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Marichev-Saigo-Maeda Differential Operator and Generalized Incomplete Hypergeometric Functions

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Abstract. This research has been made to apply Marichev-Saigo-Maeda differential operators on generalized incomplete hypergeometric functions ${}_p\gamma_q[z]$ and ${}_p\Gamma_q[z]$. Marichev-Saigo-Maeda fractional operators involving Appell function F_3 as a kernel are more general in nature. Well known fractional operators Riemann-Liouville, Weyl, Erdelyi-Kober and Saigo fractional operators are special cases of these more generalized operators.

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1. Introduction

Incomplete gamma functions $\gamma(s;\chi)$ and $\Gamma[s;\chi]$ are defined as

$$\gamma(s;\chi) = \int_0^{\chi} u^{s-1}e^{-u}du \quad (\Re(s) > 0, \chi \geq 0) \tag{1. 1}$$

and

$$\Gamma[s;\chi] = \int_{\chi}^{\infty} u^{s-1}e^{-u}du \quad (\Re(s) > 0, \chi \geq 0) \tag{1. 2}$$

Thus the decomposition formula for gamma function is

$$\gamma(s;\chi) + \Gamma[s;\chi] = \Gamma(s) \quad (\Re(s) > 0, \chi \geq 0) \tag{1. 3}$$

where $\Gamma(s)$ is the classical gamma function known as Euler's integral in the interval $[0,\infty)$ defined as

$$\Gamma(s) = \int_0^{\infty} u^{s-1}e^{-u}du \quad (\Re(s) > 0) \tag{1. 4}$$

See, Rainville [4], for details on gamma function. Srivastava et al. [8] has defined incomplete Pochhammer symbol in terms of incomplete gamma function.

$$(\epsilon : \chi)_n = \frac{\gamma(\epsilon + n\chi)}{\gamma(\chi)} \tag{1. 5}$$

and

$$[\epsilon:\chi]_n = \frac{\Gamma(\epsilon+n\chi)}{\Gamma(\chi)} \tag{1. 6}$$

The decomposition formula for classical Pochhammer symbol $(\epsilon)_n$ is

$$(\epsilon:\chi)_n + [\epsilon:\chi]_n = (\epsilon)_n \tag{1. 7}$$

where

$$(\epsilon)_n = \prod_{k=1}^{n}(\epsilon+k-1) \tag{1. 8}$$

$$= \epsilon(\epsilon+1)(\epsilon+2)...(\epsilon+n-1) \quad n \geq 1 \tag{1. 9}$$

$$(\epsilon)_0 = 1 \quad \epsilon \neq 0 \tag{1. 10}$$

This is the generalization of factorial function, since $n! = (1)_n$. See Rainville [4], for details on Pochhammer symbol. By the idea of incomplete Pochhammer symbol Srivastava et al. [9] introduced incomplete hypergeometric function as

$${}_p\gamma_q\left[\begin{array}{c}(a_1;\chi),a_2,...,a_p;\\ b_1,...,b_q;\end{array}\ \text{z}\right] = \sum_{n=0}^{\infty}\frac{(a_1;\chi)_n,(a_2)_n,...,(a_p)_n}{(b_1)_n,...,(b_q)_n}.\frac{z^n}{n!} \tag{1. 11}$$

and

$${}_p\Gamma_q\left[\begin{array}{c}(a_1;\chi),a_2,...,a_p;\\ b_1,...,b_q;\end{array}\ \text{z}\right] = \sum_{n=0}^{\infty}\frac{[a_1;\chi]_n,(a_2)_n,...,(a_p)_n}{(b_1)_n,...,(b_q)_n}.\frac{z^n}{n!} \tag{1. 12}$$

where incomplete parameters $(a_1;\chi)_n$ and $[a_1;\chi]_n$ are defined in the interval $[0,\chi]$ and $[\chi,\infty)$ respectively. Thus the decomposition formula for generalized hypergeometric function ${}_pF_q, (p,q \in N_0)$, is

$${}_p\gamma_q\left[\begin{array}{c}(a_1;\chi),a_2,...,a_p;\\ b_1,...,b_q;\end{array}\ \text{z}\right] + {}_p\Gamma_q\left[\begin{array}{c}(a_1;\chi),a_2,...,a_p;\\ b_1,...,b_q;\end{array}\ \text{z}\right]$$

$$= {}_pF_q\left[\begin{array}{c}a_1,...,a_p;\\ b_1,...,b_q;\end{array}\ \text{z}\right] \tag{1. 13}$$

