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Reducing Portfolio Quadratic Programming Problem into Regression Problem: Stepwise Algorithm

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Abstract. Mathematical programming can be classified into linear and non linear programming. This study involved a literature knowledge of formal theory essential for understanding of optimization and investigation of algorithms used for solution of special case of non linear programming, namely quadratic programming. The solution of quadratic objective function has been found using numerical and statistical approaches. Numerical technique is based on Cholesky decomposition algorithm and statistical approach is based on Least squares technique. The selected model chosen for the purpose of solving quadratic programming problem is related to portfolio selection in presence of transaction costs. The objective is to minimize the sum of squares of error by estimating parameters. It was not the purpose of study to discuss all algorithms but an algorithm namely stepwise algorithm has been discussed in detail. Using stepwise technique, we have reduced quadratic programming problem into regression problem and found the values of estimated parameters. This approach has efficiently solved the quadratic programming problem and gave the optimum values of unknown parameters.

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1. Introduction

The term mathematical is referred for the theory and applications of optimization. A large number of mathematical and real world problems can be formulated by means of linear problem. There are still many other situations for which linear models are insufficient, so the concept of non linear programming was introduced. Non linear programming is generalizatioof linear programming and especially well studied non linear programming is quadratic programming [5]. The Least squares technique of regression has been treated as a quadratic programming problem. A stepwise algorithm has been presented to solve the quadratic programming problem. This stepwise algorithm makes us able to achieve a simplemethod for solution of quadratic problem.

2. Quadratic Programming

Quadratic programming problem arise in many applications. In quadratic programming, objective function consists of terms of the form t_i^2, t_it_i,and t_i where T_i ($i = 1, 2...s$) are the variables to be determined. Mathematically we define it as follows

$$f(X) = C^TX + X^TVX$$

such that

$$HX \geq B, X \geq 0$$

where C, X and B are column vectors, V and H are matrices and the superscript T denotes the transpose subject to linear constraint. V is $n \times n$ symmetric and positive definite matrix.

3. Least squares method with equality constraints

A general linear regression model under restrictions in the structure of linear inequalities is represented as follows:

$$Y = X\theta + \in$$

such that

$$A\theta \geq E, \theta \geq 0$$

where X and A are matrices, Y and E are column vectors. By making use of Least squares technique, we achieved a general regression programming problem which is presented as follows:

$$Q(\theta) = (Y - X\theta)^T(Y - X\theta)$$

such that

$$a_i^T \geq E,\ \theta_i \geq 0,\ i = 1, 2, ...p$$

Here $Q(\theta)$ is quadratic function of θ.We solved the above quadratic model by using stepwise algorithm. Stepwise method is based on solution of Least squares problem with equality constraints as follows:

$$Q(\theta) = (Y - X\theta)^T(Y - X\theta)$$

such that

$$a_i^T = E,\ \theta_i \geq 0,\ i = 1, 2, ...p$$

Problems related to regression and Least squares method are solved in many fields of life. Least squares method is an important technique of regression. This method is used to fit best line from data. Least squares method minimizes the sum of squared errors from data. Portfolio optimization problem is also known as mean-variance optimization. The term mean refers to expected return of investment. Variance determines the risk related to portfolio. In reality, portfolio is investment. The objective function provides a measurement of maximum return or minimum loss. In portfolio optimization, the constraints represent a restriction on the total amount being invested. In classical work of Markowitz, transaction costs related to buying and selling of securities were not included in the model. But with the passage of time, the importance of transaction costs in construction of new portfolio as well as rebalancing the existing portfolio has been increasing. Transaction costs are not insignificant enough to be neglected. In presence of transaction costs, optimal portfolio depends on transaction cost. Transaction can be considered a dealing between two persons including money. The proportional transaction costs can be used in such a way that the resulting portfolio optimization problem is quadratic. The purpose of standard portfolio optimization is to determine the optimal allocation of limited resources among a limited set of investments. Historical data is used to find expected returns and variances of historical returns. The returns are considered as random variables that can show a pattern estimated from historical data. Return can be achieved in the form of dividend and capital gain. The profit provided by company is dividend but the profit that we achieve on selling is capital. Wolfe [7] and [4] presented a method for minimizing a quadratic function with linear inequality constraints. This procedure is comparable to simplex procedure for linear programming. This method is based on Barankin-Dorfman algorithm. In [1] , [2] and Wang et al. [6], discussed the solution of quadratic programming problem by using convex optimization and Wolfe algorithm. They found the solution of quadratic programming problems on the basis of statistical theory. McCarl et al.[3] discussed applications of quadratic programming in different fields namely statistics, economics, finance and agriculture. They proposed potentially new applications of quadratic programming. They studied some applied areas where quadratic programming is useful thus showing the characteristics of accompanying solutions. In finance, quadratic programming can be applied in portfolio selection. In agriculture, it is used in crop selection. They reviewed both methodological and functional applications of quadratic programming.

