SEM T Labelings and Deficiencies of Forests with Two Components (I)

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Abstract. A set of nodes called vertices V accompanied with the lines that bridge these nodes called edges E compose an explicit figure termed as a graph G(V, E), |V(G)| = ν and |E(G)| = ε specify its order and size respectively. A (ν, ε)-graph G determines an edge-magic total (EMT) labeling when Γ : V(G) ∪ E(G) → {1, ± ν + ε} is bijective so as the weights at every edge are the same constant (say) c i.e., for x, y ∈ V(G); Γ(x) + Γ(xy) + Γ(y) = c, independent of the choice of any xy ∈ E(G), such a number is interpreted as a magic constant. If all vertices gain the smallest of the labels then an EMT labeling is called a super edge-magic total (SEMT) labeling. If a graph G allows at least one SEMT labeling then the smallest of the magic constants for all possible distinct SEMT labelings of G describes super edge-magic total (SEMT) strength, sm(G), of G. For any graph G, SEMT deficiency is the least number of isolated vertices which when uniting with G yields a SEMT graph. In this paper, we will find SEMT labeling and deficiency of forests consisting of two components, where one of the components for each forest is generalized comb Cbτ(ℓ, ℓ, ..., ℓ)τ-times and other component is a star, bistar, comb or path respectively, moreover, we will investigate SEMT strength of aforesaid generalized comb.

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1. Preliminaries

Labeling is a technique that allots labels to the components of a graph. Total labeling gives us both components (vertices and edges) labelled. A \((\nu, \varepsilon)\)-graph \(G\) determines an edge-magic total (EMT) labeling when \(\Gamma : V(G) \cup E(G) \rightarrow \{1, \nu + \varepsilon\}\) is bijective so as the weights at every edge are the same constant (say) \(c\), such a number \(c\) is interpreted as a magic constant. If all vertices gain the smallest of the labels then an EMT labeling is called a super edge-magic total (SEMT) labeling. Kotzig and Rosa [17] and Enomoto et al. [7] first introduced the notions of EMT and SEMT graphs respectively and presented the conjectures: every tree is EMT [17], and every tree is SEMT [7].

If a graph \(G\) allows at least one SEMT labeling then the smallest of the magic constants for all possible distinct SEMT labelings of \(G\) describes super edge-magic total (SEMT) strength, \(sm(G)\), of \(G\). Avadayappan et al. first introduced the notion of SEMT strength [4] and found exact values of SEMT strength for some graphs.

In [17], the notion of EMT deficiency was proposed by authors and Figueroa-Centeno et al. [8] continued it to SEMT graphs. For any graph \(G\), the SEMT deficiency, signified as \(\mu_s(G)\), is the least number \(n\) of isolated vertices that we have to take in union with \(G\) so that the resulting graph \(G \cup nK_1\) is SEMT, the case \(+\infty\) will arise if no isolated vertex fulfils this criteria. More specifically,

\[
\mu_s(G) = \begin{cases} 
\min M(G) & \text{if } M(G) \neq \emptyset \\
+\infty & \text{if } M(G) = \emptyset 
\end{cases}
\]

where \(M(G) = \{n \geq 0 : G \cup nK_1 \text{ is a SEMT graph}\}\).

Exact values for SEMT deficiencies of several classes of graphs are provided in [9, 8]. The authors also proposed a conjecture which tells us about the confined deficiencies of the forests. In [10], an assumption was made as a special case of a previous one that says, the deficiency of each two-tree forest is not more than 1. Baig et al. [6] determined SEMT deficiencies of various forests made up of banana trees, stars etc. In [13, 21], S. Javed et al. and Ngurah et al. gave some upper bounds for SEMT deficiency of forests composed of stars, fans, combs, double fans, wheels and generalized combs. The results in [1, 2, 3, 5, 14, 15, 16, 18, 19, 20] might found useful in the aspect of examined labeling here. A general reference to graph theoretic terminologies can be found in [22]. For more review, see the recent survey of graph labelings by Gallian [12].

