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Examining Transformation Geometry Concept Definitions of Pre-Service Mathematics Teachers

Gülfem Sarpkaya Aktaş[*], and Nejla Gürefe[**]

Abstract

In this study, the knowledge of mathematics teacher candidates on the definitions about translation, reflection, and rotation transformations are investigated. The participants of the study in the descriptive survey model consist of 102 teacher candidates who have training in elementary mathematics teaching program. The data were collected by means of open-ended questions, and examined according to accuracy, existence, hierarchical concept structure and equivalence among the criteria required to be a definition. It has observed from the findings regarding the category of accuracy that teacher candidates generally defined necessary and not sufficient in translation, partially sufficient and unnecessary in reflection, and necessary and not sufficient in rotation. The explanations made by the participants are generally composed of informal expressions. Such explanations of the participants, who will be able to communicate these concepts as teachers in the future, can make concepts such as translation, reflection and rotation more difficult for students. In this respect, it is recommended that in the training for teacher candidates; practices should be included in order to provide them with conceptual information about these concepts.

Keywords: Translation, reflection, rotation, definition, teacher candidates

[*]Associate Professor, Faculty of Education, Department of Mathematics Education, Çukurova University, Turkey. Email: gulfemsarpkaya@yahoo.com
[**]Associate Professor, Faculty of Education, Department of Mathematics Education, Usak University, Turkey. Email: nejlacalik@gmail.com

Introduction

Geometry, which has a very important place in mathematics education, has a great place in education (Altun, 2004). Geometry can be summarized as to define the properties of geometrical objects in the plane and three-dimensional space, to define the concept of geometric locus and to find the relationships between them, to explain and express the transformations, to prove the geometric propositions (Baki, 2008). Transformation geometry which is included among the concepts of geometry enriches students' experiences, imagination and thinking abilities (Fletcher, 1973; Soon, 1989). Translation transformation constitutes the basis for the topic of function in high school following secondary school, and rotation transformation is necessary for understanding trigonometry (Gürbüz & Durmuş, 2009). The association of geometric transformations with other mathematical concepts in interpreting mathematical relations has an important place as it enables students to structure information effectively (Sünker & Zembat, 2012).

Another factor that influences the students' interpreting is writing the concept definitions related to mathematics (Shield & Swinson, 1997). Concept definitions are among the main components of teaching and learning process. The roles of mathematical concepts in the process of teaching and learning mathematics are; i) to introduce the components of a theory, and to determine the critical features of a concept, (ii) to form an essential part of concept acquisition, (iii) to establish a basis for understanding mathematical evidence and solving problems, and (iv) to build consensus on the meanings of mathematical concepts between mathematics educators and students, and thus, to pave the way for effective communication of mathematical ideas (Zaslavsky & Shir, 2005). In the development of geometric thinking skills and the formation of high-level geometric thinking, there is also an understanding of the definitions at a certain extent (Linchevsky, et al., 1992).

Tall and Vinner (1981) described the definition as a word sequence which explains a concept. The important points in considering the definition as a real definition can be sorted as knowing the words used in the definition, including all the features related to the concept and only giving the necessary conditions and features, having no contradiction among the features related to the definition (Borasi,1982). Being economic, understanding equivalent definitions, being hierarchical, being able to exist, and lastly stating the necessary and sufficient conditions are among the criteria to be a definition (Çakıroğlu, 2013). To explain these features, knowing the words used in the definition is similar to the fact that they are in hierarchical and axiomatic form. The use of words that students can understand according to class level will help to understand the concept definitions. For example, the definition of the concept of square is made by interpreting the concept of quadrilateral in the definition like "*the square is the quadrilateral whose all angle*

measurements and side lengths are equal." (Çakıroğlu, 2013). An example of providing all features and necessary conditions related to the concept is that when a feature is given like that the opposite sides of the rectangle are parallel; this feature is necessary to define the rectangle but it is not sufficient. The condition regarding the internal angles should also be specified. The criteria of existence among the criteria to be a definition is to be able to provide examples that meet all the conditions related to the concept. The criteria of equivalence is that different definitions can be made and understood for a concept (Çakıroğlu, 2013). It is mentioned in the literature that definitions are important in terms of increasing the experience in mathematics and illustrating the concepts (Shield & Swinson, 1997; Van Dormolen & Zaslavsky, 2003). For this reason, those who are taking on mathematics, should be able to define mathematical concepts and have experience how they can use the definitions (Vinner, 1991). The definition, structure and the formation of identification process of mathematical concepts are the elements of a mathematics teacher's content knowledge (Zazkis & Leikin, 2008). For this reason, it is expected that teachers and teacher candidates (TCs) will be able to use concept definitions effectively and to understand the relationships between different definitions. Therefore, the aim of this study is to examine the concepts of translation, reflection and rotation transformation which are among the transformation geometry subjects for TCs. In the literature, studies on transformation geometry generally examine students' success (Çetin, et al., 2015; Güven & Yılmaz, 2011), and provide the students and TCs teaching of the subject (Bulut & Boz Yaman, 2016; Demir & Kurtuluş, 2019; Korkmaz & Tutak, 2017; Mutlu & Söylemez, 2019) and detect knowledge of students or TCs about translation and reflection (Hacısalihoğlu-Karadeniz et al., 2015; Öksüz ve Gürefe, 2019; Sarpkaya-Aktaş ve Ünlü 2017; Son, 2006; Turgut et al., 2014). The study could not found that examines TCs' concept definitions of translation, reflection and rotation in this way. In this respect, it was thought that the study was important. In line with the purpose of the research, the problems of the research were detected as:

1. Are mathematics teacher candidates able to make definitions of translation, reflection, and rotation transformation accurately?
2. What are the features of being a definition within the definitions of the concepts of translation, reflection, and rotation transformation of mathematics TCs?

Geometric Transformations

Geometric transformations have three different transformations such as translation, reflection, and rotation. Translation transformation is a distance-preserving function, and is called isometry in mathematics (Zembat, 2013). In the movement made by the translation, the objects move in a certain direction. Since the vectors determine the directions (Argün, Arıkan, Bulut, & Halıcıoğlu, 2014), the parameter of the translation

transformation is a vector. Thus, a vector corresponds to each translation, on the contrary a translation corresponds to each vector (Argün et al., 2014). A geometric object or plane part, and its images under translation transformation are equal on the basis of distance and angle measurements. The translation transformation must be performed on all points. A translation transformation that accepts the zero vector as a parameter matches itself with the relevant polygon. In other words, the translation movement corresponding to the zero vector keeps all points in the plane constant. Changing the direction of the vector affects the direction of movement of the image points. Changing the size of the vector affects the distance between the points in the plane and the image points (Argün et al., 2014). According to the Turkish Language Institution, the translation is defined as "the movement of an object which emerges with all its points drawing equal, parallel, and corresponding ways.

