On Upper and Lower Contra-Continuous Fuzzy Multifunctions

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Abstract. This paper is devoted to the concepts of fuzzy upper and fuzzy lower contra-continuous, contra-irresolute and contra semi-continuous multifunctions. Several characterizations and properties of these multifunctions along with their mutual relationships are established in $L$-fuzzy topological spaces. Later, composition and union between these multifunctions have been studied.

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1. INTRODUCTION AND PRELIMINARIES

Kubiak [17] and Sostak [28] introduced the notion of ($L$-)fuzzy topological space as a generalization of $L$-topological spaces (originally called ($L$-)fuzzy topological spaces by Chang [8] and Goguen [10]). It is the grade of openness of an $L$-fuzzy set. A general approach to the study of topological type structures on fuzzy powersets was developed in [11-13,17,18,28-30].

Berge [7] introduced the concept multimapping $F : X \rightarrow Y$ where $X$ and $Y$ are topological spaces and Popa [24,25] introduced the notion of irresolute multimapping. After Chang introduced the concept of fuzzy topology [8], continuity of multifunctions in fuzzy topological spaces have been defined and studied by many authors from different view points (e.g. see [3,4,21-23]). Tsiporkova et. al., [31,32] introduced the Continuity of fuzzy multivalued mappings in the Chang's fuzzy topology [8]. Later, Abbas et al., [1] introduced the concepts of fuzzy upper and fuzzy lower semi-continuous multifunctions in $L$-fuzzy topological spaces.
Throughout this paper, nonempty sets will be denoted by $X$, $Y$ etc.. Let a complete lattice $L = (L, \leq, \lor, \land)$ be a complete distributive complete lattice with an order-reversing involution on it, and with a smallest element $\bot$ and largest element $\top$. The family of all $L$-fuzzy sets in $X$ is denoted by $L^X$ and $L_0 = L - \{0\}$. For $\alpha \in L$, $\alpha(x) = \alpha$ for all $x \in X$. The complement of an $L$-fuzzy set $\lambda$ is denoted by $\lambda^c$. This symbol $\circ$ for a multifunction. All other notations are standard notations of $L$-fuzzy set theory.

**Definition 1.1.** [1] Let $F : X \rightharpoonup Y$, then $F$ is called a fuzzy multifunction ($FM$, for short) iff $F(x) \in L^Y$ for each $x \in X$. The degree of membership of $y$ in $F(x)$ is denoted by $F(x)(y) = G_F(x, y)$ for any $(x, y) \in X \times Y$.

The domain of $F$, denoted by dom($F$) and the range of $F$, denoted by rng($F$), for any $x \in X$ and $y \in Y$, are defined by:

$$\text{dom}(F)(x) = \bigvee_{y \in Y} G_F(x, y) \quad \text{and} \quad \text{rng}(F)(y) = \bigvee_{x \in X} G_F(x, y).$$

**Definition 1.2.** [1] Let $F : X \rightharpoonup Y$ be a FM. Then $F$ is called:

1. Normalized iff for each $x \in X$, there exists $y_0 \in Y$ such that $G_F(x, y_0) = \top$.
2. A crisp iff $G_F(x, y) = \top$ for each $x \in X$ and $y \in Y$.

**Definition 1.3.** [1] Let $F : X \rightharpoonup Y$ be a FM. Then,

1. The image of $\lambda \in L^X$ is an $L$-fuzzy set $F(\lambda) \in L^Y$ defined by:

$$F(\lambda)(y) = \bigvee_{x \in X} [G_F(x, y) \land \lambda(x)].$$

2. The lower inverse of $\mu \in L^Y$ is an $L$-fuzzy set $F^l(\mu) \in L^X$ defined by:

$$F^l(\mu)(x) = \bigvee_{y \in Y} [G_F(x, y) \land \mu(y)].$$

3. The upper inverse of $\mu \in L^Y$ is an $L$-fuzzy set $F^u(\mu) \in L^X$ defined by:

$$F^u(\mu)(x) = \bigwedge_{y \in Y} [G_F(x, y) \lor \mu(y)].$$

**Theorem 1.4.** [1] Let $F : X \rightharpoonup Y$ be a FM. Then,

1. $F(\lambda_1) \leq F(\lambda_2)$ if $\lambda_1 \leq \lambda_2$.
2. $F^l(\mu_1) \leq F^l(\mu_2)$ and $F^u(\mu_1) \leq F^u(\mu_2)$ if $\mu_1 \leq \mu_2$.
3. $F^l(\mu) = (F^u(\mu))^c$.
4. $F^u(\mu) = (F^l(\mu))^c$.
5. $F(F^u(\mu)) \leq \mu$ if $F$ is a crisp.
6. $F(F(\lambda)) \geq \lambda$ if $F$ is a crisp.

**Definition 1.5.** [1] Let $F : X \rightharpoonup Y$ and $H : Y \rightharpoonup Z$ be two $FM$ s. Then the composition $H \circ F$ is defined by: $((H \circ F)(x))(z) = \bigvee_{y \in Y}[G_F(x, y) \land G_H(y, z)]$.

**Theorem 1.6.** [1] Let $F : X \rightharpoonup Y$ and $H : Y \rightharpoonup Z$ be two $FM$ s. Then we have the following:

1. $(H \circ F) = F(H)$.
2. $(H \circ F)^u = F^u(H^u)$.
3. $(H \circ F)^l = F^l(H^l)$. 

Theorem 1.7. [1] Let \( F_i : X \rightarrow Y \) be a FM. Then,
\begin{enumerate}
    \item \( (\bigcup_{i \in I} F_i)(\lambda) = \bigvee_{i \in I} F_i(\lambda) \).
    \item \( (\bigcup_{i \in I} F_i)^{\prime}(\mu) = \bigvee_{i \in I} F_i^{\prime}(\mu) \).
    \item \( (\bigcup_{i \in I} F_i)^{\prime\prime}(\mu) = \bigwedge_{i \in I} F_i^{\prime\prime}(\mu) \).
\end{enumerate}

Definition 1.8. [13,17,20,28] An \( L \)-fuzzy topological space (\( L \)-fts, in short) is a pair \((X, \tau)\), where \( X \) is a nonempty set and \( \tau : L^X \rightarrow L \) is a mapping satisfying the following properties:
\begin{enumerate}
    \item \( \tau(\emptyset) = \tau(\top) = \top \).
    \item \( \tau(\lambda \land \lambda_2) \geq \tau(\lambda_1) \land \tau(\lambda_2) \), for any \( \lambda_1, \lambda_2 \in L^X \).
    \item \( \tau(\bigvee_{i \in I} \tau(\lambda_i)) \geq \bigwedge_{i \in I} \tau(\lambda_i) \), for any \( \{\lambda_i\}_{i \in I} \subseteq L^X \).
\end{enumerate}
Then \( \tau \) is called an \( L \)-fuzzy topology on \( X \). For every \( \lambda \in L^X \), \( \tau(\lambda) \) is called the degree of openness of the \( L \)-fuzzy set \( \lambda \).