Incomplete hypergeometric functions are from the class of Fox Wright functions [2]. See, for details on hypergeometric function and incomplete hypergeometric function,Rainville [4], Choi and Agarwal [1], Srivastava and Agarwal [9], Srivastava et al. [8]. Srivastava and Agarwal [9] applied fractional integral operators containing Appell function F_3 in their kernel to incomplete hypergeometric functions${}_p\gamma_q$ and ${}_p\Gamma_q$. Srivastava et al. [10] applied integral operators on family of incomplete hypergeometric function containing Gauss hypergeometric function as a kernel. Choi and Agarwal [1] investigated the integral transforms like Beta transform, Laplace transform, Mellin transform, Whittaker transform, K-transform and Hankel transform of incomplete hypergeometric functions${}_p\gamma_q$and${}_p\Gamma_q$. Nadir et al. [3] investigated the integral transforms of Mittag-Leffler function. Also integral transforms with generalized Mittag-Leffler functions were studied in [7] in connection with fractional powers of Bessel operators.

Marichev-Saigo-Maeda fractional operators associated with Appell function F_3 as a kernel are more general in nature. Well-known fractional operators like Saigo operators [5], Riemann-Liouville, Weyl and Erdelyi-kober differential operators are the particular cases of these generalized operators. Appell function of first kind F_3 is basically two-variable hypergeometric function defined as

$$F_3(\rho,\rho^/,\tau,\tau^/;\omega;x;y)=\sum_{m,n=0}^{\infty}\frac{(\rho)_m(\rho^/)_n(\tau)_m(\tau^/)_n}{(\omega)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1. 14}$$

$$=\sum_{m=0}^{\infty}\frac{(\rho)_m(\tau)_m}{(\omega)_m}{}_2F_1\left[\begin{array}{c}\rho^/,\tau^/\\ \omega+m\end{array};\text{y}\right]\frac{x^m}{m!} \tag{1. 15}$$

See, for details on Appell function, Rainville [4].

Let $\rho,\rho^/,\tau,\tau^/,\omega\in\mathbf{C},x>0$, then the generalized fractional integro-differential operators involving Appell function F_3 are defined due to Saigo and Maeda [6] by the followings.

$$\left(I_{0+}^{\rho,\rho^/,\tau,\tau^/,\omega}f\right)(x)$$

$$=\frac{x^{-\rho}}{\Gamma(\omega)}\int_0^x t^{-\rho^/}(x-t)^{\omega-1}F_3\left(\rho,\rho^/,\tau,\tau^/;\omega;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)dt, \tag{1. 16}$$

$(\Re(\omega)>0)$;

$$=\frac{d^n}{dx^n}\left(I_{0+}^{\rho,\rho^/,\tau+\rho,\tau^/,\omega+\rho}f\right)(x), \tag{1. 17}$$

$(\Re(\omega)\leq 0; n=[-\Re(\omega)]+1)$;

$$\left(I_{0-}^{\rho,\rho^/,\tau,\tau^/,\omega}f\right)(x)$$

$$=\frac{x^{-\rho^/}}{\Gamma(\omega)}\int_x^\infty t^{-\rho}(t-x)^{\omega-1}F_3\left(\rho,\rho^/,\tau,\tau^/;\omega;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)dt \tag{1. 18}$$

$(\Re(\omega)>0)$;

$$=(-1)^n\frac{d^n}{dx^n}\left(I_{0-}^{\rho,\rho^/,\tau,\tau^/+\rho,\omega+\rho}f\right)(x), \tag{1. 19}$$