Reflection Transformation is a transformation that converts all points in the plane to all points in the plane and preserves the distance between points (Zembat, 2007). For Martin (1982), if the point P is above the d line for reflection transformation according to a line, it is equal to P; if the point P is out of the line, and d is at the perpendicular bisector of the PQ line segment, it is equal to the point Q. Since d line plays a key role, it is the parameter, and it is essential to find the image of the given point under reflection transformation (Yavuzsoy Köse, 2013). Argün, Arıkan, Bulut and Hacıoğlu (2014) pointed out that in order for the two points to be symmetrical with respect to a line or plane, the line segments connecting points should be perpendicular to the axis of symmetry or to the plane of symmetry, and the distances of points to this axis or plane should be equal. They refer to reflection transformations according to their points of reference as plane symmetry, line symmetry, and point symmetry. The reflection transformation is essentially an isometry. Isometry is a concept that expresses equality in the measurement. Isometry can be defined as transformations that preserve distances between points. As a result of the distance-preservation, it also preserves the shapes and the dimensions of objects (Kappraff, 2001, Yavuzsoy Köse, 2013). The points (line, plane, or point) on the parameter in the reflection transformation are constant, unchanging points. With the reflection transformation, the side lengths, angles, angle measurements, their distances to the symmetry line, and girth and area measurements of the geometric shapes are preserved. The only feature that is not preserved under the reflection transformation is the direction of the angles. (Yavuzsoy Köse, 2013).

The Turkish Language Institution defines as following;

1. According to a constant point A, for a point R, it is the point R' which provides the vectorial equation of AR '= - R'.

2. According to a constant point A, for a point R, it is the reflection of R vis-à-vis the point R's orthogonal projection on A.
3. According to a constant plane A for a point R, it is the reflection of R vis-à-vis the point R's orthogonal projection on A.

Rotation transformation is defined rotation as a transformation that matches all points in the plane again with the points in the plane with the help of a center point and angle (Martin, 1982). The parameters in this transformation are center point and angle (Zembat, 2013).

Moving the objects around a given point at a given angle is called the rotation transformation. A rotation is also defined as a combination of reflections vis-à-vis to two intersecting lines. The center of this rotation is the orthocenter of the intersecting lines, the angle is twice the angle between the two intersecting lines, and the direction of rotation depends on the composition series of reflections. Rotation and translation transformations preserve the direction (Argün et al., 2014). According to the Turkish Language Institution, it is defined as *the movement of a shape with an unchanging form around its own axis*.

In this study, the definitions of prospective teachers about these concepts were investigated.

Method

The descriptive survey model of qualitative research designs is used in the study. Survey researches are "the researches that identify the characteristics of the participants such as their opinions, interests, skills, abilities, attitudes related to a subject or event" (Büyüköztürk, Çakmak-Kılıç, Akgün, Karadeniz & Demirel, 2011, p.177). In this research, descriptive survey model is preferred as it is aimed to evaluate the definitions for the rotation transformations made by secondary school mathematics teacher candidates according to the criteria to be definition.

Study group

The research was conducted with 102 mathematics TCs. These pre-service teachers are 3rd grade students in the same class of the same university and 4th grade students in the same classroom of the same university. These prospective teachers took the analytical geometry lesson in the 3rd grade of the university and in this course. The prospective teachers were given sufficient information about the reflection, rotation, and symmetry of transformation geometry in this lesson. They also took special teaching methods lesson in the third grade and they learned the definitions of these concepts and how to transfer the concepts to secondary school students.

Data Collection Process

Data were collected through a form of open-ended questions. In the form*, "Do you define translation?"*, *"Do you define reflection?", "Do you define rotation?"* and *"You associate with each other the concepts of translation, symmetry and rotation."* questions were included. It is appropriate to use open-ended questions in qualitative researches as it allows the researcher the opportunity to capture and understand the perspective of the sample without limiting the categories of questions previously prepared (Patton, 2002). Open-ended interview questions were written on paper and distributed to the participants, and participants are asked to answer these questions. Mathematics teacher candidates are given one lesson time to answer the questions.

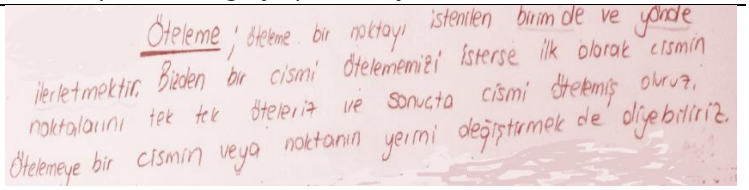
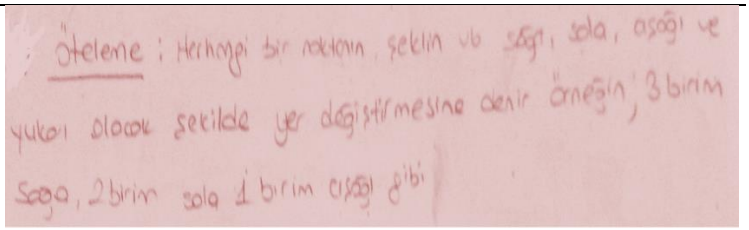
Data Analysis

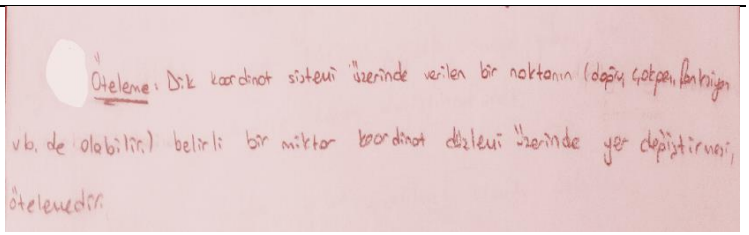
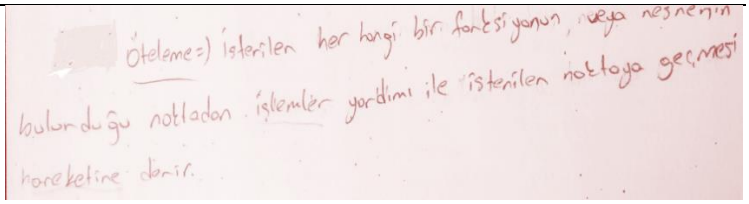
The descriptive analysis method is used to analyze the qualitative data obtained in the study. Analyses is made by considering the features which must be included in the mathematical concept definitions as stated in the literature (Borasi, 1982; Zazkis & Leikin, 2008; Van Dermelon & Zaslavsky, 2003; Vinner, 1991; Winicki-Landman & Leikin, 2000; Çakıroğlu, 2013). The accuracy, existence, equivalence and hierarchical structure of the definitions are evaluated.