A mapping \( f : (X, \tau) \rightarrow (Y, \eta) \) is said to be continuous with respect to \( L \)-fuzzy topologies \( \tau \) and \( \eta \) if \( f^{-1}(\mu) \geq \eta(\mu) \) for each \( \mu \in L^Y \).

Theorem 1.9. [9,14,16,20] Let \((X, \tau)\) be an \( L \)-fts. Then for each \( \lambda \in L^X \), \( r \in L_0 \) we define \( L \)-fuzzy operators \( C_\tau \) and \( I_\tau : L^X \times L_0 \rightarrow L^X \) as follows:
\[
    C_\tau(\lambda, r) = \bigwedge\{\mu \in L^X : \lambda \leq \mu, \tau(\mu) \geq r\},
\]
\[
    I_\tau(\lambda, r) = \bigvee\{\mu \in L^X : \mu \leq \lambda, \tau(\mu) \geq r\}.
\]
For \( \lambda, \mu \in L^X \) and \( r, s \in L_0 \) the operator \( C_\tau \) satisfies the following statements:
\begin{enumerate}
    \item \( C_\tau(\bot, r) = \bot \).
    \item \( \lambda \leq C_\tau(\lambda, r) \).
    \item \( C_\tau(\lambda, r) \lor C_\tau(\mu, r) = C_\tau(\lambda \lor \mu, r) \).
    \item \( C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r) \).
    \item \( C_\tau(\lambda, r) = \lambda \iff \tau(\lambda^c) \geq r \).
    \item \( C_\tau(\lambda^c, r) = (I_\tau(\lambda, r))^c \) and \( I_\tau(\lambda^c, r) = (C_\tau(\lambda, r))^c \).
\end{enumerate}

Definition 1.10. [6,14,27] Let \((X, \tau)\) be an \( L \)-fts. Then for each \( \lambda, \mu \in L^X \) and \( r \in L_0 \), \( \lambda \) is called:
\begin{enumerate}
    \item \( r \)-fuzzy semi-open (\( r \)-fso, in short) iff \( \lambda \leq C_\tau(I_\tau(\lambda, r), r) \).
    \item \( r \)-fuzzy semi-closed (\( r \)-fsc, in short) iff \( I_\tau(C_\tau(\lambda, r), r) \leq \lambda \).
\end{enumerate}

Theorem 1.11. [14] Let \((X, \tau)\) be an \( L \)-fts. Then for each \( \lambda \in L^X \), \( r \in L_0 \) we define \( L \)-fuzzy operators \( SC_\tau \) and \( SI_\tau : L^X \times L_0 \rightarrow L^X \) as follows:
\[
    SC_\tau(\lambda, r) = \bigwedge\{\mu \in L^X : \lambda \leq \mu, \mu \text{ is } r \text{-fsc}\},
\]
\[
    SI_\tau(\lambda, r) = \bigvee\{\mu \in L^X : \mu \leq \lambda, \mu \text{ is } r \text{-fso}\}.
\]

Theorem 1.12. [11] Let \( F : X \rightarrow Y \) be a FM between two \( L \)-fts \( X, (X, \tau), (Y, \eta) \) and \( \mu \in L^Y \). Then we have the following:
\begin{enumerate}
    \item \( F \) is FLS-continuous iff \( \tau(F^l(\mu)) \geq \eta(\mu) \).
    \item \( F \) is normalized, then \( F \) is FUS-continuous iff \( \tau(F^u(\mu)) \geq \eta(\mu) \).
    \item \( F \) is FLS-continuous iff \( \tau((F^u(\mu))^c) \geq \eta(\mu^c) \).
\end{enumerate}
(4) If $F$ is normalized, then $F$ is FU$S$-continuous iff $\tau((F^l(\mu))^c) \geq \eta(\mu^c)$. 

**Definition 1.13.** [2] Let $F : X \rightarrow Y$ be a FM between two L-f ts $(X, \tau), (Y, \eta)$ and $r \in L_0$. Then $F$ is called:

1. FU$W$-continuous (resp. FL$W$-continuous) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ (resp. $x_t \in F^l(\mu)$) for each $\mu \in L_Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in L_X, \tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \land dom(F) \leq F^u(C_\eta(\mu, r))$ (resp. $\lambda \leq F^l(C_\eta(\mu, r))$).

2. FU$W$-continuous (resp. FL$W$-continuous) if it is FU$W$-continuous (resp. FL$W$-continuous) at every $x_t \in dom(F)$.

**Proposition 1.14.** [2] If $F$ is normalized implies $F$ is FU$W$-continuous at an L-fuzzy point $x_t \in dom(F)$ if $x_t \in F^u(\mu)$ for each $\mu \in L_Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in L_X, \tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^u(C_\eta(\mu, r))$.

2. **Fuzzy Upper and Lower Contra-Continuous Multifunctions**

**Definition 2.1.** Let $F : X \rightarrow Y$ be a FM between two L-f ts $(X, \tau), (Y, \eta)$ and $r \in L_0$. Then $F$ is called:

1. Fuzzy upper contra-continuous (FU$C$-continuous, in short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L_Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in L_X, \tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \land dom(F) \leq F^u(\mu)$.

2. Fuzzy lower contra-continuous (FL$C$-continuous, in short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in L_Y$ and $\eta(\mu^c) \geq r$ there exists $\lambda \in L_X, \tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$.

3. FU$C$-continuous (resp. FL$C$-continuous) if it is FU$C$-continuous (resp. FL$C$-continuous) at every $x_t \in dom(F)$.

**Proposition 2.2.** If $F$ is normalized implies $F$ is FU$C$-continuous at an L-fuzzy point $x_t \in dom(F)$ if $x_t \in F^u(\mu)$ for each $\mu \in L_Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in L_X, \tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^u(\mu)$.

**Remark 2.3.** The notions of FU$C$-continuous multifunctions and FU$S$-continuous multifunctions are independent as shown in the following Examples 2.6 and 2.7.

**Theorem 2.4.** Let $F : X \rightarrow Y$ be a FM between two L-f ts $(X, \tau), (Y, \eta)$ and $\mu \in L_Y$, then the following are equivalent:

1. $F$ is FL$C$-continuous.
2. $\tau(F^l(\mu)) \geq r$, if $\eta(\mu^c) \geq r$.
3. $\tau(F^u(\mu))^c \geq r$, if $\eta(\mu) \geq r$.

