Fundamentals of Heat Transfer

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Figure taken from: http://heatexchanger-design.com/2011/10/06/heat-exchangers-6/ Dated: 17-Jan-2012
Thermal contact resistance [3]

(a) Ideal (perfect) thermal contact

(b) Actual (imperfect) thermal contact
Thermal contact conductance for aluminum plates with different fluids at the interface for a surface roughness of 10 \( \mu \text{m} \) and interface pressure of 1 atm (from Fried, Ref. 5)

<table>
<thead>
<tr>
<th>Fluid at the Interface</th>
<th>Contact Conductance, ( h_c ), W/m(^2) \cdot °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>3640</td>
</tr>
<tr>
<td>Helium</td>
<td>9520</td>
</tr>
<tr>
<td>Hydrogen</td>
<td>13,900</td>
</tr>
<tr>
<td>Silicone oil</td>
<td>19,000</td>
</tr>
<tr>
<td>Glycerin</td>
<td>37,700</td>
</tr>
</tbody>
</table>
General heat conduction equation

It takes into account, unsteady-state condition, all the three dimensions, and any heat generation within the material.
Coordinate systems [3]

(a) Rectangular coordinates
(b) Cylindrical coordinates
(c) Spherical coordinates
General conductivity equation in rectangular coordinates [10]
General conductivity equation in rectangular coordinates

\[
\text{(Rate of heat in)} - \text{(Rate of heat out)} + \text{(Rate of heat generation)} - \text{(Rate of heat consumption)} = \text{(Rate of heat accumulation)}
\]

\[
(q_{x|x} + q_{y|y} + q_{z|z}) - (q_{x|x+\Delta x} + q_{y|y+\Delta y} + q_{z|z+\Delta z}) + q' \times (\Delta x\Delta y\Delta z) - 0 = \rho c_p (\Delta x\Delta y\Delta z) \frac{\partial T}{\partial t}
\]

Applying Fourier’s law of heat conduction, it may be shown that

\[
\left( -k_x \Delta y \Delta z \frac{\partial T}{\partial x}\bigg|_x - k_y \Delta x \Delta z \frac{\partial T}{\partial y}\bigg|_y - k_z \Delta x \Delta y \frac{\partial T}{\partial z}\bigg|_z \right) + q' \times (\Delta x\Delta y\Delta z) = \rho c_p (\Delta x\Delta y\Delta z) \frac{\partial T}{\partial t}
\]
General conductivity equation in rectangular coordinates
(Errors and omissions are expected)

\[
k_x \left( \frac{\partial T}{\partial x} \bigg|_{x+\Delta x} - \frac{\partial T}{\partial x} \bigg|_x \right) \left( \frac{\partial T}{\partial x} \bigg|_{y+\Delta y} - \frac{\partial T}{\partial x} \bigg|_y \right) + k_y \left( \frac{\partial T}{\partial x} \bigg|_{z+\Delta z} - \frac{\partial T}{\partial x} \bigg|_z \right) + q' = \rho c_p \frac{\partial T}{\partial t}
\]

By the definition of derivative (differentiation), it may be shown that

\[
\frac{\partial}{\partial x} \left( k_x \cdot \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_y \cdot \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k_z \cdot \frac{\partial T}{\partial z} \right) + q' = \rho c_p \frac{\partial T}{\partial t}
\]

Taking \( k_x = k_y = k_z = k \) and assuming \( k, \rho, \) and \( c_p \) as independent of position and temperature, it may be shown that:

\[
k \cdot \left( \frac{\partial}{\partial x} \left( \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial T}{\partial z} \right) \right) + q' = \rho c_p \frac{\partial T}{\partial t}
\]
General conductivity equation in rectangular coordinates

\[ k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + q' = \rho c_p \, \frac{\partial T}{\partial t} \]

Defining \( \alpha = \frac{k}{\rho c_p} \) = thermal diffusivity, it may shown that

\[ \frac{\partial T}{\partial t} = \alpha \cdot \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{q'}{\rho c_p} \]