The categorization structure specified by Zazkis and Leikin (2008) is used for the accuracy criteria. For the analysis, first of all, the definitions of reflection, translation and rotation transformation in the literature are examined in detail and the critical features related to the concepts are determined according to these definitions. When examining the correctness of the definitions provided by the teachers we distinguished between appropriate and inappropriate statements. The appropriate were categorized as necessary and sufficient, the inappropriate were categorized as necessary and not sufficient, partially necessary and not sufficient and not necessary and not sufficient. There are some critical properties which the concepts have. Critical features for translation transformation are 'translating unit direction and object points', 'unchanging size of the shape, area, volume and direction'. The critical features for the reflection transformation are 'the length, form, size and volume of the shape do not change', 'only the direction changes', 'the distances of the shape and its image to the line of symmetry are equal', 'it is symmetrical with respect to the line or plane'. The critical properties for rotation transformation are identified as 'size and volume of the shape do not change', 'it preserves 'angle', 'point of reference', 'direction''. Correct logical structure of a definition includes all ofthe critical properties for the concept that is necessary and sufficient. Inappropriate categories show that the people have some logical difficulties. Such difficulties may relate to the lack of understanding of the concept of definition and its critical features. For the reliability, the following coding examples are also presented in Table 1. For example, defining a translation as "It means changing the location of any point, object etc. to the directions of right, left, up, down. For

example, 3 units to the right, 2 units to the left, 1 unit to the down." includes necessary but not sufficient conditions and, as such, may indicate misunderstanding of the role of definition in mathematics: The direction concept which is one of the critical features of translation transformation have been mentioned but only as to the right, left, down, up. The remaining, unchanging properties, and that the parameter of the translation transformation is vector, have not been mentioned. Another category is partially necessary and not sufficient. Some of the explanations of TCs included some of the critical features, but if their definitions include non-critical features, they were evaluated as partially necessary and not sufficient. Not necessary and not sufficient category does not include critical properties, but includes inappropriate explanation, shape and drawing.

Table 1
Examples of Definition of Translation for the Category of Accuracy

Appropriate	Necessary and Sufficient	 [Translation: The translation is to advance a point in the desired unit and direction. If we are asked to translate an object, we firstly translate the points of the object one by one, and consequently we will have translated the object. We can also call the translation of an object or point as the change of location.] TC14 *Explanation:* Since the unit direction and translation of the points of the object which are the critical properties of translation transformation, are present, it is considered necessary and sufficient.
Inappropriate	Necessary Not sufficient	 [Translation: It means changing the location of any point, object etc. to the directions of right, left, up, down. For example, 3 units to the right, 2 units to the left, 1 unit to the down.] TC8 *Explanation:* The direction concept which is one of the critical features of translation transformation have been mentioned but only as to the right, left, down, up. The remaining, unchanging properties, and that

	the parameter of the translation transformation is vector, have not been mentioned.
Partially necessary and not sufficient	 [Translation: Changing the location of a given point (it can be line, polygon, function etc.) on the vertical coordinate system at a certain extent on the coordinate plane is called as translation.] TC56 Explanation: The change of location which is one of the critical properties of the translation transformation are mentioned, but this definition includes non-critical features.
Not necessary and not sufficient	 [Translation: The movement of any given function or object from the point where it is located to the desired point by the means of operations.] TC89 *Explanation:* Although in the definition there is a movement action which is one of the requirements of translation transformation, there is no information about according to what this movement will be made. The definition includes the word process but it is not clear what is meant by the process.

Other categories are also identified by drawing on the works of Van Dormolen and Zaslavsky (2003) and Çakıroğlu (2013).

Hierarchical concept structure: Each concept can be defined as a special case of a more general concept. This situation leads to a hierarchical structure between concepts that embraces each other.

Existence: The existence of the concept in the definitions must be able to be proved or to be given as an example for a special case. In order for an expression to be a definition, first of all the phenomenon that it tries to define must be able to exist, or it should be able to give at least one example to indicate the existence of the concept.

Equivalence: If more than one definition is made for the same concept, it should be possible to prove that each is equivalent.

The answers of the mathematics teacher candidates are analyzed according to the categories mentioned above. The researchers independently have worked on the student responses, and they have analyzed which answers fall into which category. In cases where there is a difference in the categories, answers and categories are discussed, and a consensus is reached. Consistency between researchers in classification is calculated as 89% by using Miles and Huberman's (1994) reliability coefficient [Reliability = Agreements (Agreements + Disagreements)]. Since this ratio is more than 80%, it is a sufficient rate for reliability (Lombard, Snyder-Duch and Bracken, 2002).

Results

This section includes the findings and interpretations in order to evaluate the definitions made by the mathematical TCs for the concepts of transformation geometry.

Table 2
Frequencies related to the definition of the concepts of translation, reflection and rotation

Categories	Sub categories		Translation (f)	Reflection (f)	Rotation (f)
Accuracy	Appropriate	Necessary and sufficient	13	5	1
	Inappropriate	Necessary and not sufficient	62	35	51
		Partially necessary and not sufficient	24	40	31
		Neither necessary nor not sufficient	3	22	15
		Null	-	-	4
Existence	Acceptable example or explanation		12	18	10
	Unacceptable example or explanation		2	2	3
Hierarchical concept structure	Point		17	16	26
	Point of reference		-	1	2
	Direction		16	4	4
	Unit		11	-	1
	Location		11	3	5
	Direction		19	3	-
	Unit shift		8	-	-
	Axis		1	12	13
	Coordinate plane (system), coordinat		16	7	10
	Geometrical shape		27	10	8

	Object	-	-	1
	x and y axis	4	6	1
	z axis	1	-	-
	Length	1	-	-
	Area	2	-	1
	Volume	2	1	2
	Up-down, right-left	20	-	-
	Number	1	-	-
	Movement	9	-	2
	Measurement	1	-	1
	Constant	1	-	-
	Ration	3	-	3
	Line	2	14	2
	Axis (line, symmetry, reflection axis)	-	15	1
	Image	2	23	3
	Symmetry	-	7	-
	Scalar multiplication	-	3	--
	Sign	-	2	-
	Symmetry board or mirror	-	13	-
	Distance	5	15	1
	Similar piece	-	1	-
	Exact same	-	2	-
	Translation	-	1	4
	Dimension (identical)	6	5	6
	Origin	-	1	2
	Equal	-	1	-
	Angle	-	-	43
	Circular motion	-	-	1
	Degree	-	-	15
	Clockwise or counterclockwise	-	-	17
	Holding constant	-	-	1
	One’s own axis	-	-	1
	Revolution	-	-	1
	Edge	-	-	1
Equivalence		8	29	6