4. Materials and Methods

The major aim of this study was to minimize the variance by estimating transaction buying cost parameters. At first, to achieve this aim, a quadratic function is formed subject to linear constraints. For precise estimation of competence of model, assumptions of convexity are checked by means of Hessian matrix. The major goal is to find the transaction buying costs on two securities stock and gold respectively. By applying Cholesky method, A and B matrices are developed and converted the quadratic problem into least squares problem with linear constraints in form of equalities. By using least squares method, simultaneous equations are solved to estimate values of decision variables. Then after finding values, the evaluated values are applied on annual returns of two securities to find best optimal solution .Stepwise algorithm and quadratic programming problem as discussed by [2] are as follows

Step 1.Determine A and B matrices from quadratic form of objective function of quadratic Problem and transfer the quadratic programming problem to Least squares problem.

Step2.. Solve this Least squares problem over R_r^+ . Start from counter n=1. Represent the solution by θ^* .

Step 3. Now check that whether $S\theta^* \geq C$. If "yes" , θ^* is the solution of quadratic programming problem, stop; if "no" , form an index set $G_1 = i : s_i^t\theta \prec C_i$.

Step4 . For all elements of G_1 create and solve corresponding Least squares problem over R^r with a single linear constraint $s_i^t\theta = C_i, i \in G_1$. Represent the solution by θ^i .Set up an index set $J_1 = i : S\theta^i \geq C,\ i \in G_1$.

Step 5. Note whether J_1 is a non-empty set. If "yes", then θ^i ,$i \in J_1$ is the solution of quadratic programming problem and stop the process; if "no", set up an index sets $G_1 \setminus J_1 = i{:}s_i^t\theta \prec C_i, i \in G_1$.

Step 6. Increase the counter by 1, that is$n = n + 1$. Set up the index set. $G_n = i_1, i_2, ...i_n : s_i^t\theta^{i_1,i_2...i_n} \prec C_{i_r}; i_1, i_2, ...i_{n-1} \in G_{n-1}\backslash J_{n-1}$.

Step7. For all members $i_1, i_2, ...i_n$ of G_n.Create and solve corresponding Least squares problem over R_r^+ with the linear constraints $s_{i_n}^t\theta^{i_1,i_2...i_{n-1}} = C_{i_n}, i_1, i_2, ...i_n \in G_n$.Represent the solution by $\theta^{i_1,i_2...i_n}$.

Step8. $J_n = i_1, i_2, ...i_n : S\theta^{i_1,i_2...i_n} \geq C, i_1, i_2, ...i_n \in G_n$.

Step 9. Check whether $J_n \neq \phi$.If "yes", then $\theta^{i_1,i_2...i_r}$, $i_1, i_2, ...i_n \in J_n$, is the solution of quadratic programming problem and stop ; if "no", return to steps 6-9 till an optimal solution is found. Using the above concepts, we have the following results.