$(\Re(\omega)\leq 0; n=[-\Re(\omega)]+1)$;

$$\left(D_{0+}^{\rho,\rho^/,\tau,\tau^/,\omega}f\right)(x)=\left(I_{0+}^{-\rho^/,-\rho,-\tau^/,\tau,-\omega}f\right)(x) \tag{1. 20}$$

$$=\frac{d^n}{dx^n}\left(I_{0+}^{-\rho^/,-\rho,-\tau^/+n,-\tau,-\omega+n}f\right) \tag{1. 21}$$

$(\Re(\omega)>0; n=[\Re(\omega)]+1)$;

$$\left(D_{0-}^{\rho,\rho^/,\tau,\tau^/,\omega}f\right)(x)=\left(I_{0-}^{-\rho^/,-\rho,-\tau^/,\tau,-\omega}f\right)(x) \tag{1. 22}$$

$$=(-1)^n\frac{d^n}{dx^n}\left(I_{0-}^{-\rho^/,-\rho,-\tau^/,-\tau+n,-\omega+n}f\right)(x) \tag{1. 23}$$

$(\Re(\omega)>0; n=[\Re(\omega)]+1)$;
Our findings are based on the formulas of fractional integrals given in (1.16) and (1.18) with the evaluation of power series defined by Saigo and Maeda [6]. Saigo and Maeda [6]

established left hand-sided integral formulas of power function as

$$\left(I_{0+}^{\rho,\rho',\tau,\tau',\omega}x^{\sigma-1}\right)=\frac{\Gamma(\sigma)\Gamma(\sigma+\omega-\rho-\rho'-\tau)\Gamma(\sigma+\tau'-\rho')}{\Gamma(\sigma+\omega-\rho-\rho')\Gamma(\sigma+\omega-\rho'-\tau)\Gamma(\sigma+\tau')}x^{\sigma-\rho-\rho'+\omega-1} \tag{1. 24}$$

where $\Re(\omega)>0, \Re(\sigma)>\max[0,\Re(\rho+\rho'+\tau-\omega),\Re(\rho'-\tau')]$

Saigo and Maeda [6] established right hand-sided integral formulas of power function as

$$\left(I_{0-}^{\rho,\rho',\tau,\tau',\omega}x^{\sigma-1}\right)$$

$$=\frac{\Gamma(1+\rho+\rho'-\omega-\sigma)\Gamma(1+\rho+\tau'-\omega-\rho)\Gamma(1-\tau-\sigma)}{\Gamma(1-\sigma)\Gamma(1+\rho+\rho'-\tau'-\omega-\sigma)\Gamma(1+\rho-\tau-\sigma)}x^{\sigma-\rho-\rho'+\omega-1} \tag{1. 25}$$

where $\Re(\omega)>0, \Re(\sigma)<1+\min[\Re(-\tau),\Re(\rho+\rho'-\omega),\Re(\rho'+\tau'-\omega)]$

Saigo [5] defined fractional calculus operators $\left(I_{0+}^{\rho,\tau,\omega}f\right)(x)$, $\left(I_{0-}^{\rho,\tau,\omega}f\right)(x)$, $\left(D_{0+}^{\rho,\tau,\omega}f\right)(x)$ and $\left(D_{0-}^{\rho,\tau,\omega}f\right)(x)$ in terms of the Gauss hypergeometric function when $\rho,\tau,\omega\in\mathbf{C}$ and $x>0$.

$$\left(I_{0+}^{\rho,\tau,\omega}f\right)(x)=\frac{x^{-\rho-\tau}}{\Gamma(\rho)}\int_0^x(x-t)^{\rho-1}{}_2F_1\left(\rho+\tau,-\omega;\rho;1-\frac{t}{x}\right)f(t)dt \tag{1. 26}$$

$(\Re(\rho)>0)$;

$$\left(I_{0-}^{\rho,\tau,\omega}f\right)(x)=\frac{1}{\Gamma(\rho)}\int_0^\infty(t-x)^{\rho-1}t^{-\rho-\tau}{}_2F_1\left(\rho+\tau,-\omega;\rho;1-\frac{t}{x}\right)f(t)dt \tag{1. 27}$$