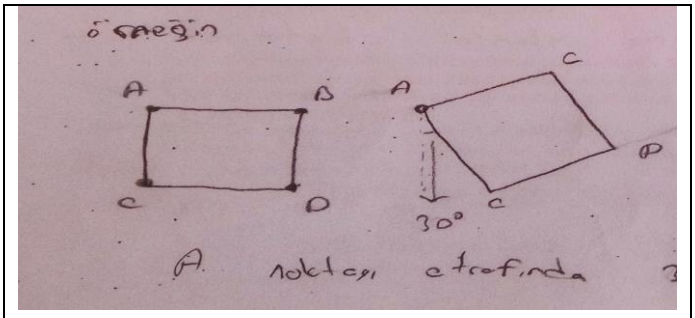
The evaluation of the concept definitions in line with the category of accuracy are given in Table 2. The accuracy of the definitions is determined according to the critical properties for the translation, reflection and rotation transformations.

It is seen that 13 of the definitions of translation transformation made by the mathematics teacher candidates are in the group of necessary and sufficient in the relevant subcategories. Since TC74's definition as *"moving only in a certain direction without disrupting its value, the structure of the shape"*, which ensures all the critical properties of translation transformation, can be considered appropriate and correct. Similarly, TC36's definition as *"it is a process of shifting an object in a particular unit in a particular direction. The size, volume and length of the object are preserved in translation"* is also a definition which meets the critical properties. The definition of 62 TCs is included in the category of necessary and not sufficient definitions which is conformed with the concept of translation transformation. The definitions which can be given as an example to this classification are TC72's definition as *"It means changing the location of a point or an object in one point by moving it in a certain direction in a certain unit"* and TC40's definition as *"If the point A (a.b) on a coordinate axis is moved in the right, left, up, and downward direction with a certain unit, and the location of the point A changes, this situation is called as translation."*. The critical feature indicating inadequacy in the definition of TC72 is that the direction of translational transformation, that's, the vector is not specified, and that the properties, which are remained constant and unchanged as a result of translation transformation are not specified. The inadequacy in the definition of TC40 is that the unchanging properties under the translation transformation are not specified, and the direction vector is ignored by taking the direction as only up, down, right and left. In the partially necessary and not sufficient category, the definitions of 24 TC are included. Since TC70's definition as *"Change of location for an object is a translation"* contains the phrase *"change of location"* which is just one critical feature of the translation transformation, it is included in this category. In addition, 3 TC have made definitions which could be included in neither necessary category nor not sufficient category. For example, since the words of *"function"* and *"operations"*, which are referred in the TC89's definition as *"The movement of any chosen function or object that passes from the point where it is located to the intended point with the help of the operations, is called as translation",* are not concepts related to translation transformation, they are included in this category.

The definitions made by the TCs for translation transformation are generally classified as necessary and not sufficient, and no definition has been found to indicate that translation transformation is a function. In the definition in which the term of function passes, it is defined as the object to be transformed rather than the mathematical meaning of the function. While teacher candidates make definition, they usually use the phrases as shifting, moving and changing location.

For the concept of reflection transformation, it is seen that 5 TCs can make an appropriate definition. For example, TC73' definition as *"It is the image of the object at an equal distance from the line. It is the change of its direction without changing its form"* can be given. TC 23's definition as *"the formation of the image on the opposite side in relation to the axis of a symmetry, and the exact overlap of the two shapes when folded over the axis of symmetry"* is not mentioned when the critical features are considered, but it is considered to be a suitable definition in which the reflection transformation concept is visually represented. It can be said that in the necessary and not sufficient category of definitions of the 35 TCs for reflection transformation, 40 of them are partly necessary and not sufficient, and 22 of them are neither necessary and nor not sufficient category. If one each example from the definitions of these categories is given respectively, TC 34 has stated *"it is the image-taking of given polygon, line, ray closed half-line etc. on an axis. It is considered as the image in front of the mirror. The dimension of the shape remains the same as in the way its coordinates are reflected."*. It is seen from its interpretation here that TC confines the shape with a geometric shape whose reflection will be made, that is, privatizes it and takes the mirror as a reflection axis, and states that there is no change in the size of the reflected object. TC 79 also states for the reflection that *"the presence of the location of the shape or point on the coordinate plane at the same points according to the x or y axis but at different marks, its change of location."*.TC79 has stated that the reflection is in relation to x and y axis, and the mark of the shape, whose reflection is taken, will change. TC31 has also stated in a similar way with its statements of *"the state of an object relative to the x-axis or the y-axis"* that the reflection would be based on the x or y axis, but has not said anything about the situation in relation to the axis. What is noteworthy here is that most of the definitions of TCs are included in the category of inappropriate definitions.

For the concept of rotation transformation, only one TC has been able to make a appropriate definition. TC1 is teacher candidate who has made the appropriate definition and has defined the rotation transformation as *"it is the rotation of the shape around a certain point of reference at the specified angle while all the features of the shape are held fixed. This point of reference is kept constant. While the features of the shape such as angle, edge, size are preserved, the direction changes."*. Then he has noted the example given in figure 1 below.