In this paper, we formulated the results on SEMT labeling and deficiency of forests consisting of two components, where one component in each forest is a generalized comb and the other component is a star, bistar, comb or path respectively. Moreover, SEMT strength of a generalized comb \(Cb_r(\ell, \ell, \ldots, \ell)\) has also been discussed here. The values of parameters of the star, bistar, comb and path are totally dependant on the parameters involved in the generalized comb.
A star on \( n \) vertices is isomorphic to \( K_{1,n-1} \). When we join two stars \( K_{1,g}, K_{1,h} \) through a bridge, where \( g, h \geq 1 \) and \( g + h = n - 2 \), the resulting tree is termed a bistar \( BS(g, h) \). The graph \( P_n \) denotes the path of order \( n \) and size \( n-1 \), with vertices labelled from \( x_1 \) to \( x_n \) along \( P_n \). The comb \( Cb_n \) is an acyclic graph consisting of \( P_n \) together with \( n-1 \) new pendant vertices \( y_1, y_2, ..., y_{n-1} \) adjacent to \( x_2, x_3, ..., x_n \) respectively, thus the new edges obtained are \( \{x_{i+1}y_i : i \in \{1, n-1\}\} \). A generalized comb is basically a detailing (or subdivision) of a comb’s pendant vertices hanging from the main horizontal path to form \( \tau \) hanging paths of order \( \ell_i \), this is denoted by \( Cb_{\tau}(\ell_1, \ell_2, ..., \ell_\tau) \). When \( \ell_1 = \ell_2 = ... = \ell_\tau = \ell \), then a generalized comb transforms into a balanced generalized comb \( Cb_{\tau}(\ell, \ell, ..., \ell) \), which can precisely be denoted as \( Cb_{\tau}(\ell, \ell, ..., \ell) \), as elaborated in fig. 1.

The following Lemma is an elementary tool for proving graphs to be SEMT. It will be used as a base in each result presented in this work.

**Lemma 2.1.** [11] A \((\nu, \varepsilon)\)-graph \( G \) is SEMT if and only if \( \exists \) a bijective map \( \Gamma : V(G) \rightarrow \{1, \nu\} \) s.t. the set of edge-sums

\[
S = \{\Gamma(l) + \Gamma(m) : lm \in E(G)\}
\]

constructs \( \varepsilon \) consecutive \( Z^+ \). In that case, \( G \) can extend to a SEMT labeling of \( G \) with magic constant \( c = \nu + \varepsilon + \min(S) \) and

\[
S = \{c - (\nu + \varepsilon), c - (\nu + \varepsilon) + 1, ..., c - (\nu + 1)\}.
\]

To understand the lemma 2.1, we consider an example, see fig. 2, where it is shown that if a graph constitutes consecutive edge-sums then its super edge-magicness is assured.
It can be seen easily that the following result about SEMT graphs also holds i.e.,

Note. [4] Let \( c(\Gamma) \) be a magic constant of a SEMT labeling \( \Gamma \) of \( G(V,E) \), then we end up on this statement:

\[
\varepsilon \ c(\Gamma) = \sum_{v \in V} \deg_G(v)\Gamma(v) + \sum_{p \in E} \Gamma(p), \quad \varepsilon = |E(G)| \quad (2.1)
\]

For a single graph, many SEMT labelings might exist and of-course for a different labeling, there will be a different magic constant. Hereafter, we are going to find the bounds for magic constants of SEMT labelings of the generalized comb.

We can see that, \( Cb_{\tau}(\ell_1, \ell_2, \ldots, \ell_\tau), \ell_1 = \ell_2 = \ldots = \ell_\tau; \ \tau \geq 2 \) has \( \tau\ell + 1 \) vertices and \( \tau\ell \) edges. From these vertices, \( \tau - 1 \) vertices have degree 3, \( \varepsilon + 1 - 2\tau \) vertices have degree 2, and the remaining \( \tau + 1 \) vertices have degree 1, see fig 1. Consider that \( Cb_{\tau}(\ell_1, \ell_2, \ldots, \ell_\tau) \) has an EMT labeling with magic constant “\( c \)”, then \( \varepsilon c \) where \( \varepsilon = \tau\ell \), can not be less than the sum we achieve when we allocate the degree-3 vertices with lowest \( \tau - 1 \) labels, the \( \varepsilon + 1 - 2\tau \) next lowest labels to degree-2 vertices, and \( \tau + 1 \) next lowest labels to degree-1 vertices; in other words:

\[
\varepsilon \ c \geq 3 \sum_{i=1}^{\tau-1} i + 2\sum_{i=\tau}^{\varepsilon-\tau} i + \sum_{i=\varepsilon-\tau+1}^{\varepsilon+1} i + \sum_{i=\varepsilon+1}^{2\varepsilon+1} i
\]