Proof. (1) $\Rightarrow$ (2) Let $x_t \in dom(F), \mu \in L_Y, \eta(\mu^c) \geq r$ and $x_t \in F^l(\mu)$ then, there exists $\lambda \in L_X, \tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$ and hence $x_t \in I_r(F^l(\mu), r)$. Therefore, we obtain $F^l(\mu) \leq I_r(F^l(\mu), r)$. Thus $\tau(F^l(\mu)) \geq r$.

(2) $\Rightarrow$ (3) Let $\mu \in L_Y$ and $\eta(\mu) \geq r$ hence by (2),

$$\tau(F^l(\mu^c)) = \tau((F^u(\mu))^c) \geq r.$$

(3) $\Rightarrow$ (2) It is similar to that of (2) $\Rightarrow$ (3).
(2) ⇒ (1) Let \( x_t \in \text{dom}(F), \mu \in L^Y, \eta(\mu^c) \geq r \) with \( x_t \in F^d(\mu) \) we have by (2), \( \tau(F^d(\mu)) \geq r \). Let \( F^d(\mu) = \lambda \) (say) then, there exists \( \lambda \in L^X, \tau(\lambda) \geq r \) and \( x_t \in \lambda \) such that \( \lambda \leq F^d(\mu) \). Thus \( F \) is FLC-continuous.

We state the following result without proof in view of above theorem.

**Theorem 2.5.** Let \( F : X \rightarrow Y \) be a FM and normalized between two L-fts's \((X, \tau), (Y, \eta)\) and \( \mu \in L^Y \), then the following are equivalent:

1. \( F \) is FUC-continuous.
2. \( \tau(F^u(\mu)) \geq r \), if \( \eta(\mu^c) \geq r \).
3. \( \tau((F^d(\mu))^c) \geq r \), if \( \eta(\mu) \geq r \).

**Example 2.6.** Let \( X = \{x_1, x_2\}, \ Y = \{y_1, y_2, y_3\} \) and \( F : X \rightarrow Y \) be a FM defined by \( G_F(x_1, y_1) = 0.1, \ G_F(x_1, y_2) = \top, \ G_F(x_1, y_3) = \bot, \ G_F(x_2, y_1) = 0.5, \ G_F(x_2, y_2) = \bot \) and \( G_F(x_2, y_3) = \top \). We assume that \( \top = 1 \) and \( \bot = 0 \). Define \( L \)-fuzzy topologies \( \tau : L^X \rightarrow L \) and \( \eta : L^Y \rightarrow L \) as follows:

\[
\tau(\lambda) = \begin{cases} 
\top, & \text{if } \lambda \in \{\bot, \top\}, \\
\frac{1}{2}, & \text{if } \lambda \in (0.5, 0.6), \\
\bot, & \text{otherwise},
\end{cases}
\]

\[
\eta(\mu) = \begin{cases} 
\top, & \text{if } \mu \in \{\bot, \top\}, \\
\frac{1}{2}, & \text{if } \mu = 0.5, \\
\bot, & \text{otherwise}.
\end{cases}
\]

(1) \( F \) is FUC-continuous but not FUS-continuous because \( \eta(0.4) = \frac{1}{3} \) in \((Y, \eta)\), \( F^u(0.4) = 0.4 \) and \( \tau(F^u(0.4)) = \bot \). Hence, \( \tau(F^u(0.4)) \not\geq \eta(0.4) \).

(2) \( F \) is FLC-continuous but not FLS-continuous because \( \eta(0.4) = \frac{1}{3} \) in \((Y, \eta)\), \( F^l(0.4) = 0.4 \) and \( \tau(F^l(0.4)) = \bot \). Hence, \( \tau(F^l(0.4)) \not\geq \eta(0.4) \).

**Example 2.7.** Let \( X = \{x_1, x_2\}, \ Y = \{y_1, y_2, y_3\} \) and \( F : X \rightarrow Y \) be a FM defined by \( G_F(x_1, y_1) = 0.1, \ G_F(x_1, y_2) = \top, \ G_F(x_1, y_3) = \bot, \ G_F(x_2, y_1) = 0.5, \ G_F(x_2, y_2) = \bot \) and \( G_F(x_2, y_3) = \top \). We assume that \( \top = 1 \) and \( \bot = 0 \). Define \( L \)-fuzzy topologies \( \tau : L^X \rightarrow L \) and \( \eta : L^Y \rightarrow L \) as follows:

\[
\tau(\lambda) = \begin{cases} 
\top, & \text{if } \lambda \in \{\bot, \top\}, \\
\frac{1}{2}, & \text{if } \lambda \in (0.4, 0.5), \\
\bot, & \text{otherwise},
\end{cases}
\]

\[
\eta(\mu) = \begin{cases} 
\top, & \text{if } \mu \in \{\bot, \top\}, \\
\frac{1}{2}, & \text{if } \mu = 0.5, \\
\bot, & \text{otherwise}.
\end{cases}
\]

(1) \( F \) is FUS-continuous but not FUC-continuous because \( \eta(0.4) = \frac{1}{3} \) in \((Y, \eta)\), \( F^u(0.4) = 0.4 \) and \( \tau((F^u(0.4))^c) = \bot \). Thus, \( \tau((F^u(0.4))^c) \not\geq \frac{1}{3} \).

(2) \( F \) is FLS-continuous but not FLC-continuous because \( \eta(0.4) = \frac{1}{3} \) in \((Y, \eta)\), \( F^l(0.4) = 0.4 \) and \( \tau(F^l(0.4))^c) = \bot \). Thus, \( \tau(F^l(0.4))^c) \not\geq \frac{1}{3} \).
Definition 2.8. Let \((X, \tau)\) be an \(L\)-fts. Then for each \(\lambda \in L^X\) and \(r \in L_0\) we define \(L\)-fuzzy operator \(\text{Ker}_\tau : L^X \times L_0 \rightarrow L^X\) as follows:

\[
\text{Ker}_\tau(\lambda, r) = \bigwedge \{ \mu \in L^X : \lambda \leq \mu, \tau(\mu) \geq r \}.
\]

Lemma 2.9. For \(\lambda\) in an \(L\)-fts \((X, \tau)\), if \(\tau(\lambda) \geq r\) then \(\lambda = \text{Ker}_\tau(\lambda, r)\).

Theorem 2.10. Let \(F : X \rightarrow Y\) be a \(FM\) between two \(L\)-fts \((X, \tau)\) and \((Y, \eta)\). If \(C_\tau(F^u(\mu), r) \leq F^u(\text{Ker}_\eta(\mu, r))\) for any \(\mu \in L^Y\), then \(F\) is \(FLC\)-continuous.