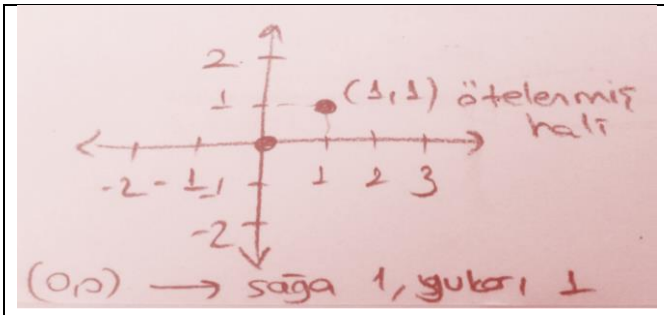


A around point

Figure 1-*An example of the definition of TC1 in its necessary and sufficient category for rotation transformation*

It is seen that most of the definitions related to the rotation transformation made by the mathematics TCs are in the necessary and not sufficient category (51). TC26 has defined the rotation transformation as *"moving a given shape clockwise or counterclockwise in accordance with the given degree."*. In this definition, the reference point of the rotation transformation that's one of the critical properties is not specified, but it is also not stated that there will be no change in the size of the shape, hence the measurement properties such as volume and area. In TC57's definition as *"the rotation of the shape or object from its one point within the identified size without disrupting its structure (e.g. 40^o)"* lacks *"direction"* as one of critical properties. Although TC mentions the center of rotation in this definition, it takes this center as a point on the central object. However, the center of rotation may be at any location on or outside the shape (Van de Walle, Karp & Bay-Williams, 2016). The definition of TC31 is considered partly necessary and not sufficient. It is seen that TC 31, who makes a definition for rotation transformation as *"how many degrees an object changes location. For example; turning 90 degrees clockwise. The shape of the object changes during the rotation."* makes emphasis only on angle and direction from the relevant critical properties related to the rotation transformation. In addition, it is identified that there is a misinformation about changing the shape of the object during the rotation. Since TC63 emphasizes a single-angle feature from critical properties by defining as *"rotation is the rotation of a shape with a certain ratio and angle*", it is considered as a definition in the partially necessary and not sufficient category. Since TC432's definition as *"it is the change of location of any point along the desired line at the desired degree"* does not reflect any critical feature of the rotation transformation, it is identified as a definition in the category of neither necessary nor sufficient. It is seen that the definitions of TCs related to rotation transformation are in the necessary and not sufficient category, and they make inappropriate definitions.

The evaluation of the concept definitions in regard to the category of existence is presented in Table 2. It must be possible to show the availability of the features that form the definition in the existence property of the definition. This feature can be provided by giving an example to explain the concept. There are acceptable or unacceptable examples of mathematics teacher candidates in their definitions of translation, reflection and rotation transformation. These are numerically 12-2, 18-2 and 10-3 respectively. Considering the acceptable examples for translation transformation, examples are given through coordinate system and adding units to axis. TC22 gives the example of *"translation is through moving the object 3 units to right and 4 units to up…"*. TC has set out the example that translation transformation will be realized by moving it at a certain unit in a certain direction. TC30 is an example of the way that point (0,0) is translated to (1,1) in Figure 2. Here, it is possible to say that an acceptable example of coordinate system is created. In the TC31's statement as *"For example, it is like showing the points of an object in the coordinate system and translating the points in the direction of (2,3)"* since the direction of (2,3) is not mathematically a valid discourse, it is categorized as an unacceptable example.



shifted

right 1, up 1

Figure 2-*TC30's acceptable example for translation transformation*

In the findings of the reflection, TC71 who considers reflection that the objects are to be in the same position, explained with the example of *"He calls Berk. Mert comes to class in the morning. Berk does in the afternoon, so one goes to school and the other goes home. When they meet each other, you become the reflection of each other."* In the TC's example for reflection, he has overlooked the features that the object and its reflection are to be of the same dimensions and have the same distances to any axis, and made a statement having no relation with the concept. This example has also been evaluated in the inappropriate example category.

For the reflection transformation, the number of acceptable examples or explanations included in the definitions made by the TCs is 18 and the number of unacceptable example or explanation is 2. Examples of the reflection of a geometric shape or point in the coordinate system are given when illustrating and explaining the reflection transformation.

The equivalence category evaluations of the concept definitions are given in Table 2. If more than one definition of the same concept is made in the equivalence feature of the definition, it should be proved that each is equivalent. TCs have used alternative definitions at most while defining the concept of reflection and at least while defining the concept of rotation. In general, it is identified that a small number of TCs make alternative definitions in explaining concepts. For the translation transformation, it is seen that the explanations in the equivalence category are in a way of moving the object and associating with other transformations. TC 84 has made an explanation that *"it is to take an object, and put it in another location in the same way"*. With this explanation, TC 84 has made emphasis on what actually changes and remains fixed in the translation, has indicated that the position of the object has changed and that its shape has never changed. In alternative definitions of reflection transformation, more emphasis was given to equality and some expressed this as using *"mirror"* as the axis of symmetry. For example, TC82 has made an explanation *as "Let's suppose that a shape is mirrored at the desired location. If it gives the same shape when it is folded into two from the axis, it is the reflection of the first shape."* and has thought the mirror as the axis of the symmetry, has explained the fact that the figure is the same with the image reflected from the axis as the reflection. It is observed that the differences were not mentioned when trying to express the similarities between the shape and its image in the reflection. TC77 has also emphasized the equality of figure and image by making the explanation as *"the object will stick on it when it is folded."* However, in the definition, how and according to what the object is folded, that's the axis of symmetry is mentioned. TC29 has made a definition related to the rotation transformation as *"We rotate the shape in the coordinate system clockwise or counterclockwise to 900,1800 2700 and 3600. According to the zones when we rotate 900, if we are rotating counterclockwise, from the zone I to the zone II, I change the position of x and y and put minus before the one which takes the lead.".* In the TC29's this definition, the concept of angle critical which is necessary for rotation transformation is mentioned. The reason for this to be considered as equivalent to the rotation transformation is the definition of the coordinate system, and only the 4 angle values. In addition, the TC has also addressed in the definition how the components of the point will change with the fact that any point (x, y) located in the Zone I of the coordinate system is rotated 90 degrees counterclockwise.

The evaluations of concept definitions on the hierarchical concept structure are given in Table2. Among the features of the hierarchical concept structure of the definition, each concept can be defined as a special case of a more general concept. The criteria of hierarchy in definitions requires the consideration of hierarchical relations between these concepts. While defining translation, reflection, and rotation, it is observed that the TCs used point, geometric shapes and objects in transformation. However, points are mostly used in the rotation, and geometric shape is used in the translation. Yet, in all

