Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 51(1)(2019) pp. 127-139

Applying Homotopy Type Techniques to Higher Order Boundary Value Problems

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Received: 12 April, 2018 / Accepted: 30 May, 2018 / Published online: 25 October, 2018

Abstract. In this paper, various homotopy approaches such as Homotopy Perturbation Method (HPM) and Optimal Homotopy Asymptotic Method (OHAM) are applied to solve the greater order multipoint boundary value problems. The problem 4.1 is solved by OHAM and HPM and their results are compared with their exact solutions and problem 4.2 is solved by OHAM only and its results are compared with its exact solution and Vibrational Iteration Method VIM has taken from references [29, 30] The problem 4.3 is solved by OHAM and its result is compared with its exact solution and variational iteration method homotopy perturbation (VIMHP) taken from reference [26]. The new analytical algorithm OHAM provides us with a convenient way to control the convergence of approximation series and allows adjustment of convergence regions where necessary. It has a very good agreement with the exact solution of the concerned problem. It is a parameter free method and shows reliability, so that the results reveal the effectiveness of OHAM which leads to better accuracy.

Key Words: OHAM, HPM, Higher-order multipoint boundary value problems.

1. INTRODUCTION

The majority of the real world problems are in general investigated by modeling certain differential equations. Such mathematical model equations play a vital role in different challenging applications. It is often desirable to describe the behavior of real-life systems, whether it is physical or sociological in terms of these models. In this article, we investigate a boundary value problem of the kind.

$$F(z^{(m)}, z^{(m-1)}, \cdots, z^{(1)}, z) = 0, G\left(z, \frac{dz}{d\eta}\right) = 0 \tag{1. 1}$$

For the solution of such boundary problem we consider the differential equation of m^{th} order as;

$$b_m(\eta)\frac{d^m z}{d\eta^m} + b_{m-1}(\eta)\frac{d^{m-1}z}{d\eta^{m-1}} + \cdots + b_1(\eta)\frac{dz}{d\eta} + b_0(\eta)z = 0. \tag{1. 2}$$

With a solution $f(\eta, z, d_1, d_2, \cdots d_m) = 0$, and $d_1, d_2, \cdots, d_m$ are real finite constants. The boundary conditions are given in problem concerned and $F$is the continuous function over the restricted domain [c, d]. The problem of the above type has been investigated by the authors [4, 10] on account of being important in hydromagnetic and hydrodynamic stability. The 5^{th} order boundary value problem happens in a mathematical model of the viscoelastic flow. The 8^{th} order ordinary differential equation governs the physics of several hydrodynamic consistency problems. Where an infinite plane layer of fluids is animated from beneath and is subject to the achievement of turning around unsteadiness cliques in. When this unsteadiness cliques in as over constancy, it is modeled by an 8^{th} order ODE. When a magnetic field is also applied, the constancy is uneasy. To restrain the stability conditions 10th order, 11^{th}th order differential equations are required to restrain the stability. In literature, many methods are used for the solutions of such type of differential equations. Out of these, some are given as ADM [2,3,36], VIM [14], HPM [12,13] and HAM homotopy analysis method [18]. The HPM in which the homotopy and perturbation techniques are combined to overcome the restrictions of large and small parameters in the problem. It concerns a vast multiplicity of non-linear problems successfully. Newly Herisanu and Marinca [16,21,22,23] presented OHAM for the approximate solution of the nonlinear problem of the lean film flows of 4^{th}-grade fluid down vertical cylinders. Through their research work, they have applied this appreciate the performance of non-linear mechanical vibration of an electrical machine. They have also applied this technique to the solution of nonlinear differential equation arising in steady-state flows of the 4^{th} -grade fluids past a stretching plate and for the solution of nonlinear equations arising in the temperature transfer. It offers a source for controlling the convergence of a series solution and bend convergence area when it is demanded. The relevant work can be seen in [1,6,8,9,11,17,18,20,27] and [31-35]. We use this technique for finding approximate numerical solutions of some sophisticated order boundary value problems. The results of OHAM are correlated not only with exact solutions, but with other literature, methods like HPM, VIM, and VIMHP. The errors are linked to the present results to check the accuracy of the proposed approach.