Proof. Suppose that \(C_\tau(F^u(\mu), r) \leq F^u(\text{Ker}_\eta(\mu, r))\) for any \(\mu \in L^Y\). Let \(\nu \in L^Y\) and \(\eta(\nu) \geq r\) by Lemma 2.9, we have \(C_\tau(F^u(\nu), r) \leq F^u(\text{Ker}_\eta(\nu, r)) = F^u(\nu)\). This implies that \(C_\tau(F^u(\nu), r) = F^u(\nu)\) and hence \(\tau((F^u(\nu))^\circ) \geq r\). Thus, by Theorem 2.4(3), \(F\) is \(FLC\)-continuous.

Theorem 2.11. Let \(F : X \rightarrow Y\) be a \(FM\) and normalized between two \(L\)-fts \((X, \tau)\) and \((Y, \eta)\). If \(C_\tau(F^l(\mu), r) \leq F^l(\text{Ker}_\eta(\mu, r))\) for any \(\mu \in L^Y\), then \(F\) is \(FUC\)-continuous.

Proof. Suppose that \(C_\tau(F^l(\mu), r) \leq F^l(\text{Ker}_\eta(\mu, r))\) for any \(\mu \in L^Y\) and \(\eta(\nu) \geq r\) by Lemma 2.9, we have \(C_\tau(F^l(\nu), r) \leq F^l(\text{Ker}_\eta(\nu, r)) = F^l(\nu)\). This implies that \(C_\tau(F^l(\nu), r) = F^l(\nu)\) and hence \(\tau((F^l(\nu))^\circ) \geq r\). Thus, by Theorem 2.5(3), \(F\) is \(FUC\)-continuous.

Theorem 2.12. Let \(\{F_i\}_{i \in \Gamma}\) be a family of \(FLC\)-continuous between two \(L\)-fts \((X, \tau)\) and \((Y, \eta)\). Then \(\bigcup_{i \in \Gamma} F_i\) is \(FLC\)-continuous.

Proof. Let \(\mu \in L^Y\) and \(\eta(\mu^c) \geq r\) then \((\bigcup_{i \in \Gamma} F_i)^l(\mu) = \bigvee_{i \in \Gamma} (F_i^l(\mu))\) by Theorem 1.7(2). Since \(\{F_i\}_{i \in \Gamma}\) is a family of \(FLC\)-continuous between two \(L\)-fts \((X, \tau)\) and \((Y, \eta)\), then \(\tau(F_i^l(\mu)) \geq r\) for each \(i \in \Gamma\). Then for each \(\mu \in L^Y\) and \(\eta(\mu^c) \geq r\), we have \(\tau((\bigcup_{i \in \Gamma} F_i)^l(\mu)) = \tau(\bigvee_{i \in \Gamma} F_i^l(\mu)) \geq \bigwedge_{i \in \Gamma} \tau(F_i^l(\mu)) \geq r\). Hence \(\bigcup_{i \in \Gamma} F_i\) is \(FLC\)-continuous.

Theorem 2.13. Let \(F_1\) and \(F_2\) be two normalized \(FUC\)-continuous between two \(L\)-fts \((X, \tau)\) and \((Y, \eta)\). Then \(F_1 \cup F_2\) is \(FUC\)-continuous.

Proof. Let \(\mu \in L^Y\) and \(\eta(\mu^c) \geq r\) then \((F_1 \cup F_2)^u(\mu) = F_1^u(\mu) \wedge F_2^u(\mu)\) by Theorem 1.7(3). Since \(F_1\) and \(F_2\) be two normalized \(FUC\)-continuous between two \(L\)-fts \((X, \tau)\) and \((Y, \eta)\), then \(\tau(F_i^u(\mu)) \geq r\) for each \(i \in \{1, 2\}\). Then for each \(\mu \in L^Y\) and \(\eta(\mu^c) \geq r\), we have \(\tau((F_1 \cup F_2)^u(\mu)) = \tau(F_1^u(\mu) \wedge F_2^u(\mu)) \geq \tau(F_1^u(\mu)) \wedge \tau(F_2^u(\mu)) \geq r\). Hence \(F_1 \cup F_2\) is \(FUC\)-continuous.

Theorem 2.14. Let \(F : X \rightarrow Y\) and \(H : Y \rightarrow Z\) be two \(FM\)s and let \((X, \tau), (Y, \eta)\) and \((Z, \delta)\) be three \(L\)-fts. If \(F\) is \(FLS\)-continuous and \(H\) is \(FLC\)-continuous, then \(H \circ F\) is \(FLC\)-continuous.
Proof. Let $F$ be $FLS$-continuous, $H$ be $FLC$-continuous and $\gamma \in L^Z$, $\delta(\gamma) \geq r$. Then from Theorem 1.12(1) and Theorem 2.4(2), we have $(\gamma \circ F)(\gamma) = F^\gamma(\gamma)$ and $\tau(F^\gamma(\gamma)) \geq \eta(\gamma)$ $r$. Thus $H \circ F$ is $FLC$-continuous.

We state the following result without proof in view of above theorem.

**Theorem 2.15.** Let $F : X \rightarrow Y$ and $H : Y \rightarrow Z$ be two $FMS$ and let $(X, \tau)$, $(Y, \eta)$ and $(Z, \delta)$ be three $L$-fts. If $F$ and $H$ are normalized, $F$ is $FUS$-continuous and $H$ is $FUC$-continuous, then $H \circ F$ is $FUC$-continuous.

**Theorem 2.16.** Let $F : X \rightarrow Y$ and $H : Y \rightarrow Z$ be two $FMS$ and let $(X, \tau)$, $(Y, \eta)$ and $(Z, \delta)$ be three $L$-fts. If $H$ is normalized, $H$ is $FUS$-continuous and $F$ is $FLC$-continuous, then $H \circ F$ is $FLC$-continuous.

Proof. Let $F$ be $FLC$-continuous, $H$ be $FUS$-continuous and $\gamma \in L^Z$, $\delta(\gamma) \geq r$. Then from Theorem 1.12(2) and Theorem 2.4(3), we have $(\gamma \circ F)(\gamma) = F^\gamma(\gamma)$ and $\tau(F^\gamma(\gamma)) \geq \eta(\gamma)$ $r$. Thus $H \circ F$ is $FUC$-continuous.

We state the following result without proof in view of above theorem.

**Theorem 2.17.** Let $F : X \rightarrow Y$ and $H : Y \rightarrow Z$ be two $FMS$ and let $(X, \tau)$, $(Y, \eta)$ and $(Z, \delta)$ be three $L$-fts. If $F$ is normalized, $F$ is $FUS$-continuous and $H$ is $FLS$-continuous, then $H \circ F$ is $FUC$-continuous.

**Definition 2.18.** [5,15,19,26] An $L$-fuzzy set $\lambda$ in an $L$-fts $(X, \tau)$ is called $r$-fuzzy compact iff every family in $\{\mu : \tau(\mu) > r, \mu \in L^X\}$, where $r \in L_s$ covering $\lambda$ has a finite subcover.