transformations there is point, geometric shape or movement of any object. Whereas, only one TC mentioned the movement of an object other than geometric concepts. This finding shows that TCs do not know exactly what the object used in transformations is. In explaining each of the three concepts, the TCs used the coordinate plane, line, axis and especially x, y axis. While the reflection and rotation take place according to the plane and axis, the translation does not occur according to the axis or plane. In addition, while defining rotation, axis or plane is not a critical concept which necessarily needs to be used. It is the rotation center and rotation angle which is critical for rotation. In the study, although there are TCs which mentions rotation angle as angle and degree, only two TCs mentions the center of rotation. While the degree is meant to be an angle, it can be said that the TC that responds in this way has a problem in terms of angle rather than rotation. It cannot be said that TCs, who mention the hierarchical relationship established with the axis instead of the center of rotation for rotation, fully understand the concept. Moreover, it is also not a right approach to think that only x and y axis are used in translation and reflection, for example, a state of reflection according to a point or plane is also in question. There have been very few TCs that emphasize the concepts of area and volume related to the properties that are changing and remain constant in transformations. Yet, there is no change in the volume and area of the objects in all three transformations, and the invariance of area and volume is one of the critical features that are expected to be mentioned when defining translation, reflection and rotation. There have also been some TCs who have used the concept of dimension instead of the concepts of area and volume, and they state that the dimensions of the shapes remain same in transformations. However, the TCs that made statements in this way constitute a very small part of the participants. Among the hierarchical concepts used in relation to reflection, similar pieces, identical, and equivalent concepts are included. These three concepts actually emphasize the equality of shapes. Although the image is equal to its object in terms of area and volume, it is not correct to say that the shapes are exactly the same. In this way, the TCs who have commented in this way have overlooked the change which would take place as a result of the reflection of objects in their direction. In this place, the change in the directions of the shapes has been ignored. It is not possible to define the concept as desired in the hierarchical structure without using the concepts below.

Conclusions and Discussion

In this study, the definitions of the concepts of translation, rotation and symmetry are asked to the TCs, and these definitions made by TCs are explored according to the criteria to be a definition as accuracy, existence, hierarchical concept structure and equivalence. It is observed that among the findings for the category of accuracy the TCs have generally made definitions necessary and not sufficient in translation, partially sufficient and unnecessary in reflection, and necessary and not sufficient in rotation. The explanations made by the

TCs were generally composed of informal expressions. Such explanations of the TCs, who will transfer these concepts as teachers in the future, can make the concepts such as translation, reflection and rotation more difficult for students. Thus, NCTM-National Council of Teachers of Mathematics (2000) states that it would be more accurate to transfer the movements such as shifting, reflecting and rotating by means of formal means. Knuchel (2004) also stated the fact that the studies especially related to reflection and its features are given to students through the formal ways, is very important to make better sense of mathematics and even the life. By this way, it is stated that students can realize doing mathematics. Therefore, formal definitions of concepts should be paid attention in the courses given to the TCs at university level so that the TCs can formally transfer these concepts to their students. In the findings related to the category of existence, the examples used by the TCs to explain the concepts are examined and it is determined that the examples given are either acceptable or unacceptable. For example, in order to illustrate that the translation is in a certain direction and unit, the TC 22 gives a appropriate example for the translation, referring to the movements of "the object 3 units to the right, 4 units to the up". In an inappropriate example for reflection, TC has considered the reflection as objects to be in the same position. In the example, TC has overlooked the features of having the same dimensions with the object and having the same distance to any axis, and has made an explanation that has no relation to the concept. The fact that the inappropriate examples are given in this way has demonstrated that the concept is not sufficiently understood by the TCs. When examining the responses of the TCs in relation to the category of hierarchical concept structure, it is observed that some of the sub-concepts they use to explain the concepts are appropriate, some are not appropriate or incomplete. For example, TCs have used the concepts of point or geometric shape for the object which would be subject to translation, reflection, or rotation, but they have not stated that any shape other than these two can be rotated. This finding shows that TCs do not know exactly what the object used in transformations is. Or it is stated that the rotation transformation is made according to an axis in rotation, whereas it is overlooked that it is a reference point, not the axis, which is important in the rotation transformation. It is observed that from the findings of the equivalence category among the concepts of translation, reflection and rotation, mostly in reflection the TCs gave responses which would fall into the category of equivalence, that's, they tend to explain the reflection with alternative definitions. In these definitions, they have benefited more from the axis of equality and symmetry. It will be in the interest of the TCs to use the various enriched examples in classroom practices, the different explanations and representations of the concept, in order to provide the comprehension of the concepts and realize effective teaching.

The findings obtained from the research show that the TCs perceived transformation as a dynamic movement. Yet, the transformation is in a static structure as a one-to-one and onto function in the plane. Unfortunately, no TC has indicated that transformation is a function. Studies in the literature also have similarities with this finding (Edwards, 2003; Hollebrands, 2003). In our study, since the TCs will be secondary school teachers, they may have preferred the concepts that are more appropriate to the student level by not using the concept of function in explaining the concepts, and the error may be caused by this. In addition, defining the transformation of translation, rotation and reflection as a whole, and taking the images of the shapes comes into question. However, teacher candidates used the concept of image especially in defining the reflection, while only a few TCs included the concept of image in translation and rotation. Usually used in reflection may also be the result of the reflection being perceived as a mirror image. The fact that it is usually used in reflection, may also be the result of the reflection being perceived as a mirror image.

In the study, it is identified that some of the TCs cannot explain the properties of geometric objects that remain constant under transformations, or they mention or cannot mention limited critical features related to transformations. For example, in the translation, the concept of vector has never been used, while the movement has been limited to the up-down and right-left. Sarama and Clements (2009) have stated that while students realize horizontal or vertical translation, they experience problems with cross translation. Some of the TCs in the rotation has not mentioned the critical concept of the angle and the point of reference, while some others has stated that the shape of the object changed during the rotation. Laborde (1993) also stated that students could not explain the features that remain constant in the transformations, and our study also supported the findings.

It is determined that the definitions made by the TCs for translation transformation are generally in the necessary and not sufficient category. No definition has been found to indicate that the translation transformation is a function. Yanık and Flores (2009) have stated that when the transformation is interpreted as a function, it will be understood as that all points on the plane rather than a point or a shape on the plane are transformed. The TCs have explained the translation as shifting, moving and change of location. However, the translation of all points on the plane during the translation transformation, and the vector itself that is the parameter of the transformation is to be subject to the translation, a distance-preserving transformation is made (Sünker & Zembat, 2012). It can be said that TCs cannot fully interpret the translation is due to the fact that the transformation is not understandable. Sünker and Zembat (2012) also noted that in the textbook activities the translation is attributed the meaning of shifting the shapes, and this situation makes it difficult to understand the background of the translation. Van de Walle, Karp and Mr.-Williams (2016) suggested that the use of the concept of shift rather than the concept of translation is more

appropriate for the initial activities associated with translation. It can be said that it is not too bad that the TC, who will transfer translation to his/her students, also attributes the meaning of shifting to this. The concept of both shifting and translation can be used at different times during the communication of the subject.