2. FUNDAMENTALS OF OHAM:

We take the succeeding ordinary differential-equation as under:

$$E(Z(\eta)) + f(\eta) + W(Z(\eta)) = 0\,,\; G\left(Z, \frac{dZ}{d\eta}\right) = 0 \tag{2. 3}$$

Where$Z$ represents an independent variable, E is a linear operator, $Z(\eta)$ represents an un-known function, $f(\eta)$ represents a known function, W is a nonlinear operator and G represents the boundary operator. By OHAM technique, we established homotopy as follows $T(\zeta(\eta,j),j) : R \times [0,1] \to R$

$$(1-j)[E(\zeta(\eta,j)) + f(\eta)] = T(j)[E(\zeta(\eta,j)) + f(\eta) + W(\zeta(\eta,j))],$$
$$G\left(\zeta(\eta,j), \tfrac{\partial\zeta(\eta,j)}{\partial\eta}\right) = 0 \tag{2. 4}$$

. Where $\eta \in R$, $j \in [0,1]$ is the embedding parameter, $T(j)$ is the non-zero auxiliary function for $j \neq 0$, $T(0) = 0$, and $\zeta(\eta,j)$ is an unknown function. Evidently,for $j = 0$ and $j = 1$, it holds that $\zeta(\eta,0) = Z_0(\eta)$ and $\zeta(\eta,1) = Z(\eta)$. As j is from 0 to 1, for solution $\zeta(\eta,j)$ tends from $Z_0(\eta)$ to $Z_1(\eta)$, where $Z_0(\eta)$ is acquired for $j = 0$ and thus we have:

$$E(Z_0(\eta)) + f(\eta) = 0\,, G\left(Z_0, \frac{dZ_0}{d\eta}\right) = 0 \tag{2. 5}$$

We select auxiliary function $T(j)$ in a form of:

$$T(j) = \sum_{i=1}^{m} j^i A_i \tag{2. 6}$$

Where A_1, A_2 and so for their constants which are to be found later, $T(j)$ can be expressed in numerous form as examined by Marinca et.al [5,16,21,24].To obtain approximate result we can expand $\zeta(\eta,j,A_i)$ in Taylor s series about j in the following form

$$\zeta(\eta,j,A_i) = Z_0(\eta) + \sum_{m=1}^{\infty} Z_m(\eta, A_1, A_2, ..., A_m) j^m \tag{2. 7}$$

.

We substitute (2.6) and (2.7) in (2.4) and compare coefficients of the same exponents of j . We obtain the following bellow linear equations for which the integration becomes easy.

An initial order problem that is the $zero^{th}$ order problem given by (2.3) where as the 1^{st}

order problem is given by

$$E(Z_1(\eta)) + f(\eta) = A_1 W_0(Z_0(\eta)), G\left(Z_1, \frac{dZ_1}{d\eta}\right) = 0 \quad (2.\ 8)$$

And the second-order problem is investigated as

$$E(Z_2(\eta)) - E(Z_1(\eta)) = A_2 W_0(Z_0(\eta)) + A_1[E(Z_1(\eta)) + W_1(Z_0(\eta), Z_1(\eta))],$$
$$G\left(Z_2, \tfrac{dZ_2}{d\eta}\right) = 0 \quad (2.\ 9)$$

The general governing equations for $Z_m(\eta)$ is

$$E(Z_m(\eta)) - E(Z_{m-1}(\eta)) = A_m W_0(Z_0(\eta)) +$$
$$\textstyle\sum_{i=1}^{m-1} A_i[E(Z_{m-1}(\eta)) + W_{m-i}(Z_0(\eta), Z_1(\eta), ...Z_{m-1}(\eta))], m = 2, 3, ...,$$
$$G\left(Z_m, \tfrac{dZ_m}{d\eta}\right) = 0 \quad (2.\ 10)$$