**Definition 2.19.** An $L$-fuzzy set $\lambda$ in an $L$-fts $(X, \tau)$ is called $r$-fuzzy strongly $S$-closed iff every family in $\{\mu : \tau(\mu) > r, \mu \in L^X\}$, where $r \in L_s$ covering $\lambda$ has a finite subcover.

**Theorem 2.20.** Let $F : X \rightarrow Y$ be a crisp $FUC$-continuous between two $L$-fts $(X, \tau)$ and $(Y, \eta)$. Suppose that $F(x_t)$ is $r$-fuzzy strongly $S$-closed for each $x_t \in dom(F)$. If an $L$-fuzzy set $\lambda$ in an $L$-fts $(X, \tau)$ is $r$-fuzzy compact, then $F(\lambda)$ is $r$-fuzzy strongly $S$-closed.

Proof. Let $\lambda$ be $r$-fuzzy compact set in $X$ and $\{\gamma_i : \eta(\gamma_i) \geq r, i \in I\}$ be a family covering of $F(\lambda)$ i.e., $F(\lambda) \leq \bigvee_{i \in I} \gamma_i$. Since $\lambda = \bigvee_{x_t \in \lambda} x_t$, we have

$$F(\lambda) = F\left(\bigvee_{x_t \in \lambda} x_t\right) = \bigvee_{x_t \in \lambda} F(x_t) \leq \bigvee_{i \in I} \gamma_i.$$ 

It follows that for each $x_t \in \lambda$, $F(x_t) \leq \bigvee_{i \in I} \gamma_i$. Since $F(x_t)$ is $r$-fuzzy strongly $S$-closed for each $x_t \in dom(F)$, then there exists finite subset $I_{x_t}$ of $I$ such that $F(x_t) \leq \bigvee_{i \in I_{x_t}} \gamma_i = \gamma_{x_t}$. By Theorem 1.4(6), we have $x_t \leq F^\gamma(\gamma_{x_t}) \leq F^\gamma(\gamma_{x_t})$ and

$$\lambda = \bigvee_{x_t \in \lambda} x_t \leq \bigvee_{x_t \in \lambda} F^\gamma(\gamma_{x_t}).$$
From Theorem 2.5(2), we have \( \tau(F^u(\gamma_x)) \geq r \). Hence \( \{F^u(\gamma_x) : \tau(F^u(\gamma_x)) \geq r, x_t \in \lambda \} \) is a family covering the set \( \lambda \). Since \( \lambda \) is compact, there exists finite index set \( N \) such that \( \lambda \leq \bigvee_{n \in N} F^u(\gamma_{x_{tn}}) \). From Theorem 1.4(5), we have

\[
F(\lambda) \leq F(\bigvee_{n \in N} F^u(\gamma_{x_{tn}})) = \bigvee_{n \in N} F(F^u(\gamma_{x_{tn}})) \leq \bigvee_{n \in N} \gamma_{x_{tn}}.
\]

Then, \( F(\lambda) \) is \( r \)-fuzzy strongly \( S \)-closed.

**Theorem 2.21.** Let \( F : X \rightarrow Y \) be a \( FM \) between two \( L \)-fts \((X, \tau), (Y, \eta)\). If \( F \) is \( FLC \)-continuous then, \( F \) is \( FLW \)-continuous.

**Proof.** Let \( x_t \in \text{dom}(F), \mu \in L^Y, \eta(\mu) \geq r \) and \( x_t \in F^l(\mu) \). Since \( F \) is \( FLC \)-continuous, \( \eta([C_\eta(\mu, r)]^c) \geq r \) and \( x_t \in F^l[C_\eta(\mu, r)] \) then, there exists \( \lambda \in L^X, \tau(\lambda) \geq r \) and \( x_t \in \lambda \) such that \( \lambda \leq F^l[C_\eta(\mu, r)] \). Hence \( FLW \)-continuous.

We state the following result without proof in view of above theorem.

**Theorem 2.22.** Let \( F : X \rightarrow Y \) be a \( FM \) and normalized between two \( L \)-fts \((X, \tau), (Y, \eta)\). If \( F \) is \( FUC \)-continuous then, \( F \) is \( FUW \)-continuous.

**Remark 2.23.** [4,33] Let \( (X, \tau) \) and \( (Y, \eta) \) be an \( L \)-fts \( s \). An \( L \)-fuzzy sets of the form \( \lambda \times \mu \) with \( \tau(\lambda) \geq r \) and \( \eta(\mu) \geq r \) form a basis for the product \( L \)-fuzzy topology \( \tau \times \eta \) on \( X \times Y \), where for any \( (x, y) \in X \times Y, (\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(y)\} \).

**Theorem 2.24.** Let \((X, \tau)\) and \((X, \tau_i)\) be \( L \)-fts \( s \). If a \( FM \) \( F : X \rightarrow \prod_{i \in I} X_i \) is \( FLC \)-continuous (where \( \prod_{i \in I} X_i \) is the product space), then \( F \circ I \) is \( FLC \)-continuous for each \( i \in I \), where \( P_i : \prod_{i \in I} X_i \rightarrow X_i \) is the projection multifunction which is defined by \( P_k((x_i)) = \{x_i\} \) for each \( k \in I \).

**Proof.** Let \( \mu_{i_0} \in L_{X_{i_0}}^1 \) and \( \tau_{i_0}(\mu_{i_0}^c) \geq r \). Then \( (P_{i_0} \circ F)^l(\mu_{i_0}) = F^l(P_{i_0}^l(\mu_{i_0})) = F^l(\mu_{i_0} \times \prod_{i \neq i_0} X_i) \). Since \( F \) is \( FLC \)-continuous and \( \tau_i((\mu_{i_0} \times \prod_{i \neq i_0} X_i))^c \geq r \), it follows that \( \tau(F^l(\mu_{i_0} \times \prod_{i \neq i_0} X_i)) \geq r \). Then \( F \circ I \) is an \( FLC \)-continuous.

We state the following result without proof in view of above theorem.

**Theorem 2.25.** Let \((X, \tau)\) and \((X, \tau_i)\) be \( L \)-fts \( s \). If a \( FM \) \( F : X \rightarrow \prod_{i \in I} X_i \) is \( FUC \)-continuous (where \( \prod_{i \in I} X_i \) is the product space), then \( F \circ I \) is \( FUC \)-continuous for each \( i \in I \), where \( P_i : \prod_{i \in I} X_i \rightarrow X_i \) is the projection multifunction which is defined by \( P_k((x_i)) = \{x_i\} \) for each \( k \in I \).

**Theorem 2.26.** Let \((X, \tau_i)\) and \((Y, \eta_i)\) be \( L \)-fts \( s \) and \( F_i : X_i \rightarrow Y_i \) be a \( FM \) for each \( i \in I \). Suppose that \( F : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i \) is defined by \( F((x_i)) = \prod_{i \in I} F_i(x_i) \). If \( F \) is \( FLC \)-continuous, then \( F_i \) is \( FLC \)-continuous for each \( i \in I \).