5. Main Results

STANDARD FORM OF MODEL

Consider the objective function

$$minW = \frac{1}{2}X^TVX \tag{5. 1}$$

such that

$$\begin{cases} U^Tx \geq e_0 \\ \sum_{i=1}^{2} x_i = 1 - \sum_{i=1}^{2} o_i^{BS} \\ 0 \leq x_i \leq 1 \end{cases} \tag{5. 2}$$

Here expression (5.1) is objective function which minimizes the variance. In expression (5.2), first constraint represented that expected return should be greater equal minimum required expected return and second constraint expressed the sum of investments in presence of transaction costs. In expression (5.2), it represents Mean returns where U is column vector ,e denotes required minimun expected return of portfolio, O_i^{BS} isTransaction buying and selling costs ,x_i =Proportional weight of ith security in portfolio. Now we convert quadratic programming problem into matrix form as follows

$$\min\theta = [\overline{x}v - xv]\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + \frac{1}{2}\theta_1\theta_2\begin{bmatrix} v & \text{-}v \\ -v & v \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \tag{5. 3}$$

such that

$$[U\ \ U]\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \leq \widetilde{e_0} \tag{5. 4}$$

$$[1 + c^B \quad 1 + c^B]\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \tag{5. 5}$$

$$0 \leq \theta_1 \leq 1 - \overline{x}_i, \quad 0 \leq \theta_2 \leq \overline{x}_i, \quad \widetilde{e_0} = U^T\overline{x} - e_0$$

where in expression (5.3), θ_1 and θ_2 are ith security transaction buying costs. x is mean weight of two securities. To apply stepwise algorithm, we multiplied expression (5.4) with negative one. So the expression (5.4) becomes

$$[-U \;\; \text{-}U] \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \geq -\widetilde{e_0} \tag{5. 6}$$

From expression (5.3), (5.4) and (5.6), we have $b = [\overline{x}v - xv]$, $V = \begin{bmatrix} v & \text{-}v \\ -v & v \end{bmatrix}$, $R = \begin{bmatrix} -U & \text{-}U \\ 1+c^B & 1+c^B \end{bmatrix}$, $S = \begin{bmatrix} -\widetilde{e_0} \\ 0 \end{bmatrix}$, where in above expression, b represented coefficients of linear form of decision variables from objective function, V has shown variance covariance matrix, R highlighted coefficients of decision variables on the left side of inequality and S represented constants on right side of inequality. Using Cholesky decomposition method, variance covariance matrix V has been decomposed as $V = T^tT$ where T is real upper triangular matrix having positive diagonal elements. Then the matrix A is developed which is based on positive definite variance covariance matrix. Using matrix A and column vector b, column vector B=-$\frac{1}{2}$ (A^T)$^{-1}$ b is created. Now for computations of matrices A and B, we apply Cholesky decomposition method as follows: According to Cholesky decomposition method, a positive definite variance covariance matrix V can be decomposed as

$$V = \begin{bmatrix} I_{11} & 0 \\ I_{21} & I_{22} \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} \\ 0 & I_{22} \end{bmatrix}$$

From above relation, we have

$$I_{11} = \frac{\sigma_1}{\sqrt{2}}, \;\; I_{12} = \frac{-\sigma_{12}}{\sigma_1\sqrt{2}}, \;\; I_{22} = \sqrt{\frac{\sigma_2^2}{2} - \frac{\sigma_{12}^2}{2\sigma_1^2}}$$

Using the estimated values of I_{11} , I_{22} and I_{12} , we have constructed the upper triangular matrix A as follows:

$$A = \begin{bmatrix} \frac{\sigma_1}{\sqrt{2}} & \frac{-\sigma_{12}}{\sqrt{2}\sigma_1} \\ 0 & \sqrt{\frac{\sigma_2^2}{2} - \frac{\sigma_{12}^2}{2\sigma_1^2}} \end{bmatrix} \tag{5. 7}$$

Using matrix A and column vector b, we here calculated the matrix B as follows:

$$B = -\frac{1}{2}(A^T)^{-1}b \tag{5. 8}$$

$$B = \begin{bmatrix} \frac{-\overline{x}}{\sqrt{2\sigma_1^2}} \\ \frac{m(\sqrt{\sigma_1^2} - \frac{-\sigma_{12}}{\sqrt{\sigma_1^2}}}{\sqrt{2}\sqrt{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}} \end{bmatrix}$$

To find optimal solution, we converted problem into Least squares problem by putting above estimated values of matrices A and B in the following expression (5.9) which minimized the variances.