We can get the upper bound for \( \varepsilon c \) by assigning highest \( \tau - 1 \) labels to vertices of degree 3, \( \varepsilon - 2\tau + 1 \) next highest labels to vertices of degree 2, and \( \tau + 1 \) next highest labels to vertices of degree 1, in other words:

\[
\varepsilon \ c \leq 3 \sum_{i=2\varepsilon-\tau+3}^{2\varepsilon+1} i + 2\sum_{i=\varepsilon+\tau+2}^{\varepsilon-\tau+2} i + \sum_{i=\varepsilon+1}^{\varepsilon+\tau+1} i + \sum_{i=1}^{\varepsilon} i
\]
Consequently, we end up with the following result:

**Lemma 2.2.** If $Cb_\tau(\ell_1, \ell_2, \ldots, \ell_\tau); \tau \geq 2$ is EMT graph, then magic constant “$c$” is in the following interval:

$$\frac{1}{2\epsilon}(5\epsilon^2 + 2\tau^2 - 2\epsilon\tau + 7\epsilon - 2\tau + 2) \leq c \leq \frac{1}{2\epsilon}(7\epsilon^2 - 2\tau^2 + 2\epsilon\tau + 5\epsilon + 2\tau - 2); \epsilon = \tau\ell$$

By similar process, we can get following result for SEMT graphs:

**Lemma 2.3.** If $Cb_\tau(\ell_1, \ell_2, \ldots, \ell_\tau); \tau \geq 2$ is SEMT graph, then magic constant “$c$” is in the following interval:

$$\frac{1}{2\epsilon}(5\epsilon^2 + 2\tau^2 - 2\epsilon\tau + 7\epsilon - 2\tau + 2) \leq c \leq \frac{1}{2\epsilon}(5\epsilon^2 - 2\tau^2 + 2\epsilon\tau + 7\epsilon + 2\tau - 2); \epsilon = \tau\ell$$

### 3. SEMT Strength of Generalized Comb

From SEMT labeling for a generalized comb $Cb_\tau(\ell_1, \ell_2, \ldots, \ell_\tau), \ell_1 = \ell_2 = \ldots = \ell_\tau; \tau \geq 2$, [13], we have magic constant $c = 2\tau\ell + \lceil\frac{\tau\ell}{2}\rceil + 3$ and by the given lower bound of magic constants in Lemma 2.3, we have:

**Theorem 3.1.** The SEMT strength for generalized comb $G \cong Cb_\tau(\ell, \ell, \ldots, \ell), \tau \geq 2$ is, for $\epsilon = \tau\ell$:

$$\frac{5\epsilon^2 + 2\tau^2 - 2\epsilon\tau + 7\epsilon - 2\tau + 2}{2\epsilon} \leq sm(G) \leq 2\epsilon + \lceil\frac{\epsilon}{2}\rceil + 3$$

### 4. SEMT Labeling and Deficiency of Forests Formed by Generalized Comb and Star, Generalized Comb and Bistar

In this section, it is shown that the two forests, made up of two components i.e., generalized comb and star, generalized comb and bistar, are SEMT. The first result of this section can be concluded as follows:

**Theorem 4.1.** For $\ell \geq 2, \tau \geq 2$

(a): $Cb_\tau(\ell, \ell, \ldots, \ell) \cup K_{1, \omega}$ is SEMT.

(b): $\mu(Cb_\tau(\ell, \ell, \ldots, \ell) \cup K_{1, \omega}) \leq 1; (\tau, \ell) \neq (2, 2)$, where $\omega \geq 1$ and is given by $\omega = \lceil\frac{\tau\ell}{2}\rceil$.

**Proof.** (a): Consider the graph $G \cong Cb_\tau(\ell, \ell, \ldots, \ell) \cup K_{1, \omega}$.

Here $V(K_{1, \omega}) = \{y_p; 1 \leq p \leq \omega + 1\}$ and $E(K_{1, \omega}) = \{y_1y_p; 2 \leq p \leq \omega + 1\}$.