**Proof.** Let \( \mu_i \in L_{Y_i}^1 \) and \( \eta_i(\mu_i^c) \geq r \). Then \( \eta_i((\mu_i \times \prod_{i \neq j} Y_j)^c) \geq r \). Since \( F \) is \( FLC \)-continuous, it follows that \( \tau_i(F^l(\mu_i \times \prod_{i \neq j} Y_j)) \geq r \) and \( F^l(\mu_i \times \prod_{i \neq j} Y_j) = F^l(\mu_i) \times \prod_{i \neq j} X_j \). Consequently, we obtain that \( \tau_i(F^l(\mu_i)) \geq r \) for each \( i \in I \).

Thus, \( F_i \) is \( FLC \)-continuous.
We state the following result without proof in view of above theorem.

**Theorem 3.5.** Let \((X_i, \tau_i)\) and \((Y_i, \eta_i)\) be \(L\)-fts and \(F_i : X_i \rightarrow Y_i\) be a \(FM\) for each \(i \in I\). Suppose that \(F : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i\) is defined by \(F((x_i)) = \prod_{i \in I} F_i(x_i)\). If \(F\) is \(FUC\)-continuous, then \(F_i\) is \(FUC\)-continuous for each \(i \in I\).

3. Fuzzy Upper and Lower Contra-Semi-Continuous Multifunctions

**Definition 3.1.** Let \(F : X \rightarrow Y\) be a \(FM\) between two \(L\)-fts \((X, \tau), (Y, \eta)\) and \(r \in L_0\). Then \(F\) is called:

1. Fuzzy upper contra-semi-continuous (\(\text{FUCS}\)-continuous, in short) at an \(L\)-fuzzy point \(x_t \in \text{dom}(F)\) iff \(x_t \in F^u(\mu)\) for each \(\mu \in L^Y\) and \(\eta(\mu^r) \geq r\) there exists \(r\)-fso set \(\lambda \in L^X\) and \(x_t \in \lambda\) such that \(\lambda \wedge \text{dom}(F) \leq F^u(\mu)\).

2. Fuzzy lower contra-semi-continuous (\(\text{FLCS}\)-continuous, in short) at an \(L\)-fuzzy point \(x_t \in \text{dom}(F)\) iff \(x_t \in F^l(\mu)\) for each \(\mu \in L^Y\) and \(\eta(\mu^r) \geq r\) there exists \(r\)-fso set \(\lambda \in L^X\) and \(x_t \in \lambda\) such that \(\lambda \leq F^l(\mu)\).

3. \(\text{FUCS}\)-continuous (resp. \(\text{FLCS}\)-continuous) iff it is \(\text{FUCS}\)-continuous (resp. \(\text{FLCS}\)-continuous) at every \(x_t \in \text{dom}(F)\).

**Definition 3.2.** Let \(F : X \rightarrow Y\) be a \(FM\) between two \(L\)-fts \((X, \tau), (Y, \eta)\) and \(r \in L_0\). Then \(F\) is called:

1. Fuzzy upper contra-irresolute (\(\text{FUC}\)-irresolute, in short) at an \(L\)-fuzzy point \(x_t \in \text{dom}(F)\) iff \(x_t \in F^u(\mu)\) for each \(\mu \in L^Y\) is \(r\)-fsc there exists \(r\)-fso set \(\lambda \in L^X\) and \(x_t \in \lambda\) such that \(\lambda \wedge \text{dom}(F) \leq F^u(\mu)\).

2. Fuzzy lower contra-irresolute (\(\text{FLC}\)-irresolute, in short) at an \(L\)-fuzzy point \(x_t \in \text{dom}(F)\) iff \(x_t \in F^l(\mu)\) for each \(\mu \in L^Y\) is \(r\)-fsc there exists \(r\)-fso set \(\lambda \in L^X\) and \(x_t \in \lambda\) such that \(\lambda \leq F^l(\mu)\).

3. \(\text{FUC}\)-irresolute (resp. \(\text{FLC}\)-irresolute) iff it is \(\text{FUC}\)-irresolute (resp. \(\text{FLC}\)-irresolute) at every \(x_t \in \text{dom}(F)\).

**Proposition 3.3.** \(F\) is normalized implies \(F\) is \(\text{FUCS}\)-continuous (resp. \(\text{FUC}\)-irresolute) at \(x_t \in \text{dom}(F)\) iff \(x_t \in F^u(\mu)\) for each \(\mu \in L^Y\) and \(\eta(\mu^r) \geq r\) (resp. \(\mu \) is \(r\)-fsc) there exists \(r\)-fso set \(\lambda \in L^X\) and \(x_t \in \lambda\) such that \(\lambda \leq F^u(\mu)\).

**Remark 3.4.** The notions of \(\text{FUC}\)-continuous multifunctions and \(\text{FUC}\)-irresolute multifunctions are independent as shown in the following Examples 3.9 and 3.10.

The following implications hold:
1. \(\text{FUC}\)-continuous \(\Rightarrow\) \(\text{FUCS}\)-continuous \(\Rightarrow\) \(\text{FUC}\)-irresolute.
2. \(\text{FLC}\)-continuous \(\Rightarrow\) \(\text{FLCS}\)-continuous \(\Rightarrow\) \(\text{FLC}\)-irresolute.

In general the converses are not true.

**Theorem 3.5.** Let \(F : X \rightarrow Y\) be a \(FM\) between two \(L\)-fts \((X, \tau), (Y, \eta)\) and \(\mu \in L^Y\), then the following are equivalent:

1. \(F\) is \(\text{FLCS}\)-continuous.
2. \(F^l(\mu)\) is \(r\)-fso, if \(\eta(\mu^r) \geq r\).
3. \(F^u(\mu)\) is \(r\)-fsc, if \(\eta(\mu) \geq r\).
Proof. (1) ⇒ (2) Let \( x_t \in \text{dom}(F) \), \( \mu \in L^Y \), \( \eta(\mu^c) \geq r \) and \( x_t \in F^l(\mu) \) then, there exists \( r \)-fo set \( \lambda \in L^X \) and \( x_t \in \lambda \) such that \( \lambda \leq F^l(\mu) \) and hence \( x_t \in SI_r(F^l(\mu), r) \).
Therefore, we obtain \( F^l(\mu) \leq SI_r(F^l(\mu), r) \). Thus, \( F^l(\mu) \) is \( r \)-fo.

(2) ⇒ (3) Let \( \mu \in L^Y \) and \( \eta(\mu) \geq r \) hence by (1), \((F^l(\mu))^c = (F^l(\mu))^c\) is \( r \)-fo. Then, \( F^l(\mu) \) is \( r \)-fo.