$$Q(\theta) = (B - A\theta)^T(B - A\theta) \tag{5. 9}$$

Putting values of A and B in expression in (5.9), we have

$$= \frac{\overline{x}\sigma_{12}^2\sigma_{12}^2}{2\sigma_1^2} + \overline{x}\sigma_{12}\,\theta_1 - \frac{\overline{x}\theta_2\sigma_{12}^2}{2\sigma_1^2} + \frac{\theta_1^2\sigma_1^2}{2} + \frac{\theta_2^2\sigma_{12}^2}{2\sigma_1^2} - \theta_1\theta_2\sigma_{12}$$
$$+ \frac{\overline{x}^2\sigma_{12}^2(\sigma_1^2 - \sigma_{12})^2}{2\sigma_1^2(\sigma_1^2\sigma_2^2 - (\sigma_{12})^2)^2} - \frac{\overline{x}\sigma_{12}\theta_2(\sigma_1^2 - \sigma_{12})}{\sigma_{12}} + \frac{\theta_2^2(\sigma_1^2\sigma_2^2 - (\sigma_{12})^2)}{2\sigma_1^2} \quad (5.\ 10)$$

Taking partial derivative with repect to θ_1,we have

$$\theta_1 = \frac{\sigma_{12}\,\theta_2}{\sigma_1^2} - \frac{\overline{x}\sigma_{12}}{\sigma_1^2} \quad (5.\ 11)$$

Now taking partial derivative with respect to θ_2

$$\theta_2 = \frac{\sigma_{12}^2\,\theta_2}{\sigma_1^2\sigma_2^2} + \frac{\theta_1\sigma_{12}}{\sigma_2^2} + \frac{\overline{x}\sigma_{12}\,\sigma_1^2 - \sigma_{12}}{\sigma_1^2\sigma_2^2} \quad (5.\ 12)$$

Putting value of θ_2 in eq. (5.11), we have

$$\theta_1 = \frac{\sigma_{12}\,\theta_2}{\sigma_1^2} - \frac{\overline{x}\sigma_{12}}{\sigma_1^2}$$

$$\theta_1 = \frac{(\sigma_{12})^3\overline{x}}{\sigma_1^2\sigma_2^2 - (\sigma_{12})^2} + \frac{\overline{x}(\sigma_{12})^2\sigma_1^2 - \sigma_{12}}{\sigma_1^2\sigma_2^2 - (\sigma_{12})^2} - \frac{\sigma_{12}\,\overline{x}\sigma_2^2}{\sigma_1^2\sigma_2^2 - (\sigma_{12})^2} \quad (5.\ 13)$$

Putting value of θ_1 in eq. (5.12)

$$\theta_2 = \left[\frac{\overline{x}((\sigma_{12})^2}{\sigma_1^2\sigma_2^2} + \frac{\overline{x}((\sigma_{12})^4}{\sigma_1^2\sigma_2^2 - (\sigma_{12})^2} + \frac{\overline{x}((\sigma_{12})^3\sigma_1^2 - (\sigma_{12})}{\sigma_1^2\sigma_2^2 - (\sigma_{12})^2} - \frac{\overline{x}(\sigma_{12})^2}{\sigma_1^2\sigma_2^2 - (\sigma_{12})^2} + \frac{\overline{x}\sigma_{12}\,\sigma_1^2 - \sigma_{12}}{\sigma_1^2\sigma_2^2}\right. \quad (5.\ 14)$$

Expression (5.13) and (5.14) are derived values of estimated parameters θ_1 and θ_2 .The results thus obtained have been applied on the given example for which data is shown in different tables. Table 5.1 showed annual returns of stock and gold from1968-1988. Table 5.2 represented variances, covariances and means of annual returns of two securities namely stock and gold.

Table 5.1 Data for returns on two securites that is stock and gold.

Years	Stock	Gold	Years	Stock	Gold
1968	11	11	1979	19	59
1969	-9	8	1980	33	99
1970	4	-14	1981	-5	-25
1971	14	14	1982	22	4
1972	19	44	1983	23	-11
1973	-15	66	1984	6	-15
1974	-27	64	1985	32	-12
1975	37	0	1986	19	16
1976	24	-22	1987	5	22
1977	-7	18	1988	17	-2
1978	7	31			

Table.5.2 Variances and covariance of stock and gold returns.

