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# $e-$Chaotic Generalized Shift Dynamical Systems

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**Abstract.** In the following text we prove that for bijection $\varphi : \mathbb{N} \to \mathbb{N}$ and discrete set $\{1, \ldots, k\}$ with $k \geq 2$, the generalized shift dynamical system $(\{1, \ldots, k\}^{\mathbb{N}}, \sigma_\varphi)$ is $e-$chaotic, (expansive, positively expansive) if and only if $\{\{\varphi^i(n) : i \in \mathbb{Z}\} : n \in \mathbb{N}\}$ is a finite partition of $\mathbb{N}$ (or equivalently there exists $N \in \mathbb{N}$ such that $\mathbb{N} = \bigcup\{\varphi^i(\{1, \ldots, N\}) : i \in \mathbb{Z}\}$).

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## 1. Preliminares

By a *topological dynamical system* (or briefly *dynamical system*) $(X, f)$ we mean a topological space $X$ (*phase space*) and continuous map $f : X \to X$.
For a nonempty set $X$ consider two maps one-sided shift $\sigma_1 : X^{\mathbb{N}} \to X^{\mathbb{N}}$ and two-sided shift $\sigma_2 : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ with $\sigma_1((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$ (for $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$) and $\sigma_2((y_n)_{n\in\mathbb{Z}}) = (y_{n+1})_{n\in\mathbb{Z}}$ (for $(y_n)_{n\in\mathbb{Z}} \in X^{\mathbb{Z}}$), where $\mathbb{N} = \{1, 2, \ldots\}$ is the set of all natural numbers and $\mathbb{Z} = \{0, \pm1, \pm2, \ldots\}$ is the set of all integers. One-sided and two-sided shifts are well-known and studied in several branches of mathematics, like ergodic theory and dynamical systems. For arbitrary nonempty set $\Gamma$, map $\varphi : \Gamma \to \Gamma$, nonempty set $X$ with at least two elements, we call $\sigma_\varphi : X^\Gamma \to X^\Gamma$ with $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) = (x_{\varphi(\alpha)})$ (for all $(x_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$) the *generalized shift*. If $X$ is a topological space, consider $X^\Gamma$ under product (pointwise convergence) topology, so $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is continuous. Generalized shifts in the above form has been introduced in [3], and their dynamical (and non-dynamical) properties has been studied in several texts, like [1], [2] and [6].

Remark 1. *Suppose $X$ is a topological space with at least two elements and $\Gamma$ is a nonempty set, equip $X^\Gamma$ with product topology. It is well-known that* [7]*:*

- *$X^\Gamma$ is compact if and only if $X$ is compact;*
- *$X^\Gamma$ is metrizable if and only if $X$ is metrizable and $\Gamma$ is countable.*

REMARK 2. *Suppose $X$ is a nonempty set with at least two elements and $\Gamma$ is a nonempty set, then the following statements are equivalent* [3]*:*

- *$\sigma_\varphi : X^\Gamma \to X^\Gamma$ is bijective;*
- *$\varphi : \Gamma \to \Gamma$ is bijective.*

*Hence if $X$ has a topological structure, then $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is a homeomorphism if and only if $\varphi : \Gamma \to \Gamma$ is bijective.*

REMARK 3. *If $\Gamma$ is a nonempty set and $\varphi : \Gamma \to \Gamma$ is arbitrary, for $\alpha, \beta \in \Gamma$ let $\alpha \sim_\varphi \beta$ if and only if there exists $n, m \in \mathbb{N}$ with $\varphi^n(\alpha) = \varphi^m(\beta)$. Then $\sim_\varphi$ is an equivalence relation on $X$. If $\varphi : \Gamma \to \Gamma$ is bijective and $\alpha \in \Gamma$, then the equivalence class of $\alpha$ under $\varphi$ is $\frac{\alpha}{\sim_\varphi} = \{\varphi^n(\alpha) : n \in \mathbb{Z}\}$, so $\frac{\alpha}{\sim_\varphi}$ has exactly one of the following forms:*

- *there exists $m \in \mathbb{N}$ with $\frac{\alpha}{\sim_\varphi} = \{\alpha_n : 0 \leq n < m\}$ where $\alpha_i$'s are distinct $\varphi(\alpha_i) = \alpha_{i+1}$ for $i = 0, \ldots, m-1$ and $\varphi(\alpha_{m-1}) = \alpha_0$;*
- *$\frac{\alpha}{\sim_\varphi} = \{\alpha_n : n \in \mathbb{Z}\}$, $\alpha_n$s are distinct and $\varphi(\alpha_n) = \alpha_{n+1}$ for $n \in \mathbb{Z}$.*

*In addition for bijective $\varphi : \Gamma \to \Gamma$ and $\alpha, \beta \in \Gamma$ we have $\alpha \sim_\varphi \beta$ if and only if there exists $n \in \mathbb{Z}$ with $\varphi^n(\alpha) = \beta$.*

In the following text suppose $X$ is a discrete finite set with at least two elements and $\Gamma$ is a countable infinite set. So we may suppose $X = \{1, \ldots, k\}$ with discrete topology, $k \geq 2$, and $\Gamma = \mathbb{N}$, also suppose $\varphi : \Gamma \to \Gamma$ is bijective (note to Remarks 1 and 2). The main aim of this text is to study $e-$chaotic generalized shift dynamical system $(\{1, \ldots, k\}^\mathbb{N}, \sigma_\varphi)$.