(3) ⇒ (2) It is similar to that of (2) ⇒ (3).

(2) ⇒ (1) Let \( x_t \in \text{dom}(F) \), \( \mu \in L^Y \), \( \eta(\mu^c) \geq r \) with \( x_t \in F^l(\mu) \) we have by (2), \( F^l(\mu) = \lambda \) (say) is \( r \)-fo then, there exists \( r \)-fo set \( \lambda \in L^X \) and \( x_t \in \lambda \) such that \( \lambda \leq F^l(\mu) \). Thus, \( F \) is \( FLC \)-isolates.

**Theorem 3.6.** Let \( F : X \rightarrow Y \) be a \( FM \) between two \( L \)-fts \( s (X, \tau), (Y, \eta) \) and \( \mu \in L^Y \), then the following are equivalent:

1. \( F \) is \( FLC \)-isolates.
2. \( F^l(\mu) \) is \( r \)-fo, for any \( \mu \) is \( r \)-fo.
3. \( F^u(\mu) \) is \( r \)-fo, for any \( \mu \) is \( r \)-fo.

Proof. (1) ⇒ (2) Let \( x_t \in \text{dom}(F) \), \( \mu \in L^Y \), \( \mu \) be \( r \)-fo and \( x_t \in F^l(\mu) \) then, there exists \( r \)-fo set \( \lambda \in L^X \) and \( x_t \in \lambda \) such that \( \lambda \leq F^l(\mu) \) and hence \( x_t \in SI_r(F^l(\mu), r) \).
Therefore, we obtain \( F^l(\mu) \leq SI_r(F^l(\mu), r) \). Thus, \( F^l(\mu) \) is \( r \)-fo.

(2) ⇒ (3) Let \( \mu \in L^Y \) and \( \mu \) be \( r \)-fo hence by (1), \((F^l(\mu))^c = (F^u(\mu))^c\) is \( r \)-fo. Then, \( F^u(\mu) \) is \( r \)-fo.

(3) ⇒ (2) It is similar to that of (2) ⇒ (3).

(2) ⇒ (1) Let \( x_t \in \text{dom}(F) \), \( \mu \in L^Y \), \( \mu \) be \( r \)-fo with \( x_t \in F^l(\mu) \) we have by (2), \( F^l(\mu) = \lambda \) (say) is \( r \)-fo then, there exists \( r \)-fo set \( \lambda \in L^X \) and \( x_t \in \lambda \) such that \( \lambda \leq F^l(\mu) \). Thus, \( F \) is \( FLC \)-isolates.

We state the following results without proof in view of above theorems.

**Theorem 3.7.** Let \( F : X \rightarrow Y \) be a \( FM \) and normalized between two \( L \)-fts \( s (X, \tau), (Y, \eta) \) and \( \mu \in L^Y \), then the following are equivalent:

1. \( F \) is \( FUCS \)-continuous.
2. \( F^l(\mu) \) is \( r \)-fo, if \( \eta(\mu^c) \geq r \).
3. \( F^l(\mu) \) is \( r \)-fo, if \( \eta(\mu) \geq r \).

**Theorem 3.8.** Let \( F : X \rightarrow Y \) be a \( FM \) and normalized between two \( L \)-fts \( s (X, \tau), (Y, \eta) \) and \( \mu \in L^Y \), then the following are equivalent:

1. \( F \) is \( FUC \)-isolates.
2. \( F^u(\mu) \) is \( r \)-fo, for any \( \mu \) is \( r \)-fo.
3. \( F^l(\mu) \) is \( r \)-fo, for any \( \mu \) is \( r \)-fo.

**Example 3.9.** Let \( X = \{x_1, x_2\} \), \( Y = \{y_1, y_2, y_3\} \) and \( F : X \rightarrow Y \) be a \( FM \) defined by \( G_F(x_1, y_1) = 0.1, G_F(x_1, y_2) = \top, G_F(x_1, y_3) = \bot, G_F(x_2, y_1) = 0.5, G_F(x_2, y_2) = \bot \) and \( G_F(x_2, y_3) = \top \). We assume that \( \top = 1 \) and \( \bot = 0 \). Define \( L \)-fuzzy topologies \( \tau : L^X \rightarrow L \) and \( \eta : L^Y \rightarrow L \) as follows:

\[
\tau(\lambda) = \begin{cases} 
\top, & \text{if } \lambda \in \{\bot, \top\}, \\
\frac{1}{2}, & \text{if } \lambda \in \{0.5, 0.6\}, \\
\bot, & \text{otherwise}.
\end{cases}
\]
\[ \eta(\mu) = \begin{cases} \top, & \text{if } \mu \in \{\frac{1}{2}, 1\}, \\ \frac{1}{2}, & \text{if } \mu = 0.5, \\ \frac{1}{2}, & \text{if } \mu = 0.4, \\ \bot, & \text{otherwise}. \end{cases} \]

(1) \( F \) is FUCS-continuous (resp. FUC-irresolute) but not FUC-irresolute because \( 0.45 \) is \( \frac{1}{2} \)-fts in \((Y, \eta)\) and \( F^\lambda(0.45) = 0.45 \) is not \( \frac{1}{2} \)-fts.

(2) \( F \) is FLCS-continuous (resp. FLC-irresolute) but not FLC-irresolute because \( 0.45 \) is \( \frac{1}{2} \)-fts in \((Y, \eta)\) and \( F^\mu(0.45) = 0.45 \) is not \( \frac{1}{2} \)-fts.

**Example 3.10.** Let \( X = \{x_1, x_2\} \), \( Y = \{y_1, y_2, y_3\} \) and \( F : X \rightarrow Y \) be a FM defined by \( G_F(x_1, y_1) = 0.2 \), \( G_F(x_1, y_2) = \top \), \( G_F(x_1, y_3) = 0.3 \), \( G_F(x_2, y_1) = 0.5 \), \( G_F(x_2, y_2) = 0.3 \) and \( G_F(x_2, y_3) = \top \). We assume that \( \top = 1 \) and \( \bot = 0 \). Define \( L \)-fuzzy topologies \( \tau : L^X \rightarrow L \) and \( \eta : L^Y \rightarrow L \) as follows:

\[
\tau(\lambda) = \begin{cases} \top, & \text{if } \lambda \in \{\frac{1}{2}, 1\}, \\ \frac{1}{2}, & \text{if } \lambda = 0.3, \\ \bot, & \text{otherwise}, \end{cases}
\]

\[
\eta(\mu) = \begin{cases} \top, & \text{if } \mu \in \{\frac{1}{2}, 1\}, \\ \frac{1}{2}, & \text{if } \mu = 0.4, \\ \bot, & \text{otherwise}. \end{cases}
\]