		Stock	Gold
Stock	PearsonCorrelation	1	$-.159$
	(Sig2-tailed)		..490
	Sum of Squares and Cross-products	$5.498E3$	$-1.747E3$
	Covariance	274.890	-87.360
	N	21	21
Gold	PearsonCorrelation	$-.159$	1
	(Sig2-tailed)	..490	
	Sum of Squares and Cross-products	$-1.747E3$	$2.189E4$
	Covariance	-87.360	$1.094E3$
	N	21	21

Table5.3. Descriptive statistics

Performance measure	Mean	Standard Deviation	N
Stock	10.9048	16.57982	21
Gold	16.9048	33.08308	21

Here $\sigma_1^2 = 274.89, \sigma_2^2 = 1094,\ U_1 = 10.90, U_2 = 16.90.$ Here σ_1^2 and σ_2^2 represented the variances of stock and gold respectively whereas U_1,and U_2 were the means of annual returns of stock and gold respectively. For the sake of simplicity, we divided the Variances, covariance, means, minimum expected return and transaction buying cost by hundred and obtained variance covarianc matrix as follows:

$$V = \begin{bmatrix} 2.7489 & .8736 \\ -.8736 & 10.94 \end{bmatrix}$$

But from expression (5.3), we have

$$V = \frac{1}{2}\begin{bmatrix} 2.7489 & .8736 \\ -.8736 & 10.94 \end{bmatrix}$$

$$V = \begin{bmatrix} 1.3745 & \text{-0.4368} \\ \text{-0.4368} & 5.4700 \end{bmatrix}$$

To check, whether variance covariance matrix is positive definite or not, we found eigen values using MATLAB. Thus eigen values obtained are 0.3373 and 5.5072. As eigen values are greater than zero so it is shown that variance covariance matrix is positive definite. Finally, considering Hessian matrix, we checked convexity of matrix V

$$\widehat{H} = \begin{bmatrix} 1.3745 & \text{-0.4368} \\ \text{-0.4368} & 5.4700 \end{bmatrix}$$

According to given data, we have total \$800 and invested 37% of \$800 in stocks and 63% of \$800 in gold. The minimum required expected return is taken at least \$10. The annual returns obtained in twenty one years is shown in table 5.1.If traded transaction costs occurred, then we assumed that every \$100 of stock and gold portfolio traded costs \$1and \$2 respectively. To find the optimal transaction buying cost on two securities, we took traded transaction buying cost on securities \$3 and \$10 respectively. After putting values of σ_1^2, σ_2^2 and σ_{12} in eq. 5.12 and 5.13, cost parameters θ_1 and θ_2 are evaluated. The results thus obtained are multiplied by hundred.$\theta_1 = .1843 = \$18$, $\theta_2 = -.0541 = -\$5$. Then after putting values of θ_1 and θ_2 in constraints (5.4) and (5.6), we got optimal results. As expressions (5.11) and (5.12) showed that both constraints are satisfied, it meansvalues of decision variables thus obtained provided optimal solution. The results showed that variance will increase \$18 for unit increase in transaction buying cost ofstock and variance will decrease \$5 for unit decrease in transaction buying cost of gold results.

6. CONCLUSION

Regression problems and least squares algorithm are encountered in many fields of our everyday life. Many researchers make use of them to model and for solution of real world problems. Least square technique is considered an effective tool by many companies and decision makers. Reformulation of quadratic programming problem in the form of regression problem is very helpful for many researchers in the field of industry. The solution of quadratic programming problem on the basis of its relevant least squares problem appeared easy to understand. A stepwise technique to solve quadraticprogramming problem has been suggested. This method is based on the associationbetween objective function of quadratic programming and its associated least squaresproblem. This algorithm is easy to follow and is well known by many statisticians. The rate of convergence of this stepwise technique is faster than other methods such asWolfe method and simplex method etc. This algorithm reduced the number ofvariables whereas many methods of mathematical programming such as Wolf methodand simplex methods are difficult to solve and increased slack and artificial variablesin order to find optimal solution.

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