Let $\nu = |V(G)|$ and $\varepsilon = |E(G)|$, so $\nu = \tau\ell + \omega + 2$ and $\varepsilon = \tau\ell + \omega$.

Valuation $\Gamma : V(Cb_\tau(\ell, \ell, \ldots, \ell)) \to \{1, \ell + 1\}$ is described as follows:

$$\Gamma(x_{i,j}) = \left\{ \begin{array}{ll} \frac{i+1}{2} + \frac{\varepsilon(j-1)}{2} & : i, j \equiv 1(mod 2) \\
\frac{j^2}{2} - \frac{i}{2} + 1 & : i, j \equiv 0(mod 2) \end{array} \right.$$

Now consider the labeling $\Omega : V(G) \to \{1, \nu\}$.

For $1 \leq p \leq \omega + 1$

$$\Omega(y_p) = \left\{ \begin{array}{ll} \left\lfloor\frac{\tau\ell}{2}\right\rfloor + 1 & : p = 1 \\
\tau\ell + p & : p \neq 1 \end{array} \right.$$
Let $A = \lceil \frac{\tau \ell}{2} \rceil + 1$ and $B = \tau \ell + \varpi + 1$, then

$$
\Gamma(x_{i,j}) = \begin{cases} 
A + \frac{\ell (i-1) + j}{2} & ; i \equiv 0 \pmod{2}, j \equiv 1 \pmod{2} \\
A + \frac{\ell j - i - 1}{2} & ; i \equiv 1 \pmod{2}, j \equiv 0 \pmod{2}
\end{cases}
$$

$$
\Gamma(x_{1,0}) = B + 1 = \tau \ell + \varpi + 2
$$

$$
\Omega(x_{i,j}) = \Gamma(x_{i,j}); 1 \leq i \leq \ell, 0 \leq j \leq \tau.
$$

The edge-sums of $G$ induced by the above labeling $\Omega$ form consecutive integers starting from $h + 1$ and ending on $\ell + \varepsilon$, where $h = \lceil \frac{\tau \ell}{2} \rceil + 2$. Hence from Lemma 2.1, we end up on a SEMT graph with $c = \lceil \frac{\tau \ell}{2} \rceil + 2 \tau \ell + 2 \varpi + 5$.

(b): Let $\tilde{G} \equiv Cb_{\tau}(\ell, \ell, \ldots, \ell) \cup K_{1,\varpi-1} \cup K_{1}; (\tau, \ell) \neq (2, 2)$ so,

$$
V(\tilde{G}) = V(Cb_{\tau}(\ell, \ell, \ldots, \ell)) \cup V(K_{1,\varpi-1}) \cup \{z\}
$$

$$
V(K_{1,\varpi-1}) = \{y_p; 1 \leq p \leq \varpi\}
$$

$$
E(K_{1,\varpi-1}) = \{y_1 y_p; 2 \leq p \leq \varpi\}
$$

Let $\tilde{\nu} = |V(\tilde{G})| = \tau \ell + \varpi + 2$ and $\tilde{\ell} = |E(\tilde{G})| = \tau \ell + \varpi - 1$. Keeping in mind the valuation $\Gamma$ defined in (a), we describe the labeling $\tilde{\Omega} : V(\tilde{G}) \rightarrow \{1, \tilde{\nu}\}$ as

$$
\tilde{\Omega}(x_{1,0}) = \tau \ell + \varpi + 2
$$

$$
\tilde{\Omega}(z) = \tau \ell + \varpi + 1
$$

$$
\tilde{\Omega}(x_{i,j}) = \Gamma(x_{i,j}); 1 \leq i \leq \ell, 0 \leq j \leq \tau.
$$

The edge-sums of $\tilde{G}$ induced by the above labeling $\tilde{\Omega}$ form consecutive integers starting from $\tilde{h} + 1$ and ending on $\tilde{h} + \tilde{\varepsilon}$, where $\tilde{h} = \lceil \frac{\tau \ell}{2} \rceil + 2$. Hence from Lemma 2.1, we end up on a SEMT graph with $\tilde{c} = \tilde{\nu} + \tilde{\varepsilon} + \tilde{h} + 1$.