## 2. WHEN IS $(\{1, \ldots, k\}^\mathbb{N}, \sigma_\varphi)$ EXPANSIVE?

We call the dynamical system $(Y, f)$ (or briefly $f : Y \to Y$) with compact metric space $(Y, \rho)$ and homeomorphism $f : Y \to Y$, *expansive* if there exists $\mu > 0$ such that for all distinct $x, y \in Y$ there exists $n \in \mathbb{Z}$ with $\rho(f^n(x), f^n(y)) > \mu$.

REMARK 4. *For arbitrary set $Y$ we call the collection $\mathcal{F}$ of subsets of $Y \times Y$ a uniform structure in $Y$ if (let $\Delta_Y = \{(x, x) : x \in Y\}$)* [5]*:*

- *$\forall \alpha \in \mathcal{F}\ (\Delta_Y \subseteq \alpha)$;*
- *$\forall \alpha, \beta \in \mathcal{F}\ (\alpha \cap \beta \in \mathcal{F})$;*
- *$\forall \alpha \in \mathcal{F}\ \exists \beta \in \mathcal{F}\ (\beta \circ \beta^{-1} \subseteq \alpha)$;*
- *$\forall \alpha \in \mathcal{F}\ \forall \beta \subseteq Y \times Y\ (\alpha \subseteq \beta \Rightarrow \beta \in \mathcal{F})$.*

*Moreover for all $\alpha \in \mathcal{F}$ and $x \in Y$ let $\alpha[x] = \{y \in Y : (x, y) \in \alpha\}$.*
*If $\mathcal{F}$ is a uniform structure in $Y$, then $\{U \subseteq Y : \forall x \in Y \exists \alpha \in \mathcal{F}(\alpha[x] \subseteq U)\}$ is a topology on $Y$, we call it uniform topology induced from $\mathcal{F}$. We call the topological space uniformazable if there exists a uniform structure $\mathcal{F}$ in $Y$ such that uniform topology induced from $\mathcal{F}$ coincides with original topology on $Y$, and in this case we call $\mathcal{F}$ compatible uniform structure in $Y$. Every compact Hausdorff (resp. compact metric) space is uniformazable and has a unique compatible uniform structure. If $Y$ is a compact metric space, for $\varepsilon > 0$ let $\alpha_\varepsilon = \{(x, y) \in Y \times Y : \rho(x, y) \leq \varepsilon\}$, and $\mathcal{G} = \{D \subseteq Y \times Y : \exists \delta > 0(\alpha_\delta \subseteq D)\}$, then $\mathcal{G}$ is a compatible uniform structure in $Y$. It's evident that homeomorphism $f : Y \to Y$ is expansive if and only if there exists $\beta \in \mathcal{G}$ such that for all distinct $x, y \in Y$ there exists $n \in \mathbb{Z}$ with $(f^n(x), f^n(y)) \notin \beta$. Since $\mathcal{G}$ is the unique compatible uniform structure in $Y$, expansivity of homeomorphism $f : Y \to Y$ does not depends on $\rho$ and we may choose any compatible metric on $Y$.*

Consider the equivalence relation $\sim_\varphi$ on $\mathbb{N}$ as in Remark 3. We prove $\sigma_\varphi : X^{\mathbb{N}} \to X^{\mathbb{N}}$ is expansive if and only if $\frac{\mathbb{N}}{\sim_\varphi} = \{\frac{\alpha}{\sim_\varphi} : \alpha \in \mathbb{N}\}$ is finite. Also using Remark 4 equip $\{1,\dots,k\}^{\mathbb{N}}$ with metric

$$d((x_n)_{n\in\mathbb{N}},(y_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} \frac{\delta(x_n,y_n)}{2^n} \tag{*}$$

for $(x_n)_{n\in\mathbb{N}},(y_n)_{n\in\mathbb{N}} \in \{1,\dots,k\}^{\mathbb{N}}$, where:

$$\delta(z,w) = \begin{cases} 0 & z = w\,, \\ 1 & z \neq w\,. \end{cases}$$

So $(\{1,\dots,k\}^{\mathbb{N}}, d)$ is a compact metric space (metric topology on $\{1,\dots,k\}^{\mathbb{N}}$ induced from $d$, coincides with product topology on $\{1,\dots,k\}^{\mathbb{N}}$ (see [7])).