Where $W_m(Z_0(\eta), Z_1(\eta), ..., Z_{m-1}(\eta))$ is the coefficient of j^min the expansion of $W(\zeta(\eta, j))$ about j which is given by:

$$W(\zeta(\eta, j, A_i)) = W_0(Z_0(\eta)) + \sum_{m=1}^{\infty} W_m(Z_0, Z_1, ..., Z_m) j^m \quad (2.\ 11)$$

It has been noticed that convergence of (2.5) depends on auxiliary constants A_1, A_2 and so forth. If the equation (2.7) is convergent at $j = 1$,then we have

$$\zeta(\eta, A_i) = Z_0(\eta) + \sum_{m=1}^{\infty} Z_m(\eta, A_1, A_2, ..., A_m) \quad (2.\ 12)$$

.

The result of the m^{th} order approximation is specified by:

$$\tilde{Z}(\eta, A_1, A_2, ..., A_m) = Z_0(\eta) + \sum_{i=1}^{m} Z_i(\eta, A_1, A_2..., A_k) \quad (2.\ 13)$$

. Using equation (2.13) in (2.3), we get the residual of the form

$$S(\eta, A_1, A_2, ..., A_m) = E(\tilde{Z}(A_1, A_2, ..., A_m)) + f(\eta) +$$
$$W(\tilde{Z}(\eta, A_1, A_2, ..., A_m)) \quad (2.\ 14)$$

. If $S = 0$, then $\tilde{Z}$ is an exact solution. Usually, it is not occurring especially in the non-linear problem. To get the optimal values of we 1^{st} develop the Function by using the least squares method

$$K(A_1, A_2, ..., A_m) = \int_c^d S^2(\eta, A_1, A_2, ..., A_m) d\eta \quad (2.\ 15)$$

and after minimizing we get:

$$\frac{\partial K}{\partial A_1} = 0, \frac{\partial K}{\partial A_2} = 0, ... \frac{\partial K}{\partial A_m} = 0 \quad (2.\ 16)$$

Whereas using Galerkin, s technique, we solve following system easily

$$\int_{c}^{d} S\frac{\partial \tilde{Z}}{\partial A_1}d\eta = 0, \int_{c}^{d} S\frac{\partial \tilde{Z}}{\partial A_2}d\eta = 0, ... \int_{c}^{d} S\frac{\partial \tilde{Z}}{\partial A_m}d\eta = 0 \tag{2. 17}$$

One can utilize either (2.16) or (2.17) to get the constants , which ultimately leads to getting the approximate solution of the problem concerned.

3. FUNDAMENTALS OF HPM:

We show the basic idea of the homotopy perturbation method [15] and considering the nonlinear differential equation of the form:

$$C(z) - g(s) = 0,\ s \in \Omega \tag{3. 18}$$

(18) With boundary conditions

$$G\left(z, \frac{dz}{dn}\right) = 0\,, s \in \Gamma \tag{3. 19}$$

Where C is a general differential operator, $G$is a boundary operator, Z is a known analytical function, $\Gamma$is the boundary Ω of the domain . The operator C can be divided into two parts, E and W are linear and nonlinear . Equation (18) takes the form:

$$E(z) + W(z) - g(s) = 0 \tag{3. 20}$$

By the homotopy method proposed by Liao [19]. A homotopy can be constructed as $v(s,\, j) : \Omega \times [0, 1] \to R$ This satisfies:

$$T(v,\, j) = (1 - j)[E(v) - E(z_0)] + j[C(v) - g(s)] = 0 \tag{3. 21}$$

or

$$T(v,\, j) = E(v) - E(z_0) + jE(z_0) + j[W(v) - g(s)] = 0 \tag{3. 22}$$

Where $s \in \Gamma$ and $j \in [0, 1]$ is an embedding parameter, is an initial approximation of (3.18) which satisfies the boundary conditions using (3.21) we can easily guess that

$$T(v, 0) = E(v) - E(z_0) = 0 \tag{3. 23}$$

$$T(v, 1) = C(v) - g(s) = 0 \tag{3. 24}$$

And the altering technique of j from zero to one is just that of $T(v, j)$ from $E(v) - E(z_0)$ to $C(v) - g(s)$. In the Topology, this is called deformation, where $C(v) - g(s)$ and $E(v) - E(z_0)$ are called homotopic. The embedding parameter j is known initially. For $0 < j \leq 1$ equation (3.4) can be given as