In the formulation of next results, we will use the labeling $\Gamma$ provided in previous theorem 4.1.
Theorem 4.2. For $\ell, \tau \geq 2; \omega, \varpi \geq 1$, 
(a): $C_b(\ell, \ell, \ldots, \ell) \cup BS(\omega, \varpi)$ is SEMT, $(\ell, \tau) \neq (2, 2)$.
(b): $\mu_1(C_b(\ell, \ell, \ldots, \ell) \cup BS(\omega, \varpi - 1)) \leq 1; (\ell, \tau) \notin \{(3, 2), (2, 3)\}$ and $\varpi \geq 2$.
Where $\varpi$ is given by $\varpi = \left\lceil \frac{\tau - 3}{2} \right\rceil$.

Proof. (a): Consider the graph $G \cong C_b(\ell, \ell, \ldots, \ell) \cup BS(\omega, \varpi); \ell, \tau \geq 2, \omega, \varpi \geq 1$.
$V(BS(\omega, \varpi)) = \{z_{ut} : u = 1, 2; 0 \leq t \leq \rho\}$, where

$$\rho = \begin{cases} \\
\omega & ; u = 1 \\
\varpi & ; u = 2 \\
\end{cases}$$

and $E(BS(\omega, \varpi)) = \{z_{10} \leq \ell \leq \omega \} \cup \{z_{10} \leq \omega \} \cup \{z_{20} \leq \varpi \}$. Let $\nu = |V(G)|$ and $\varepsilon = |E(G)|$, so we get $\nu = \tau \ell + \omega + \varpi + 3$ and $\varepsilon = \tau \ell + \omega + \varpi + 1$.

Keeping in mind the valuation $\Gamma$ defined in Theorem 4.1 with $A = \left\lceil \frac{\tau \ell}{2} \right\rceil + \omega + 1$ and $B = \tau \ell + \omega + \varpi + 2$, we describe the labeling $\Omega : V(G) \rightarrow \{\Gamma, \nu\}$ as

$$\Omega(z_{ut}) = \begin{cases} \\
\left\lceil \frac{\tau \ell}{2} \right\rceil + t & ; u = 1, t = r, 1 \leq r \leq \omega \\
\left\lceil \frac{\tau \ell}{2} \right\rceil + \omega + 1 & ; u = 2, t = 0 \\
\tau \ell + \omega + 2 & ; u = 1, t = 0 \\
\tau \ell + \omega + 2 + t & ; u = 2, t = r, 1 \leq r \leq \varpi \\
\end{cases}$$

$\Omega(x_{i,j}) = \Gamma(x_{i,j}); 1 \leq i \leq \ell, 0 \leq j \leq \tau$

$\Omega(x_{1,0}) = B + 1 = \tau \ell + \omega + \varpi + 3$.

The edge-sums of $G$ induced by the above labeling $\Omega$ form consecutive integers starting from $h + 1$ and ending on $h + \varepsilon$, where $h = \left\lceil \frac{\tau \ell}{2} \right\rceil + \omega + 2$. Hence from Lemma 2.1, we end up on a SEMT graph.

(b): Let $\hat{G} \cong C_b(\ell, \ell, \ldots, \ell) \cup BS(\omega, \varpi - 1) \cup K_1; \ell, \tau \geq 2, \omega \geq 2$. Here $V(\hat{G}) = V(C_b(\ell, \ell, \ldots, \ell)) \cup V(BS(\omega, \varpi - 1)) \cup \{z\}$ and where $V(BS(\omega, \varpi - 1)) = \{z_{ut} : u = 1, 2; 0 \leq t \leq \rho\}$, where

$$\rho = \begin{cases} \\
\omega & ; u = 1 \\
\varpi - 1 & ; u = 2 \\
\end{cases}$$

and $E(BS(\omega, \varpi - 1)) = \{z_{10} \leq \ell \leq \omega \} \cup \{z_{10} \leq \omega \} \cup \{z_{20} \leq \varpi - 1 \}$. Let $\hat{\nu} = |V(\hat{G})|$ and $\hat{\varepsilon} = |E(\hat{G})|$, so we get $\hat{\nu} = \tau \ell + \omega + \varpi + 3$ and $\hat{\varepsilon} = \tau \ell + \omega + \varpi$. Keeping in mind the valuation $\hat{\Gamma}$ defined in Theorem 4.1 with $A$ and $B$ the same as in part (a), we describe the labeling $\hat{\Omega} : V(\hat{G}) \rightarrow \{\hat{\Gamma}, \hat{\nu}\}$ as

$$\hat{\Omega}(x_{1,0}) = B + 2, \hat{\Omega}(z) = B + 1$$

$\hat{\Omega}(x_{i,j}) = \Gamma(x_{i,j}); 1 \leq i \leq \ell, 0 \leq j \leq \tau$.