A high value of thermal diffusivity \( \alpha = \frac{k}{\rho c_p} \) could result either from a high value of the thermal conductivity, which would indicate a rapid energy transfer rate, or from a low value of the thermal heat capacity \( (\rho c_p) \). A low value of the thermal heat capacity would mean that less of the energy moving through the material would be absorbed thus more will transfer.
General conductivity equation in rectangular coordinates

(Errors and omissions are expected)

Prove that the temperature profile through a plane wall having constant thermal conductivity is

$$T = T_1 + \frac{T_2 - T_1}{r_2} r$$

Starting from the general heat conduction equation (constant $\alpha$)

$$\frac{\partial T}{\partial t} = \alpha \cdot \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{q'}{\rho c_p}$$

For the case of steady-state, one dimensional ($x$-direction), without heat generation heat conduction problem, the general heat conduction equation for constant $\alpha$ reduces to

$$\frac{d^2 T}{dx^2} = 0$$

$$\frac{d}{dx} \left( \frac{dT}{dx} \right) = 0$$
General conductivity equation in rectangular coordinates

(Integrating the above equation, it may be shown that)

$$\frac{dT}{dx} = C_1$$

(Integrating again, it may shown that)

$$T = C_1 \cdot x + C_2$$

(1)

Applying boundary conditions

B.C. 1: At $x = 0$, $T = T_1$
B.C. 2: At $x = \Delta x$, $T = T_2$
From Eq. 1 and using B.C.s., it may shown that

\[ T_1 = C_2 \]

\[ T_2 = C_1 \cdot \Delta x + C_2 \Rightarrow C_1 = \frac{T_2 - T_1}{\Delta x} \]

Inserting values of \( C_1 \) and \( C_2 \) in Eq. 1, it may shown that

\[ T = \frac{T_2 - T_1}{\Delta x} \cdot x + T_1 \]

\[ T = T_1 + \frac{T_2 - T_1}{\Delta x} \cdot x \]
General conductivity equation in rectangular coordinates (constant $k$, $c_p$ and $\rho$)

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \cdot c_p} \cdot \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{q'}{\rho \cdot c_p}$$

$$\frac{\partial T}{\partial t} = \alpha \cdot \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{q'}{\rho \cdot c_p}$$

For derivation see class notes.
General conductivity equation in rectangular coordinates (constant $k$, $c_p$ and $\rho$)

In case no heat generation, we have the following equation. This form of equation is called as Fourier’s second law of heat conduction or heat diffusion equation as the temperature variation are a function only of thermal diffusivity.

$$\frac{\partial T}{\partial t} = \alpha \cdot \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$
For steady-state condition, the general heat conduction equation can be written as the following. This form of equation is called as Poisson’s equation for heat transfer.

\[
\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = -\frac{q'}{k}
\]
General conductivity equation in rectangular coordinates (constant $k$, $c_p$ and $\rho$)

For steady-state condition and no heat generation situation, the general heat conduction equation can be written as the following. This form of equation is called as **Laplace equation for heat transfer**.

\[
\left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) = 0
\]

Using the **Laplacian operator**, it may be shown that

\[
\nabla^2 T = 0
\]
General conductivity equation in other coordinates [10]

(a) Cylindrical coordinates

(b) Spherical coordinates
General conductivity equation in other coordinates (constant $k$, $c_p$ and $\rho$)

In cylindrical coordinates:

$$\frac{\partial T}{\partial t} = \alpha \cdot \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{q'}{\rho \cdot c_p}$$

In spherical coordinates:

$$\frac{\partial T}{\partial t} = \alpha \cdot \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial^2 T}{\partial \phi^2} \right) + \frac{q'}{\rho \cdot c_p}$$
Homework problems

Workout the temperature profiles:

a) For a case of steady-state, one dimensional ($r$-direction), and without heat generation heat conduction problem in cylindrical coordinates

b) For a case of steady-state, one dimensional ($r$-direction), and without heat generation heat conduction problem in spherical coordinates
Conduction with internal heat generation (Not for exams)

Using general conductivity equation, derive an expression for internal heat generation in a plane wall with constant thermal conductivity where heat transfer occurs only in the $x$-direction.