$$v = v_0 + jv_1 + j^2 v_2 + \cdots \tag{3. 25}$$

The approximate solution of equation of (3.18) can be obtained as follows:

$$z = \lim v\, j \to 1, v_0 + v_1 + v_2 + \cdots \tag{3. 26}$$

4. NUMERICAL PROBLEMS:

4.1. **Problems:** For $\eta \in [0, 1]$, we deliberate the given boundary value problem.

$$Z^{(8)}(\eta) + \eta\, Z(\eta) + (48 + 15\,\eta + \eta^3)\, e^z = 0,$$
$$Z(0) = 0, Z''(0) = 0, Z^{(4)}(0) = -8, Z^{(6)}(0) = -24,$$
$$Z(1) = 0, Z''(1) = -4\,e, Z^{(4)}(1) = -16\,e, Z^{(6)}(1) = -36\,e \tag{4. 27}$$

Also, an exact solution to this problem is

$$Z(\eta) = \eta(1 - \eta)e^{\eta} \tag{4. 28}$$

We apply OHAM method mentioned above accordingly. The initial is the $zero^{th}$ order problem which is investigated as

$$Z_0{}^{(8)}(\eta) = 0\,,$$
$$Z_0(0) = 0, Z_0{}''(0) = 0, Z_0{}^{(4)}(0) = -8, Z_0{}^{(6)}(0) = -24, Z_0(1) = 0, Z_0{}''(1) = -4e,$$
$$Z_0{}^{(4)}(1) = -16e, Z_0{}^{(6)}(1) = -36e. \tag{4. 29}$$

First order problem is

$$Z_1{}^{(8)}(\eta, A_1) = A_1[48\, e^{\eta} + 15\,\eta\, e^{\eta} + \eta^3\, e^{\eta} + \eta\, Z_0(\eta)] + (1 + A_1)\, Z_0{}^{(8)}(\eta),$$
$$Z_1(0) = 0, Z_1{}''(0) = 0, Z_1{}^{(4)}(0) = 0, Z_1{}^{(6)}(0) = 0, Z_1(1) = 0, Z_1{}''(1) = 0,$$
$$Z_1{}^{(4)}(1) = 0, Z_1{}^{(6)}(1) = 0. \tag{4. 30}$$

From the above we get first approximate solution for by OHAM as

$$\tilde{Z}(\eta, A_1) = Z_0(\eta) + Z_1(\eta, A_1) + O(\eta^{17}) \tag{4. 31}$$

Using the OHAM technique, as mentioned above on , we get the residual as

$$S = Z^{(8)}(\eta) + \eta\, Z(\eta) + (48 + 15\,\eta + \eta^3)e^{\eta} \tag{4. 32}$$

We obtained $A_1 = -0.9999597600559361$. Using this value of A_1, the approximate solution is

$$\tilde{Z}(\eta) =$$
$$\eta - 0.499999\,\eta^3 - \tfrac{1}{3}\eta^4 - 0.125\,\eta^5 - \tfrac{1}{30}\eta^6 - 0.00694443\,\eta^7 -$$
$$0.00119043\,\eta^8 - 0.000173604\,\eta^9 - 0.0000220671\;\eta^{10} - 2.48006 \times 10^{-6}\,\eta^{11} -$$
$$2.47222 \times 10^{-7}\,\eta^{12} - 2.29635 \times 10^{-8}\,\eta^{13} - 2.18971 \times 10^{-9}\,\eta^{14}\; -$$
$$1.49114 \times 10^{-10}\,\eta^{15} + o(\eta^{16}). \tag{4. 33}$$

The following table1 displays values of OHAM, HPM solutions, exact solution and their errors. From table 1, we deduce that exact and proposed methods OHAM and HPM solution are in good agreement.