The edge-sums of $\hat{G}$ induced by the above labeling $\hat{\Omega}$ form consecutive integers starting from $h + 1$ and ending on $h + \hat{\varepsilon}$, where $\hat{h} = \left\lceil \frac{\tau \ell}{2} \right\rceil + \omega + 2 = h$. Hence from Lemma 2.1, we end up on a SEMT graph. □
5. SEMT Forests Formed by Generalized Comb and Comb, Generalized Comb and Path

The motivation of this section is to continue the work of exploring forests with two components that are SEMT. In the previous section, we have determined two forests that were SEMT and also provided the situations for their SEMT deficiencies. The next two results of this section give us SEMT labeling for the disjoint union of the generalized combs $Cb_\tau(\ell,\ell,\ldots,\ell)$ with comb $Cb_\omega$ and path $P_\omega$ respectively.

**Theorem 5.1.** For $\tau, \ell \geq 2; \omega \geq 1$

(a): $Cb_\tau(\ell,\ell,\ldots,\ell) \cup Cb_\omega$ is SEMT.

(b): $\mu_s(Cb_\tau(\ell,\ell,\ldots,\ell) \cup Cb_{\omega-1}) \leq 1; \omega \geq 2, (\ell, \tau) \neq (2, 2)$, where

$$\omega = \left\lfloor \frac{\tau\ell - 1}{2} \right\rfloor.$$

**Proof.** (a): Consider the graph $G \cong Cb_\tau(\ell,\ell,\ldots,\ell) \cup Cb_\omega$, where

$V(Cb_\omega) = \{x_p; 0 \leq p \leq \omega\} \cup \{y_q; 1 \leq q \leq \omega\}$,

$E(Cb_\omega) = \{x_p, x_{p+1}; 0 \leq p \leq \omega - 1\} \cup \{x_p, y_p; 1 \leq p \leq \omega\}$.

Let $\nu = |V(G)|$ and $\varepsilon = |E(G)|$, so we get $\nu = \tau\ell + 2\omega + 2$ and $\varepsilon = \tau\ell + 2\omega$. Keeping in mind the valuation $\Gamma$ defined in Theorem 4.1 with $A = \left\lceil \frac{\tau\ell}{2} \right\rceil + 1$ and $B = \tau\ell + 2\omega + 1$,
we describe the labeling $\Omega : V(G) \rightarrow \{1, \nu\}$ as

For $0 \leq p \leq \omega$, $1 \leq q \leq \omega$,

$$\Omega(x_p) = \begin{cases} \lceil \frac{\tau\ell}{2} \rceil + p + 1 & : p \text{ is even} \\ \tau\ell + \omega + 1 + p & : p \text{ is odd} \end{cases}$$

and

$$\Omega(y_q) = \begin{cases} \tau\ell + \omega + 1 + q & : q \text{ is even} \\ \lceil \frac{\tau\ell}{2} \rceil + q + 1 & : q \text{ is odd} \end{cases}$$

$$\Omega(x_{i,j}) = \Gamma(x_{i,j}); 1 \leq i \leq \ell, 0 \leq j \leq \tau$$

and

$$\Omega(x_{1,0}) = B + 1.$$
also
\[ \hat{\Omega}(y_q) = \begin{cases} \tau \ell + \omega + q = \Omega(y_q) - 1 & : q \text{ is even} \\ \left\lfloor \frac{\tau \ell}{2} \right\rfloor + q + 1 = \Omega(y_q) & : q \text{ is odd} \end{cases} \]

\[ \hat{\Omega}(x_{i,j}) = \Gamma(x_{i,j}), 1 \leq i \leq \ell, 0 \leq j \leq \tau \]

\[ \hat{\Omega}(x_{1,0}) = B + 2, \hat{\Omega}(z) = B + 1. \]

The edge-sums of \( \hat{G} \) induced by the above labeling \( \hat{\Omega} \) form consecutive integers starting from \( \bar{h} + 1 \) and ending on \( \bar{h} + \varepsilon \), where \( \bar{h} = \left\lfloor \frac{\tau \ell}{2} \right\rfloor + \omega + 1 \). Hence from Lemma 2.1, we end up on a SEMT graph. \[ \square \]

**Theorem 5.2.** For \( \tau \geq 2, \ell \geq 2 \)

(a)(i): \( C_b(\tau, \ell, \ldots, \ell) \cup P_{\bar{\varepsilon}} \) is SEMT.