$(\Re(\rho)>0)$;

$$\left(D_{0+}^{\rho,\tau,\omega}f\right)(x)=\left(I_{0+}^{-\rho,-\tau,\rho+\omega}f\right)(x)=\frac{d^n}{dx^n}\left(I_{0+}^{-\rho+n,-\tau-n,\omega+\rho-n}f\right)(x) \tag{1. 28}$$

$(\Re(\rho)>0; n=[\Re(\rho)+1])$

$$\left(D_{0-}^{\rho,\tau,\omega}f\right)(x)=\left(I_{0-}^{-\rho,-\tau,\rho+\omega}f\right)(x)=(-1)^n\frac{d^n}{dx^n}\left(I_{0-}^{-\rho+n,-\tau-n,\omega+\rho}f\right)(x) \tag{1. 29}$$

$(\Re(\rho)>0; n=[\Re(\rho)+1])$

Marichev-Saigo-Maeda operators [6] reduce to Saigo fractional operators [5] by the following identities

$$\left(I_{0+}^{\rho,0,\tau,\tau',\omega}f\right)(x)=\left(I_{0+}^{\omega,\rho-\omega,-\tau}f\right)(x) \qquad (\omega\in C); \tag{1. 30}$$

$$\left(I_{0-}^{\rho,0,\tau,\tau',\omega}f\right)(x)=\left(I_{0-}^{\omega,\rho-\omega,-\tau}f\right)(x) \qquad (\omega\in C); \tag{1. 31}$$

$$\left(D_{0+}^{0,\rho',\tau,\tau',\omega}f\right)(x)=\left(D_{0+}^{\omega,\rho'-\omega,\tau'-\omega}f\right)(x) \quad (\Re(\omega)>0); \tag{1. 32}$$

$$\left(D_{0-}^{0,\rho',\tau,\tau',\omega}f\right)(x)=\left(D_{0-}^{\omega,\rho'-\omega,\tau'-\omega}f\right)(x) \quad (\Re(\omega)>0); \tag{1. 33}$$

2. LEFT-SIDED GENERALIZED FRACTIONAL DIFFERENTIATIONS OF INCOMPLETE HYPERGEOMETRIC FUNCTIONS ${}_p\gamma_q$ AND ${}_p\Gamma_q$

Theorem 1: Let $\rho, \rho^{/}, \tau, \omega \in C$ such that $(\Re(\omega) > 0)$; If the condition $\Re(\omega) > \max[0, \Re(\omega - \rho - \rho^{/} - \tau^{/}), \Re(\tau - \rho)]$ is satisfied then the left sided Marichev-Saigo-Maeda operator of differentiation of incomplete hypergeometric function is given by

$$\left(D_{0+}^{\rho,\rho^{/},\tau,\tau^{/},\omega}\left(t^{\sigma-1}{}_p\gamma_q(\mu t)\right)\right)(x)$$

$$= x^{\sigma+\rho+\rho^{/}-\omega-1}\frac{\Gamma(\sigma)\Gamma(\sigma-\omega+\rho+\rho^{/}+\tau^{/})\Gamma(\sigma-\tau+\rho)}{\Gamma(\sigma-\omega+\rho+\tau^{/})\Gamma(\sigma-\omega+\rho+\rho^{/})\Gamma(\sigma-\rho)}$$

$$\times {}_{p+3}\gamma_{q+3}\left[\begin{array}{c}(a_1;\chi)_n, (a_2)_n, ..., (a_p)_n, (\sigma)_n, (\sigma-\omega+\rho+\rho^{/}+\tau^{/})_n, (\sigma-\tau+\rho)_n \\ (b_1)_n, ..., (b_q)_n, (\sigma-\omega+\rho+\rho^{/})_n, (\sigma-\omega+\rho+\tau^{/})_n, (\sigma-\tau)_n\end{array};\mu \mathrm{x}\right]$$

and

$$\left(D_{0+}^{\rho,\rho^{/},\tau,\tau^{/},\omega}\left(t^{\sigma-1}{}_p\Gamma_q(\mu t)\right)\right)(x)$$

$$= x^{\sigma+\rho+\rho^{/}-\omega-1}\frac{\Gamma(\sigma)\Gamma(\sigma-\omega+\rho+\rho^{/}+\tau^{/})\Gamma(\sigma-\tau+\rho)}{\Gamma(\sigma-\omega+\rho+\tau^{/})\Gamma(\sigma-\omega+\rho+\rho^{/})\Gamma(\sigma-\rho)}$$