4.2. **Problem:** We take the eleventh order boundary value problem in the domain $\eta \in [0,1]$ [29,30]:

$$Z^{(11)}(\eta)+22(5+\eta)e^{\eta}-Z(\eta)=0,$$
$$Z(0)=0, Z(1)=0, Z^{(1)}(0)=1, Z^{(1)}(1)=-2e, Z^{(2)}(0)=-1, Z^{(2)}(1)=-6e,$$
$$Z^{(3)}(0)=-5,\ Z^{(3)}(1)=-12e,\ Z^{(4)}(0)=-11,\ Z^{(4)}(1)=-20e,\ Z^{(5)}(0)=-19 \tag{4. 34}$$

Moreover exact solution of a problem is:

$$Z(\eta)=(1-\eta^2)e^{\eta} \tag{4. 35}$$

. Using the OHAM technique, $zero^{th}$ order problem is

$$Z_0{}^{(11)}(\eta)=0, Z_0(0)=0, Z_0(1)=0, Z_0{}^{(1)}(0)=1,$$
$$Z_0{}^{(1)}(1)=-2e, Z_0{}^{(2)}(0)=-1, Z_0{}^{(2)}(1)=-6e,$$
$$Z_0{}^{(3)}(0)=-5,\ Z_0{}^{(3)}(1)=-12e,\ Z_0{}^{(4)}(0)=-11,$$
$$Z_0{}^{(4)}(1)=-20e,\ Z_0{}^{(5)}(0)=-19 \tag{4. 36}$$

First order problem is

$$Z_1{}^{(11)}(\eta,A_1)=110e^{\eta}A_1+22e^{\eta}\eta A_1-A_1Z_0(\eta)+(1+A_1)Z_0{}^{(11)}(\eta),$$
$$Z_1(0)=0, Z_1(1)=0, Z_1^{(1)}(0)=0, Z_1{}^{(1)}(1)=0, Z_1{}^{(2)}(0)=0, Z_1{}^{(2)}(1)=0,$$
$$Z_1{}^{(3)}(0)=0,\ Z_1{}^{(3)}(1)=0,\ Z_1{}^{(4)}(0)=0,\ Z_1{}^{(4)}(1)=0,\ Z_1{}^{(5)}(0)=0 \tag{4. 37}$$

. Second order problem is

$$Z_2{}^{11)}(\eta,A_1)=-A_1Z_1(\eta)+(1+A1)Z_1{}^{(11)}(\eta),$$
$$Z_2(0)=0, Z_2(1)=0, Z_2{}^{(1)}(0)=0, Z_2{}^{(1)}(1)=0, Z_2{}^{(2)}(0)=0, Z_2{}^{(2)}(1)=0,$$
$$Z_2{}^{(3)}(0)=0,\ Z_2{}^{(3)}(1)=0,\ Z_2{}^{(4)}(0)=0,\ Z_2{}^{(4)}(1)=0,\ Z_2{}^{(5)}(0)=0 \tag{4. 38}$$

Now using the above equations the second approximate solution by OHAM for is given by

$$\tilde{Z}(\eta,A_1)=Z_0(\eta)+Z_1(\eta,A_1)+Z_2(\eta,A_1) \tag{4. 39}$$

. Following the technique of OHAM on the domain , we use the residual of the form

$$S=\tilde{Z}^{(11)}(\eta)+22e^{\eta}(5+\eta)-\tilde{Z}(\eta) \tag{4. 40}$$

. For this problem, we obtained the value of $A_1=-0.9994747251387355$ to obtain the approximate solution as:

$$\tilde{Z}(\eta)=1+\eta-\tfrac{\eta^2}{2}-\tfrac{5\eta^3}{6}-\tfrac{11\eta^4}{24}-$$
$$\tfrac{19\eta^5}{120}-0.0402778\eta^6-0.00813492\eta^7-0.00136409\eta^8-0.000195657\eta^9-$$
$$0.000024526\eta^{10}-2.73068\times10^{-6}\eta^{11}-2.73485\times10^{-7}\eta^{12}-$$
$$2.48915\times10^{-8}\eta^{13}-2.0762\times10^{-9}\eta^{14}-$$
$$1.59826\times10^{-10}\eta^{15}+O(\eta^{16}). \tag{4. 41}$$