Only 5 TCs have been able to make appropriate definitions for the concept of reflection transformation. In this definition, features such as that it is equal distance to line, and it changes direction while it remains in constant form, have been mentioned. In spite of this, most TCs gave partially necessary and not sufficient answers. The definitions that are made for reflection are generally composed of phrases as "reflection in water", "image in mirror". Bassarear (1995) stated that reflection can be seen in nature and in man-made substances that are part of our daily life. The findings that were obtained in this study has supported this view. Bintaş, Altun and Arslan (2003) have also found that students use similar concepts for reflection. In addition, there were teacher candidates who indicated that the figures in the image were identical, and they ignored the aspect of changing quantity. Some has also stated that the mark of the image obtained from the reflection will be different. However, there is no mark of the image, there are markings of the coordinate points of the image, and they do not necessarily have to be always different. At this point, it can be said that in reflection transformation, TC is not aware of the fact that the reflection of all points of the figure is taken, the reflection of the image is directly taken. Some of the TCs has indicated that the reflection would be based only on the x and y axis. However, reflection cannot only be based on these axis, but it can also be based on line, point and plane. Hacısalihoğlu Karadeniz, Baran, Bozkuş & Gündüz (2015) revealed that TCs cannot identify the symmetry of the shape when the axis of symmetry is oblique. In one aspect, the findings obtained in our study has supported this finding. The source of the problems in the definitions of the TCs may be that the teaching practices given to them are not conceptual but rather with limited examples. In the teaching process, giving conceptual information about these concepts to TCs, providing more diverse examples, using concrete materials, enriched activities and different models can help to eliminate the problems.

While only one TC has made a correct definition with regard to rotation transformation, the majority of them made definitions in the necessary and not sufficient category. It is identified that in some of the definitions made by the TCs, that they did not receive any reference points in which they would make rotation, and they rotated the shape in its entirety. In other words, some TCs have mentioned that the shape would be rotated at a certain angle, but they have not indicated around its which point it would be rotated, and the center of rotation. Some studies in the literature (Bazan, 2017; Turgut, Yenilmez & Anapa, 2014) also put forward that in a similar way students experience difficulties in identifying the center of rotation of the shape. There have been some TCs stating that the rotation has taken place at a certain angle, but it is identified that they have

misunderstandings about the angle of rotation. The angle of rotation is limited to certain angles, such as 900, 1800, 2700 and 3600, and the angles of rotation other than these angles have never been mentioned. One of TCs even wanted to point out that the rotation would only take place at a 360-degree angle, explaining as "an object, which moves from one point, returns to the same point." However, in rotation, rotating a shape with a desired angle is in question. This situation can be the result of the x and y axis in the coordinate plane shown with angles of 900, 1800, 2700 and 3600. Similarly, Clements and Battista (1992) have also stated that students experience difficulties in terms of the angle of rotation. As a source of this situation, they have showed the evaluation of the concept of angle with a static approach. There are also those who explain the rotation as translation at a certain angle, which shows that TCs consider translation and rotation in the same sense.

As a result, it is observed that TCs have experienced difficulties in defining the concepts of translation, reflection and rotation. In the future, candidates, who will teach these concepts in their classes, should be further trained in courses related to mathematics in undergraduate program.

References

Altun, M. (2004). *Matematiköğretimi.* İstanbul: Alfa Yayıncılık.

Argün, Z., Arıkan, A, Bulut, S. & Halıcıoğlu, S. (2014). *Temelmatematikkavramlarınınkünyesi.* Ankara: Gazi Kitabevi.

Baki, A. (2008). *Kuramdanuygulamayamatematikeğitimi (Genişletilmiş 4. Basım).* Ankara: Harf Eğitim Yayıncılığı.

Bassarear, T. (1995). *Mathematics for elementary school teachers*. Houhton Mifflin Compony, Boston New York.

Bazan, T. (2017). Ortaokul öğrencilerinin dönüşüm geometrisinde yaşadıkları öğrenme güçlükleri. (Yayınlanmamış Yüksek lisans tezi), Dokuz Eylül Üniversitesi, İzmir.

Bintaş, J., Altun, M. & Arslan, K. (2001). *Gerçekçimatematikeğitimiilesimetriöğretimi.* Matematikçiler Derneği Matematik Sempozyumu. Milli Kütüphane, Ankara.

Borasi, R. (1992). Learning mathematics through Inquiry. Portsmouth, NH: Heinemann.

Bulut, S., Boz Yaman, B. & Yavuz, F. D. (2016). 7. sınıf matematik ders kitaplarında dönüşüm geometrisi işlenişinin öğretim programları açısından Değerlendirilmesi. *İlköğretim Online*, 15(4), 1164-1190.

Büyüköztürk, Ş., ÇakmakKılıç, E., Akgün, Ö., Karadeniz, Ş. & Demirel, F. (2011). *Bilimselaraştırmayöntemleri* (10. baskı). Ankara: Pegem Akademi Yayıncılık.

Clements, D. H. & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420–464). New York: Macmillan.

Çakıroğlu, E. (2013). Matematikkavramlarınıntanımlanması. In Zembat, İ. Ö; Özmantar M.F.; Bingölbali E.; Şandır H. ve Delice A. (Eds.) *Tanımları ve tarihsel gelişimleriyle matematiksel kavramla*r (1.baskı). Ankara: Pegem Akademi Yayıncılık.

Çetin, İ., Erdoğan, A. & Yazlık, D. Ö. (2015). Geogebra ile öğretimin sekizinci sınıf öğrencilerinin dönüşüm geometrisi konusundaki başarılarına etkisi. *Uluslararası Türk Eğitim Bilimleri Dergisi*, 4, 84-92.

Demir, Ö. & Kurtuluş, A. (2019). Dönüşüm geometrisi öğretiminde 5e öğrenme modelinin 7. sınıf öğrencilerinin van hiele dönü-şüm geometrisi düşünme düzeylerine etkisi .*Eskişehir Osmangazi Üniversitesi Sosyal Bilimler Dergisi,* Eskişehir Osmangazi Üniversitesi Sosyal Bilimler Dergisi Armağan Özel Sayisi, 1279-1299.

Edwards, L. (2003). *The nature of mathematics as viewed from cognitive science*. Paper presented at the third congress of the European Society for Research in Mathematics, Bellaria, Italy.

Fletcher, T. J., (1973). *Some lessons in Mathematics: A handbook on the Teaching of Modern Mathematics,* The University Pres. Cambridge.

Gürbüz, K. & Durmuş, S. (2009). The qualifıcation levels of in-service elementary school teachers on the sub-learning strands, namely transformational geometry, geometric objects, pattern and tesselations, *AbantİzzetBaysal University Journal of Faculty of Education,9*(1), 1-22.