$$\times {}_{p+3}\Gamma_{q+3}\left[\begin{array}{c}(a_1;\chi)_n, (a_2)_n, ..., (a_p)_n, (\sigma)_n, (\sigma-\omega+\rho+\rho^{/}+\tau^{/})_n, (\sigma-\tau+\rho)_n \\ (b_1)_n, ..., (b_q)_n, (\sigma-\omega+\rho+\rho^{/})_n, (\sigma-\omega+\rho+\tau^{/})_n, (\sigma-\tau)_n\end{array};\mu \mathrm{x}\right]$$

Proof. By using the incomplete version of hypergeometric function ${}_p\gamma_q$ (1.11) and applying the differential operator (1.21) and changing the order of integration and summation due to uniform convergence of the series, we have

$$\left(D_{0+}^{\rho,\rho^{/},\tau,\tau^{/},\omega}\left(t^{\sigma-1}{}_p\gamma_q(\mu t)\right)\right)(x) = \left(\frac{d}{dx}\right)^m\sum_{m=0}^{\infty}\frac{(a_1;\chi)_n, (a_2)_n, ..., (a_p)_n}{(b_1)_n, ..., (b_q)_n}\frac{(\mu)^n}{n!}$$

$$\times\left(I_{0+}^{-\rho,-\rho^{/},-\tau^{/}+m,-\tau,-\omega+m}\left(t^{\sigma+n-1}\right)\right)(x)$$

Now using definition (1.24), and replacing σ by $\sigma + n$, we get

$$\left(D_{0+}^{\rho,\rho^{/},\tau,\tau^{/},\omega}\left(t^{\sigma-1}{}_p\gamma_q(\mu t)\right)\right)(x) = \sum_{n=0}^{\infty}\frac{(a_1;\chi)_n, (a_2)_n, ..., (a_p)_n}{(b_1)_n, ..., (b_q)_n}\frac{\mu^n}{n!}$$

$$\times\frac{\Gamma(\sigma+n)\Gamma(\sigma-\omega+\rho+\rho^{/}+\tau^{/}+n)\Gamma(\sigma-\tau+\rho+n)}{\Gamma(\sigma-\omega+m+\rho+\rho^{/}+n)\Gamma(\sigma-\omega+\rho+\tau^{/}+n)\Gamma(\sigma-\rho+n)}$$

$$\times\left(\frac{d}{dx}\right)^m\left(x^{\sigma+\rho+\rho^{/}-\omega+m+n-1}\right)$$

Since $D^m x^n = \frac{\Gamma(n+1)x^{n-m}}{\Gamma(n-m+1)}, n \geq m$
we have

$$\left(D_{0+}^{\rho,\rho^{/},\tau,\tau^{/},\omega}\left(t^{\sigma-1}{}_p\gamma_q(\mu t)\right)\right)(x) = \sum_{n=0}^{\infty}\frac{(a_1;\chi)_n, (a_2)_n, ..., (a_p)_n}{(b_1)_n, ..., (b_q)_n}\frac{\mu^n}{n!}$$

$$\times\frac{\Gamma(\sigma+n)\Gamma(\sigma-\omega+\rho+\rho^{/}+\tau^{/}+n)\Gamma(\sigma-\tau+\rho+n)}{\Gamma(\sigma-\omega+m+\rho+\rho^{/}+n)\Gamma(\sigma-\omega+\rho+\tau^{/}+n)\Gamma(\sigma-\rho+n)}$$

$$\times \frac{\Gamma(\sigma+\rho+\rho^{/}-\omega+m+n)}{\Gamma(\sigma+\rho+\rho^{/}+n-\omega)}\left(x^{\sigma+\rho+\rho^{/}-\omega+n-1}\right)$$

Hence, we get the required result. For the second assertion of Theorem 1 again using the incomplete version of hypergeometric function (1.12) and using the definition of (1.21) and (1.24), we get the result. □