4.3. **Problem:** Considering the nonlinear boundary value problem in the domain $[0, 1]$ Shahid et al. [29, 30]:

$$Z^{(12)}(\eta) - 2\,e^{\eta} Z^2(\eta) - Z^{(3)}(\eta) = 0,$$
$$Z(0) = 1,\ Z(1) = \tfrac{1}{e},\ Z^{(2)}(0) = 1,\ Z^{(2)}(1) = \tfrac{1}{e},\ Z^{(4)}(0) = 1,\ Z^{(4)}(1) = \tfrac{1}{e}$$
$$Z^{(6)}(0) = 1,\ Z^{(6)}(1) = \tfrac{1}{e},\ Z^{(8)}(0) = 1,\ Z^{(8)}(1) = \tfrac{1}{e},$$
$$Z^{(10)}(0) = 1,\ Z^{(10)}(1) = \tfrac{1}{e} \tag{4. 42}$$

Where the exact solution of the problem is as bellow:

$$Z(\eta) = e^{-\eta}. \tag{4. 43}$$

Here in OHAM technique, the $zero^{th}$ order problem is:

$$Z_0{}^{(12)}(\eta) = 0,\ Z_0(0) = 1,\ Z_0(1) = \tfrac{1}{e},\ Z''{}_0(0) = 1,\ Z''{}_0(1) = \tfrac{1}{e},$$
$$Z_0''''\ (0) = 1, Z_0''''\ (1) = \tfrac{1}{e}, Z_0{}^{(6)}(0) = 1,\ Z_0{}^{(6)}(1) = \tfrac{1}{e},\ Z_0{}^{(8)}(0) = 1,$$
$$Z_0{}^{(8)}(1) = \tfrac{1}{e},\ Z_0{}^{(10)}(0) = 1,\ Z_0{}^{(10)}(1) = \tfrac{1}{e} \tag{4. 44}$$

,

First order problem is:

$$Z_1{}^{(12)}(\eta, A_1) = -2e^{\eta} A_1 Z_0{}^2(\eta) - A_1 Z_0{}^{(3)}(\eta) + (1 + A_1) Z_0{}^{(12)}(\eta),\ Z_1(0) = 0,$$
$$Z_1(1) = 0,\ Z_1{}^{(2)}(0) = 0,\ Z_1{}^{(2)}(1) = 0,\ Z_1{}^{(4)}(0) = 0\,,$$
$$Z_1{}^{(4)}(1) = 0, Z_1{}^{(6)}(0) = 0, Z_1{}^{(6)}(1) = 0,\ Z_1{}^{(8)}(0) = 0,\ Z_1{}^{(8)}(1) = 0,$$
$$Z_1{}^{(10)}(0) = 0,\ Z_1{}^{(10)}(1) = 0 \tag{4. 45}$$

.

Now we use the above equations to get 1^{st}-order approximate solutions by using OHAM for $q = 1$ which is

$$\tilde{Z}(\eta, A_1) = Z_0(\eta) + Z_1(\eta, A_1) \tag{4. 46}$$

, Using the proposed technique with the domain $a = 0, b = 1$, we use the residual as follows.

$$S = \tilde{Z}^{(12)}(\eta) - 2e^{\eta} \tilde{Z}^{2}(\eta) - \tilde{Z}^{(3)} \tag{4. 47}$$

Accordingly, $A_1 = -2.305759098580603 \times 10^{-13}$ is obtained. Using this value of A_1 The approximate solution is:

$$Z(\eta) = 1 - 1.\eta + \tfrac{\eta^{\eta}}{2} - 0.166662\,\eta^3 + \tfrac{\eta^4}{24} - 0.00833552\,\eta^5$$
$$+ \tfrac{\eta^6}{720} - 0.000\,\eta^7 + \tfrac{\eta^8}{40320} - 2.82949 \times 10^{-6}\eta^9 +$$
$$\tfrac{\eta^{10}}{3628800} - 1.5836 \times 10^{-8}\eta^{11} + 4.8138 \times 10^{-22}\eta^{12} -$$
$$3.70287 \times 10^{-23}\eta^{13} + 2.64418 \times 10^{-24}\eta^{14} -$$
$$1.76306 \times 10^{-25}\eta^{15} + O(\eta^{16}). \tag{4. 48}$$