Güven, B. & Yılmaz Kaleli, G. (2011). Dinamik geometri yazilimlarinin dönüşüm geometrisi konusunda sinif öğretmeni adaylarinin başarilarina etkisi. 5th International Computer Instructional Technologies Symposium, Gaziantep.

Hacısalihoğlu-Karadeniz, M., Baran, T., Bozkuş, F. & Gündüz, N. (2015). Difficulties of prospective elementary mathematics teachers' regarding to reflection symmetry. *Turkish Journal of Computer and Mathematics Education, 6*(1), 117-138.

Hollebrands, K. F. (2003). High school students' understandings of geometric transformations in the context of a technological environment. *The Journal of Mathematical Behavior, 22*(1), 55-72.

Knuchel, C. (2004). Teaching symmetry in the elementary curriculum. *The Montana Mathemnatics Enthusiast, 1*(1), 3-8.

Korkmaz, E. & Tutak, T. (2017). Dönüşüm Geometrisi Konularının Gerçekçi Matematik Eğitimi Etkinlikleriyle İşlenmesinin Öğrenci Başarısına ve Matematik Tutumuna Etkisi. *Disiplinlerarası Eğitim Araştırmaları Dergisi*, *1*(2), 30-42.

Laborde, C. (1993). The computer as part of the learning environment: the case of geometry. In C. Keitel, & K. Ruthven (Eds.), *Learning from computers: mathematics education and technology* (pp. 48–67). Berlin: Springer.

Linchevsky, L., Vinner, S. & Karsenty, R. (1992). To be or not to be minimal? Student teachers' views about definitions in geometry. In W. Geeslin and K. Graham (Eds.), *Proceedings of the 16th Annual Conference of the International Group for the Psychology of Mathematics Education* (pp. 48-55). Durham, NH: PME.

Lombard, M., Snyder-Duch, J. & Bracken,C.C. (2002). Content analysis in mass communication: Assessment and reporting of inter coder reliability. *Human Communication Research, 28,* 587- 604.

Martin, G. (1982). Transformation Geometry: An Introduction to symmetry. New York: Springer-Verlag.

Miles, B. M. & Huberman, A. M. (1994). *Qualitative data analvsis* (2nd ed.). London: Sage Publications.

Mutlu, Y. & Söylemez, İ. (2019). Ortaokul 8. Sınıf Öğrencilerine Geogebra Matematik Yazılımı ile Dönüşüm Geometrisi Konusunun Öğretimi. *Eğitim ve Teknoloji*1(2) 163-172.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Öksüz, H. & Gürefe, N. (2019). Ortaokul 8. sınıf öğrencilerinin öteleme ve yansıma konusunda sahip olduğu bilgiler. *4. Uluslararası Türk Bilgisayar ve Matematik Eğitimi (TÜRKBİLMAT-4) Sempozyumu*, 26-28 September, 1082-1090.

Patton, M. Q. (2002). *Qualitative research & evaluation methods.* USA: Sage Publications.

Sarama, J. & Clements, D. H. (2009). Building blocks and cognitive building blocks. *American Journal of Play, 1*(3), 313-337.

Sarpkaya-Aktaş, G. & Ünlü, M. (2017). Understanding of eight grade students about transformation geometry: perspectives on students' mistakes. *Journal of Education and Training Studies, 5*(5), 103-119.

Shield, M. J. & Swinson, K. V. (1997). Encouraging learning in mathematics through writing. *The Australian Mathematics Teacher,53*(1), 4–9.

Son, J. W. (2006). Investıgatıng preservıce teachers' understandıng and strategıes on a student's errors of reflectıve symmetry. *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education, Charles University Prague* 5, 145-152.

Soon, Y. P. (1989). An investigation of van Hiele-like levels of learning in transformation geometry of secondary school students in Singapore. http://wwwlib.umi.com/ dissertations/fullcit/8915764 on 14.02.2018.

Sünker, S. & Zembat, Ö. İ. (2012). Ötelemedönüşümünün Wingeom-trortamındavektörleryardımıylaöğretimi. *Elementary Education Online, 11*(1), 173-194.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.

TDK dictionary http://www.tdk.gov.tr/index.php?option=com_bts&arama= kelime&guid =TDK.GTS.5b6814ba834900.44195021on 6.08.2018.

Turgut, M., Yenilmez, K. & Anapa, P. (2014). Symmetry and rotation skills of prospective elementary mathematics teachers. *Bolema: Boletim de Educação Matemática*, 28(48), 383–402.

Winicki-Landman, G. & Leikin, R. (2000). On equivalent and non-equivalent definitions: Part 1. *For the Learning of Mathematics, 20*(1), 17–21.

Van Dormolen, J. & Zaslavsky, O. (2003). The many facets of a definition: The case of periodicity. *The Journal of Mathematical Behavior,* 22(1), 91-106.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In David Tall (Ed.), *Advanced Mathematical Thinking*, (pp. 65–81). Kluwer, Dordrecht, The Netherlands.

Yanik, H. B. & Flores, A. (2009). Understanding rigid geometric transformations: Jeff's learning path for translation. *The Journal of Mathematical Behavior*, *28*(1), 41- 57.

Yavuzsoy-Köse, N. (2013) Geometrikdönüşümlerdenbiri: Yansıma Dönüşümünü Anlamak. In Zembat, İ. Ö; Özmantar M. F.; Bingölbali E.; Şandır H. veDelice A. (Eds.) *Tanımlarıvetarihselgelişimleriylematematikselkavramlar (1.baskı).* Ankara: Pegem Akademi Yayıncılık.

Zaslavsky, O. & Shir, K. 2005. Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education, 36*(4): 317–346.

Zazkis, R. & Leikin, R. (2008). Exemplifying definitions: A case of a square. *Educational Studies in Mathematics, 69,* 131-148.

Zembat, İ. Ö. (2007). The main tenets of direct instruction and constructivism: the case of translations. *Gazi University Journal of Gazi Educational Faculty, 27*(1), 195-213.

Zembat, İ. Ö. (2013). Geometrikdönüşümlerdenötelemevefarklıanlamaları. In Zembat, İ. Ö; Özmantar M.F.; Bingölbali E.; Şandır H. veDelice A. (Eds.) *Tanımları vetarihselgelişimleriylematematikselkavramlar (1.baskı).* Ankara: Pegem Akademi Yayıncılık.

Zembat, İ. Ö. (2013a). Geometrik Dönüşümlerden Dönmeve Özellikleri. In Zembat, İ. Ö; Özmantar M. F.; Bingölbali E.; Şandır H. veDelice A. (Eds.) *Tanımlarıvetarihselgelişimleriylematematikselkavramlar* (1. baskı). Ankara: Pegem Akademi.