Corollary 2.1. *Let* $\rho,\tau,\omega \in C$*be such that* $(\Re(\omega)>0)$ *;If the condition* $\Re(\rho)>0, \Re(\sigma)> -\min[\Re(\tau),\Re(\omega)]$*is satisfied then by the relation (1.32) and (1.24) the left sided Saigo differential operator of incomplete hypergeometric function is given by*

$$\left(D_{0+}^{\rho,\tau,\omega}\left(t^{\sigma-1}{}_{p}\gamma_{q}(\mu t)\right)\right)(x) = x^{\sigma+\tau-1}\frac{\Gamma(\sigma)\Gamma(\sigma+\rho+\tau+\omega)}{\Gamma(\sigma+\tau)\Gamma(\sigma+\omega)}$$

$$\times {}_{p+2}\gamma_{q+2}\left[\begin{array}{c}(a_1;\chi)_n,(a_2)_n,...,(a_p)_n,(\sigma)_n,(\sigma+\rho+\tau+\omega)_n\\(b_1)_n,...,(b_q)_n,(\sigma+\tau)_n,(\sigma+\omega)_n\end{array};\mu x\right]$$

and

$$\left(D_{0+}^{\rho,\tau,\omega}\left(t^{\sigma-1}{}_{p}\Gamma_{q}(\mu t)\right)\right)(x) = x^{\sigma+\tau-1}\frac{\Gamma(\sigma)\Gamma(\sigma+\rho+\tau+\omega)}{\Gamma(\sigma+\tau)\Gamma(\sigma+\omega)}$$

$$\times {}_{p+2}\Gamma_{q+2}\left[\begin{array}{c}(a_1;\chi)_n,(a_2)_n,...,(a_p)_n,(\sigma)_n,(\sigma+\rho+\tau+\omega)_n\\(b_1)_n,...,(b_q)_n,(\sigma+\tau)_n,(\sigma+\omega)_n\end{array};\mu x\right]$$

3. RIGHT-SIDED GENERALIZED FRACTIONAL DIFFERENTIATION OF INCOMPLETE HYPERGEOMETRIC FUNCTIONS ${}_{p}\gamma_{q}$ AND ${}_{p}\Gamma_{q}$

Theorem 2: Let $\rho,\rho^{/},\tau,\omega \in C$ be a complex number such that $(\Re(\omega)>0)$; If the condition $\Re(\sigma)<1+\min[0,\Re(\tau^{/}),\Re(\omega-\rho-\rho^{/}-\tau^{/}),\Re(\omega-\rho^{/}-\tau)]$ is satisfied then the right-sided Marichev-Saigo-Maeda operator of differentiation of incomplete hypergeometric function is given by

$$\left(D_{0-}^{\rho,\rho^{/},\tau,\tau^{/},\omega}\left(t^{\sigma-1}{}_{p}\gamma_{q}(\tfrac{\mu}{t})\right)\right)(x) = x^{\sigma-\rho-\rho^{/}+\omega-1}$$

$$\times\frac{\Gamma(1-\sigma+\omega-\rho-\rho^{/})\Gamma(1-\sigma+\omega-\rho^{/}-\tau)\Gamma(1-\sigma-\tau^{/})}{\Gamma(1-\sigma)\Gamma(1-\sigma-\rho^{/}+\tau^{/})\Gamma(1-\sigma-\rho-\rho^{/}-\tau+\omega)}$$

$$\times {}_{p+3}\gamma_{q+3}\left[\begin{array}{c}[a_1;\chi]_n,(a_2)_n,...,(a_p)_n,\Delta\\(b_1)_n,...,(b_q)_n,\Delta^{/}\end{array};\tfrac{\mu}{x}\right]$$

and

$$\left(D_{0-}^{\rho,\rho^{/},\tau,\tau^{/},\omega}\left(t^{\sigma-1}{}_{p}\Gamma_{q}(\tfrac{\mu}{t})\right)\right)(x) = x^{\sigma-\rho-\rho^{/}+\omega-1}$$