5. Conclusion

In this article, we proposed some new techniques of homotopy perturbation methods, which are so-called OHAM and HPM to obtain an approximate solution of some benchmark problems of linear and non-linear differential equations. From the numerical test problems, we have witnessed that the proposed techniques are reliable alternatives for the problems under consideration as the approximate solutions are accurate and efficient. We have seen that the convergence is fast in the proposed techniques, therefore it provides a reliable platform to solve highly challenging problems occurring in science and engineering.

6. Acknowledgments

The authors are thankful to the anonymous reviewers and the Editor-in-Chief for the productive comments that led to positive enhancement in the paper.

7. Authors Contribution:

Muhammad Naeem contributed in the solution of the problems, formation of graphs and graphical discussion, Sher Muhammad and Syed Asif Hussain contributed in tables preparation and their comparisons with already published results, Liaqat Ali Contributed writing of paper in the Latex format, Zaheer Ud Din and Liaqat ali thoroughly checked the manuscript again and again for removing the mistakes.

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TABLE 1. Displays values of OHAM, HPM solutions, exact solution and their errors.which deduce that exact and proposed methods OHAM and HPM solution are in good agreement for problem-1.

Z	Exact sol.	OHAM sol.	HPM sol.	Error OHAM	Error HPM
0.	0.0	1.	0.0	0.0	0.0
0.1	0.0994654	0.0994653	0.0994654	4.9×10^{-11}	1.1×10^{-14}
0.2	0.195424	0.195424	0.195424	9.4×10^{-11}	2.9×10^{-14}
0.3	0.28347	0.28247	0.28347	1.3×10^{-10}	3.5×10^{-12}
0.4	0.358038	0.358038	0.358038	1.5×10^{-10}	1.4×10^{-12}
0.5	0.41218	0.41218	0.41218	1.6×10^{-10}	1.3×10^{-12}
0.6	0.437309	0.437308	0.437308	1.5×10^{-10}	9.6×10^{-12}
0.7	0.422888	0.422888	0.422888	1.3×10^{-10}	5.2×10^{-11}
0.8	0.356087	0.356086	0.356087	9.4×10^{-11}	2.3×10^{-10}
0.9	0.221364	0.221384	0.221365	4.9×10^{-10}	8.5×10^{-10}
1.	0.000000	-1.9×10^{-12}	2.7×10^{-6}	1.9×10^{-15}	2.7×10^{-9}

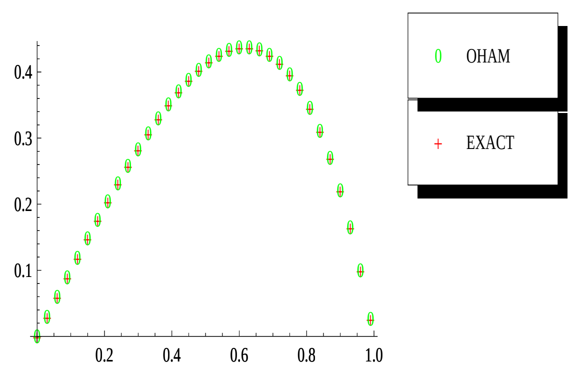

FIGURE 1. The red stars curve is the graph of exact solution, the green circles curve is the graph of OHAM solution which are coincident upon each other for problem-1.

TABLE 2. Indicates the conclusions of the values obtained by applying the OHAM technique mentioning the absolute error of the methods and the exact solution of the problem concerned. The absolute error of the OHAM solution is comparatively less than the absolute error of the VIM solution, this means that the OHAM solution is more effective than the VIM [29, 30] solution in the middle of discretized points for problem-2.