We can obtain the followings:

\[
SC_{\tau}(\lambda, r) = \begin{cases} \bot, & \text{if } \lambda = \bot, \ r \in L_\emptyset, \\ \frac{1}{2}, & \text{if } 0.3 \leq \lambda \leq 0.7, \ \bot < r \leq \frac{1}{2}, \\ \top, & \text{otherwise}, \end{cases}
\]

\[
SC_{\eta}(\lambda, r) = \begin{cases} \bot, & \text{if } \lambda = \bot, \ r \in L_\emptyset, \\ \frac{1}{2}, & \text{if } 0.4 \leq \lambda \leq 0.6, \ \bot < r \leq \frac{1}{2}, \\ \top, & \text{otherwise}. \end{cases}
\]

(1) \( F \) is FUCS-continuous (resp. FUC-irresolute) but not FUC-continuous because \( \eta(0.4) = \frac{1}{2} \) in \((Y, \eta)\), \( F^\lambda(0.4) = 0.4 \) and \( \tau([F^\lambda(0.4)]^\epsilon) \nsubseteq \frac{1}{2} \).

(2) \( F \) is FLCS-continuous (resp. FLC-irresolute) but not FLC-continuous because \( \eta(0.4) = \frac{1}{2} \) in \((Y, \eta)\), \( F^\mu(0.4) = 0.4 \) and \( \tau([F^\mu(0.4)]^\epsilon) \nsubseteq \frac{1}{2} \).

**Theorem 3.11.** Let \( F : X \rightarrow Y \) be a FM between two \( L \)-fts \((X, \tau), (Y, \eta)\) and \( \mu \in L^Y \). Suppose that one of the following properties hold:

(1) \( SC_{\tau}(F^\lambda(\mu), r) \leq F^\lambda(I_\eta(\mu(\mu), r)) \).

(2) \( F^\lambda(C_\eta(\mu), r) \leq SI_{\tau}(F^\lambda(\mu), r) \).

Then \( F \) is FLCS-continuous.

**Proof.** (1) \( \Rightarrow \) (2) Let \( \mu \in L^Y \) hence by (1), we obtain \( [SI_{\tau}(F^\lambda(\mu), r)]^\epsilon = SC_{\tau}([F^\lambda(\mu)]^\epsilon, r) = SC_{\tau}(F^\lambda(\mu^\epsilon), r) \leq F^\lambda(I_\eta(\mu^\epsilon, r)) = [F^\lambda(C_\eta(\mu, r))]^\epsilon \). Then, we obtain \( F^\lambda(C_\eta(\mu, r)) \leq SI_{\tau}(F^\lambda(\mu), r) \).

Suppose that (2) holds. Let \( \mu \in L^Y \) and \( \eta(\mu^\epsilon) \geq r \) then by (2), we have \( F^\lambda(\mu) \leq SI_{\tau}(F^\lambda(\mu), r) \). Thus \( F^\lambda(\mu) \) is \( f.r.s.o. \). Then from Theorem 3.5(2), \( F \) is FLCS-continuous.
Theorem 3.12. Let \( F : X \rightarrow Y \) be a \( FM \) between two \( L \)-fts’s \((X, \tau), (Y, \eta)\) and \( \mu \in L^Y \). Suppose that one of the following properties hold:
1. \( SC^r(I^c_\eta(\mu), r) \leq F^c(I^c_\eta(\mu), r) \).
2. \( F^c(SC^r(\mu, r)) \leq SI_r(F^c(\mu), r) \).

Then \( F \) is \( FLC \)-irresolute.

Proof. \((1) \Rightarrow (2)\) Let \( \mu \in L^Y \) hence by (1), we obtain \([SI_r(F^c(\mu), r)]^c = SC^r([F^c(\mu)]^c, r) = SC^r(F^c(\mu^c), r) \leq F^c(SI^c_\eta(\mu^c, r)) = [F^c(SC^r_\eta(\mu, r))]^c \). Then, we obtain

\[
F^c(SC^r_\eta(\mu, r)) \leq SI_r(F^c(\mu), r).
\]

Suppose that (2) holds. Let \( \mu \in L^Y \) and \( \mu \) be \( r-fsc \) then by (2), we have \( F^c(\mu) \leq SI_r(F^c(\mu), r) \). Thus \( F^c(\mu) \) is \( r-fso \). Then from Theorem 3.6(2), \( F \) is \( FLC \)-irresolute.

We state the following results without proof in view of above theorems.

Theorem 3.13. Let \( F : X \rightarrow Y \) be a \( FM \) and normalized between two \( L \)-fts’s \((X, \tau), (Y, \eta)\) and \( \mu \in L^Y \). Suppose that one of the following properties hold:
1. \( SC^r(F^c(\mu), r) \leq F^c(I^c_\eta(\mu), r) \).
2. \( F^c(SC^r(\mu, r)) \leq SI_r(F^c(\mu), r) \).

Then \( F \) is \( FUCS \)-continuous.

Theorem 3.14. Let \( F : X \rightarrow Y \) be a \( FM \) and normalized between two \( L \)-fts’s \((X, \tau), (Y, \eta)\) and \( \mu \in L^Y \). Suppose that one of the following properties hold:
1. \( SC^r(F^c(\mu), r) \leq F^c(I^c_\eta(\mu), r) \).
2. \( F^c(SC^r(\mu, r)) \leq SI_r(F^c(\mu), r) \).

Then \( F \) is \( FUC \)-irresolute.

Theorem 3.15. Let \( F : X \rightarrow Y \) and \( H : Y \rightarrow Z \) be two \( FM \)-s and let \((X, \tau), (Y, \eta)\) and \((Z, \delta)\) be three \( L \)-fts-s. If \( H \) is normalized, \( H \) is \( FUS \)-continuous and \( F \) is \( FLCS \)-continuous, then \( H \circ F \) is \( FLCS \)-continuous.

Proof. Let \( F \) be \( FLCS \)-continuous, \( H \) be \( FUS \)-continuous and \( \gamma \in L^Z, \delta(\gamma^c) \geq r \). Then from Theorem 1.12(4) and Theorem 3.5(2), we have \( (H \circ F)^c(\gamma) = F^c(H^c(\gamma)) \) and \( F^c(H^c(\gamma)) \) is \( r-fso \) with \( \eta((H^c(\gamma))^c) \geq r \). Thus, \( H \circ F \) is \( FLCS \)-continuous.

We state the following result without proof in view of above theorem.

Theorem 3.16. Let \( F : X \rightarrow Y \) and \( H : Y \rightarrow Z \) be two \( FM \)-s and let \((X, \tau), (Y, \eta)\) and \((Z, \delta)\) be three \( L \)-fts-s. If \( F \) is normalized, \( F \) is \( FUCS \)-continuous and \( H \) is \( FLS \)-continuous, then \( H \circ F \) is \( FUCS \)-continuous.

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