(a)(ii): \( C_b(\ell, \ell, \ldots, \ell) \cup P_{\bar{\omega} - 1} \) is SEMT.

(b)(i): \( i \mathcal{M}(C_b(\ell, \ell, \ldots, \ell) \cup P_{\bar{\omega} - 2}) \leq 1; (\ell, \tau) \neq (2, 2) \).

(b)(ii): \( \mathcal{M}(C_b(\ell, \ell, \ldots, \ell) \cup P_{\bar{\omega} - 3}) \leq 1; (\ell, \tau) \neq (2, 2) \) where

\[ \bar{\omega} = \begin{cases} \tau \ell \quad ; \tau, \ell \equiv 1(\text{mod} \ 2) \\ \tau \ell - 1 \quad ; \text{otherwise} \end{cases} \]

**Proof.** (a): Consider the graph \( G \cong C_b(\ell, \ell, \ldots, \ell) \cup P_{\omega} \), where \( V(P_{\omega}) = \{ x_p; 1 \leq p \leq t \} \) and \( E(P_{\omega}) = \{ x_p x_{p+1}; 1 \leq p \leq t - 1 \} \). Let \( \nu = |V(G)| \) and \( \varepsilon = |E(G)| \), so we get \( \nu = \tau \ell + t + 1 \) and \( \varepsilon = \tau \ell + t - 1 \), where

\[ t = \begin{cases} \bar{\omega} \quad ; \text{for (a)(i)} \\ \bar{\omega} - 1 \quad ; \text{for (a)(ii)} \end{cases} \]

Keeping in mind the valuation \( \Gamma \) defined in Theorem 4.1 with

\[ A = \begin{cases} \left\lfloor \frac{\tau \ell}{2} \right\rfloor + \left\lfloor \frac{\tau \ell - 1}{2} \right\rfloor ; \text{for (a)(i)} \\ \left\lfloor \frac{\tau \ell}{2} \right\rfloor + \left\lfloor \frac{\tau \ell - 1}{2} \right\rfloor ; \text{for (a)(ii)} \end{cases} \]

and for (a)(i)

\[ B = \begin{cases} 2\tau \ell \quad ; \ell \text{ is odd, } \tau \text{ is odd} \\ 2\tau \ell - 1 \quad ; \text{otherwise} \end{cases} \]

and for (a)(ii)

\[ B = \begin{cases} 2\tau \ell - 1 \quad ; \ell \text{ is odd, } \tau \text{ is odd} \\ 2\tau \ell - 2 \quad ; \text{otherwise} \end{cases} \]

We describe the labeling \( \Omega : V(G) \to \{ 1, \nu \} \) as

\[ \Omega(x_p) = \begin{cases} \left\lfloor \frac{\tau \ell}{2} \right\rfloor + r \quad ; p = 2r - 1; 1 \leq r \leq \left\lfloor \frac{\nu + 1}{2} \right\rfloor \\ \tau \ell + \left\lfloor \frac{\nu}{2} \right\rfloor + \ell \quad ; p = 2\ell, 1 \leq \ell \leq \left\lfloor \frac{\nu}{2} \right\rfloor, \text{ for (a)(i)} \\ \tau \ell + \left\lfloor \frac{\nu}{2} \right\rfloor + 1 - \ell \quad ; p = 2\ell, 1 \leq \ell \leq \left\lfloor \frac{\nu}{2} \right\rfloor, \text{ for (a)(ii)} \end{cases} \]

furthermore,

\[ \Omega(x_{i,j}) = \Gamma(x_{i,j}); 1 \leq i \leq \ell, 0 \leq j \leq \tau \]

\[ \Omega(x_{1,0}) = B + 1. \]

The edge-sums of \( G \) induced by above labeling \( \Omega \) form consecutive integers

For (a)(i) \( \{ \bar{h} + 1, \bar{h} + \varepsilon \} \).
For \( (a)(ii) \) \( \{h, h + \varepsilon - 1\} \)
where
\[
h = \begin{cases} 
\tau \ell + 2 & \text{; } \tau \text{ is odd, } \ell \text{ is odd} \\
\tau \ell + 1 & \text{; otherwise}
\end{cases}
\]

Hence from Lemma 2.1, we end up on a SEMT graph.