Z	Exact sol.	OHAM sol.	Error OHAM	Error VIM
0.	1.000000	1.000000	0.0	0.0
0.1	1.09412	1.09412	0.0	6.4×10^{-15}
0.2	1.17255	1.17255	0.0	2.4×10^{-13}
0.3	1.22837	1.22837	$.24 \times 10^{-15}$	1.4×10^{-12}
0.4	1.25313	1.25313	5.5×10^{-14}	3.8×10^{-12}
0.5	1.23654	1.23654	3.8×10^{-13}	6.1×10^{-12}
0.6	1.16616	1.16616	1.8×10^{-12}	6.2×10^{-12}
0.7	1.02701	1.02701	6.7×10^{-12}	3.8×10^{-12}
0.8	0.801195	0.801195	2.8×10^{-11}	1.1×10^{-12}
0.9	0.467325	0.467325	5.7×10^{-11}	9.9×10^{-14}
1.	0.000000	1.4×10^{-10}	1.9×10^{-15}	6.4×10^{-14}

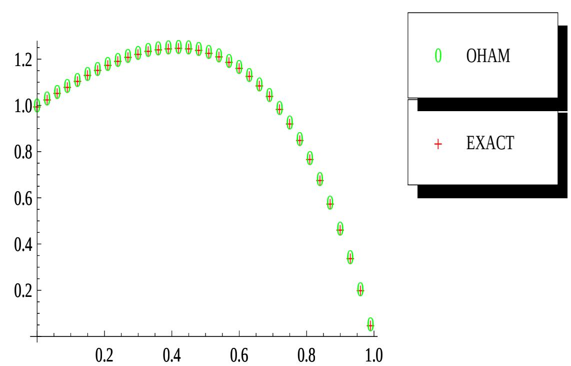

FIGURE 2. The red stars curve graph displays exact solution, the green circles graph displays approximate OHAM solutions. From the above two graphs, we conclude that there exists similarity between the two graphs and hence OHAM method is another alternative besides VIM for problem-2

TABLE 3. shows the exact solution, the series OHAM solution with errors obtained by using VIMHP [26] and OHAM. We observe that better accuracy can be found by evaluating more component of . In this table, we see that OHAM and VIMPH agreed at the discretized points 0 to 0.9 and nodal point 1, our proposed approach of OHAM is having upper hand over the VIMHP method for problem-3

Z	Exact sol.	OHAM sol.	Error OHAM	Error VIMHP
0.	1.000000	1.000000	0.0	0.0
0.1	0.904837	0.904837	2.6×10^{-7}	1.6×10^{-7}
0.2	0.818731	0.81873	5.0×10^{-7}	3.0×10^{-7}
0.3	0.740818	0.740818	6.9×10^{-7}	4.2×10^{-7}
0.4	0.67032	10.670319	8.1×10^{-7}	4.9×10^{-7}
0.5	0.606531	0.60653	8.5×10^{-7}	5.2×10^{-7}
0.6	0.548812	0.548811	8.1×10^{-7}	4.9×10^{-7}
0.7	0.496585	0.496585	6.9×10^{-7}	4.2×10^{-7}
0.8	0.449329	0.449328	5.0×10^{-7}	3.0×10^{-7}
0.9	0.40657	0.406569	2.6×10^{-7}	1.6×10^{-7}
1.	0.367879	0.367879	1.9×10^{-14}	2.0×10^{-10}

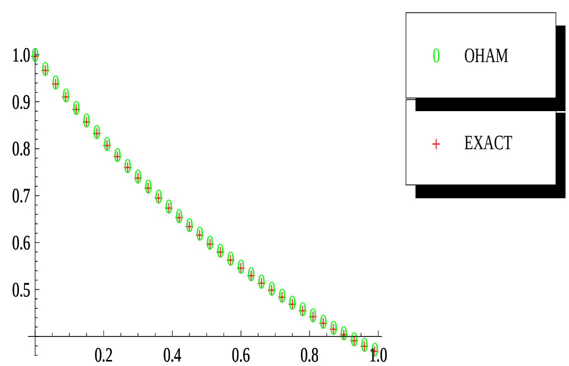

FIGURE 3. the red stars curve shows the graph of exact solution and the green circles graph shows the OHAM solution, where the two types of graphs coincide for problem-3.