\left(2\sum_{j=1}^{s}\zeta(\mathbf{t}, k_j)\right). \tag{136}$$

Examples:

$$
\begin{aligned}
\tau_1 &= e^{c_1 X(k_1)} \circ 1 = (1 + c_1 X(k_1)) \circ 1 = 1 + c_1 e^{2\zeta(\mathbf{t}, k_1)}, \\
\tau_2 &= e^{c_2 X(k_2)} e^{c_1 X(k_1)} \circ 1 \\
&= (1 + c_2 X(k_2))(1 + c_1 X(k_1)) \circ 1 \\
&= 1 + c_1 e^{2\zeta(\mathbf{t}, k_1)} + c_2 e^{2\zeta(\mathbf{t}, k_2)} + c_1 c_2 A_{12} e^{2(\zeta(\mathbf{t}, k_1) + \zeta(\mathbf{t}, k_2))}, \\
\tau_N &= e^{c_N X(k_N)} \ldots e^{c_2 X(k_2)} e^{c_1 X(k_1)} \circ 1 \\
&= (1 + c_N X(k_N)) \cdots (1 + c_2 X(k_2))(1 + c_1 X(k_1)) \circ 1 \\
&= e^{c_N X(k_N)} \tau_{N-1}.
\end{aligned}
$$

§2.6.2 Vertex operator of the KP(II) equation

Rewrite the bilinear KP(II) (27) as (121):

$$(4D_1D_3 - D_1^4 - 3D_2^2)\tau\cdot\tau = 0. \tag{137}$$

Its NSS(37) is

$$\tau_N = \sum_{J\subset I}\left[\left(\prod_{i\in J} c_i\right)\left(\prod_{\substack{i,j\in J\\ i<j}} A_{ij}\right)\exp\left(\sum_{i\in J}\xi_i\right)\right], \tag{138a}$$

where $c_i \in \mathbb{R}$,

$$\xi_j = \sum_{i=0}^{\infty}(p_j^i - q_j^i)t_i, \quad e^{a_{ij}} = A_{ij} = \frac{(p_i-p_j)(q_i-q_j)}{(p_i-q_j)(q_i-p_j)}, \tag{138b}$$

I stands for the set $I = \{1, 2, \cdots, N\}$, J is a subset of I.

Theorem

For the τ function defined by (138a), there is

$$\tau_{N+1} = e^{c_{N+1} X(p_{N+1}, q_{N+1})} \tau_N, \tag{139}$$

where

$$X(p,q) = e^{\xi(\mathbf{t},p)-\xi(\mathbf{t},q)}\, e^{-(\xi(\widetilde{\partial},p^{-1})-\xi(\widetilde{\partial},q^{-1}))}, \tag{140a}$$

$$\xi(\mathbf{t},k) = \sum_{j=0}^{\infty} k^j t_j, \quad \mathbf{t} = (t_1, t_2, t_3, \cdots), \tag{140b}$$

$$\widetilde{\partial} = \left(\partial_1, \frac{\partial_2}{2}, \frac{\partial_3}{3}, \cdots\right), \quad \partial_j = \partial_{t_j}. \tag{140c}$$

Further reading:

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Thank You!

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