(b): Let \( \hat{G} \cong Cb_{7}(\ell, \ldots, \ell) \cup P_{t} \cup K_{1}\), where \( V(P_{t}) = \{x_{p}; 1 \leq p \leq t\} \), \( V(K_{1}) = \{z\} \) and \( E(P_{t}) = \{x_{p}x_{p+1}; 1 \leq p < t - 1\} \). Let \( \nu = |V(\hat{G})| \) and \( \varepsilon = |E(\hat{G})| \), so we get
\[
\nu = \tau \ell + t + 2 \text{ and } \varepsilon = \tau \ell + t - 1 , \text{ where}
\]
\[
t = \begin{cases} 
\varpi - 2 & \text{; } \text{for } (b)(i) \\
\varpi - 3 & \text{; } \text{for } (b)(ii)
\end{cases}
\]
Keeping in mind the valuation \( \Gamma \) defined in Theorem 4.1 with
\[
A = \begin{cases} 
\left\lceil \frac{\tau}{2} \right\rceil + \left\lfloor \frac{\tau - 1}{2} \right\rfloor & \text{; } \text{for } (b)(i) \\
\left\lceil \frac{\tau}{2} \right\rceil + \left\lfloor \frac{\tau - 3}{2} \right\rfloor & \text{; } \text{for } (b)(ii)
\end{cases}
\]
and for \( b(i) \)
\[
B = \begin{cases} 
2\tau \ell - 2 & \text{; } \tau \text{ is odd, } \ell \text{ is odd} \\
2\tau \ell - 3 & \text{; otherwise}
\end{cases}
\]
and for \( b(ii) \)
\[
B = \begin{cases} 
2\tau \ell - 3 & \text{; } \tau \text{ is odd, } \ell \text{ is odd} \\
2\tau \ell - 4 & \text{; otherwise}
\end{cases}
\]
We describe the labeling \( \hat{\Omega} : V(\hat{G}) \to \{1, \nu\} \) as
\[
\hat{\Omega}(x_p) = \begin{cases} 
\left\lceil \frac{r}{2} \right\rceil + r & : p = 2r - 1; 1 \leq r \leq \left\lceil \frac{\ell + 1}{2} \right\rceil \\
\tau \ell - 1 + \left\lceil \frac{r}{2} \right\rceil + \ell & : p = 2\ell, 1 \leq \ell \leq \left\lfloor \frac{\tau}{2} \right\rfloor, \text{for } (b(i)) \\
\tau \ell - 2 + \left\lceil \frac{r}{2} \right\rceil + \ell & : p = 2\ell, 1 \leq \ell \leq \left\lfloor \frac{\tau}{2} \right\rfloor, \text{for } (b(ii))
\end{cases}
\]

Furthermore,
\[
\hat{\Omega}(x_{i,j}) = \Gamma(x_{i,j}); 1 \leq i \leq \ell, 0 \leq j \leq \tau \\
\hat{\Omega}(x_{1,0}) = B + 2, \hat{\Omega}(z) = B + 1.
\]

The edge-sums of \( \hat{G} \) induced by above labeling \( \hat{\Omega} \) form consecutive integers

For \((b)(i)\) \{\(\hat{h} + 1, \hat{h} + \varepsilon\}\}

For \((b)(ii)\) \{\(\hat{h}, \hat{h} + \varepsilon - 1\}\}

where
\[
\hat{h} = \begin{cases} 
\tau \ell + 1 & : \tau \text{ is odd, } \ell \text{ is odd} \\
\tau \ell & : \text{otherwise}
\end{cases}
\]

Hence from Lemma 2.1, we end up on a SEMT graph.

\[\square\]

REFERENCES