$$\times\frac{\Gamma(1-\sigma+\omega-\rho-\rho^{/})\Gamma(1-\sigma+\omega-\rho^{/}-\tau)\Gamma(1-\sigma-\tau^{/})}{\Gamma(1-\sigma)\Gamma(1-\sigma-\rho^{/}+\tau^{/})\Gamma(1-\sigma-\rho-\rho^{/}-\tau+\omega)}$$

$$\times {}_{p+3}\Gamma_{q+3}\left[\begin{array}{c}[a_1;\chi]_n,(a_2)_n,...,(a_p)_n,\Delta\\(b_1)_n,...,(b_q)_n,\Delta^{/}\end{array};\tfrac{\mu}{x}\right]$$

where $\Delta=(1-\sigma+\omega-\rho-\rho^{/})_n,(1-\sigma+\omega-\rho^{/}-\tau)_n,(1-\sigma+\tau^{/})_n$
and $\Delta^{/}=(1-\sigma)_n,(1-\sigma-\rho^{/}+\tau^{/})_n,(1-\sigma-\rho-\rho^{/}-\tau+\omega)_n$

Proof. Parallel to the theorem 1, using incomplete version of hypergeometric functions (1.11) and (1.12) and applying Marchieve-Saigo-Maeda operator of right-hand sided (1.23) and (1.25) and changing the order of integration and summation due to the uniform convergence of the series, we get the required result. □

Corollary 3.1. *Let $\rho, \tau, \omega \in C$be such that $(\Re(\omega) > 0)$;If the condition $\Re(\sigma) > -\min[0, \Re(-\tau - n), \Re(\rho + \omega)], n = [\Re(\rho)] + 1$is satisfied then by the relation (1.33) the right sided Saigo differential operator of incomplete hypergeometric function is given by*

$$\left(D_{0-}^{\rho,\tau,\omega}\left(t^{\sigma-1}{}_p\gamma_q(\frac{\mu}{t})\right)\right)(x) = x^{\sigma+\tau-1}\frac{\Gamma(1-\sigma-\tau)\Gamma(1-\sigma+\omega+\tau)}{\Gamma(1-\sigma)\Gamma(1-\sigma+\omega-\tau)}$$
$$\times_{p+2}\gamma_{q+2}\left[\begin{array}{c}[a_1;\chi]_n, (a_2)_n, ..., (a_p)_n, (1-\sigma-\tau)_n, (1-\sigma+\tau+\omega)_n, \\ (b_1)_n, ..., (b_q)_n, (1-\sigma)_n, (1-\sigma+\omega-\tau)_n\end{array} ;\frac{\mu}{x}\right]$$

and

$$\left(D_{0-}^{\rho,\tau,\omega}\left(t^{\sigma-1}{}_p\Gamma_q(\frac{\mu}{t})\right)\right)(x) = x^{\sigma+\tau-1}\frac{\Gamma(1-\sigma-\tau)\Gamma(1-\sigma+\omega+\tau)}{\Gamma(1-\sigma)\Gamma(1-\sigma+\omega-\tau)}$$
$$\times_{p+2}\Gamma_{q+2}\left[\begin{array}{c}[a_1;\chi]_n, (a_2)_n, ..., (a_p)_n, (1-\sigma-\tau)_n, (1-\sigma+\tau+\omega)_n, \\ (b_1)_n, ..., (b_q)_n, (1-\sigma)_n, (1-\sigma+\omega-\tau)_n\end{array} ;\frac{\mu}{x}\right]$$

4. Remarks

Marichev-Saigo-Maeda differential operator coincides with the classical Riemann Liouville fractional differential operator and Erdelyi-Kober fractional differential operator. Upon setting $\tau = -\rho$, our results deduce to Riemann Liouville fractional differential operator (see, Srivastava et al. [9]) and upon setting $\tau = 0$ our results deduce to Erdelyi-Kober fractional differential operator (see, Srivastava et al. [9]).

5. Conclusions

We conclude that our results in this paper are more general in nature and capable to yield fractional differential form for incomplete hypergeometric functions defined in (1.11) and (1.21). Also our results unifies with several fractional differential operators such as Saigo operator, Riemann Liouville fractional differential operator, the Weyl fractional differential operator and the Erdelyi-Kober fractional differential operators.

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