LEMMA 2.1. *If $\frac{\mathbb{N}}{\sim_\varphi} = \{\frac{\alpha_1}{\sim_\varphi},\dots,\frac{\alpha_s}{\sim_\varphi}\}$, then for all distinct $x,y \in \{1,\dots,k\}^{\mathbb{N}}$ there exists $n \in \mathbb{N}$ such that $d(f^n(x),f^n(y)) \geq \frac{1}{2^{\max(\alpha_1,\dots,\alpha_s)}}$ (consider metric $d$ on $\{1,\dots,k\}^{\mathbb{N}}$ as in (*)).*

*Proof.* Consider distinct $x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in \{1,\dots,k\}^{\mathbb{N}}$. There exists $m \in \mathbb{N}$ with $x_m \neq y_m$, there exists $r \in \{1,\dots,s\}$ with $m \in \frac{\alpha_r}{\sim_\varphi}$, so there exists $k \in \mathbb{Z}$ with $\varphi^k(\alpha_r) = m$. Suppose $(z_n)_{n\in\mathbb{N}} := \sigma_\varphi^k(x) = (x_{\varphi^k(n)})_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}} := \sigma_\varphi^k(y) = (y_{\varphi^k(n)})_{n\in\mathbb{N}}$. Thus

$$\begin{aligned} d(\sigma_\varphi^k(x),\sigma_\varphi^k(y)) &= d((z_n)_{n\in\mathbb{N}},(w_n)_{n\in\mathbb{N}}) \\ &\geq \frac{\delta(z_{\alpha_r},w_{\alpha_r})}{2^{\alpha_r}} = \frac{\delta(x_{\varphi^k(\alpha_r)},y_{\varphi^k(\alpha_r)})}{2^{\alpha_r}} \\ &= \frac{\delta(x_m,y_m)}{2^{\alpha_r}} = \frac{1}{2^{\alpha_r}} \geq \frac{1}{2^{\max(\alpha_1,\dots,\alpha_s)}} \end{aligned}$$

which completes the proof. □

COROLLARY 2.1. *If $\frac{\mathbb{N}}{\sim_\varphi}$ is finite, then $\sigma_\varphi : \{1,\dots,k\}^{\mathbb{N}} \to \{1,\dots,k\}^{\mathbb{N}}$ is expansive.*

*Proof.* If $\frac{\mathbb{N}}{\sim_\varphi} = \{\frac{\alpha_1}{\sim_\varphi},\dots,\frac{\alpha_s}{\sim_\varphi}\}$ choose $\mu \in (0, \frac{1}{2^{\max(\alpha_1,\dots,\alpha_s)}})$. By Lemma 2.1 for all distinct $x,y \in \{1,\dots,k\}^{\mathbb{N}}$ there exists $n \in \mathbb{Z}$ with $d(\sigma_\varphi^n(x),\sigma_\varphi^n(y)) \geq \frac{1}{2^{\max(\alpha_1,\dots,\alpha_s)}} > \mu$ which leads to the desired result by Remark 4. □

LEMMA 2.2. *If $\frac{\mathbb{N}}{\sim_\varphi}$ is infinite, then $\sigma_\varphi : \{1,\dots,k\}^{\mathbb{N}} \to \{1,\dots,k\}^{\mathbb{N}}$ is not expansive.*

*Proof.* Consider $\mu > 0$, then there exists $N \in \mathbb{N}$ such that $\displaystyle\sum_{n>N} \frac{1}{2^n} < \mu$. Since $\frac{\mathbb{N}}{\sim_\varphi}$ is infinite, there exists $m \in \mathbb{N}$ such that $\frac{m}{\sim_\varphi} \neq \frac{k}{\sim_\varphi}$ for all $k \in \{1,\dots,N\}$, i.e. $m \in \mathbb{N} \setminus (\frac{1}{\sim_\varphi} \cup \cdots \cup \frac{N}{\sim_\varphi})$, and $\frac{m}{\sim_\varphi} \subseteq \mathbb{N} \setminus (\frac{1}{\sim_\varphi} \cup \cdots \cup \frac{N}{\sim_\varphi}) \subseteq \mathbb{N} \setminus \{1,\dots,N\}$ let $x_n = y_n = 1$ for all $n \in \mathbb{N} \setminus \{m\}$, $x_m = 1$ and $y_m = 2$. For $x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}}$ for all $r \in \mathbb{Z}$ we have:

$$\begin{aligned} d(\sigma_\varphi^r(x),\sigma_\varphi^r(y)) &= d((x_{\varphi^r(n)})_{n\in\mathbb{N}},(y_{\varphi^r(n)})_{n\in\mathbb{N}}) \\ &\leq \sum_{x_{\varphi^r(n)} \neq y_{\varphi^r(n)}} \frac{1}{2^n} = \sum_{\varphi^r(n)=m} \frac{1}{2^n} \leq \sum_{n \sim_\varphi m} \frac{1}{2^n} \leq \sum_{n>N} \frac{1}{2^n} < \mu\,. \end{aligned}$$

Hence we have:

$$\forall \mu > 0\, \exists x \neq y\, \forall r \in \mathbb{Z}\, (d(f^r(x),f^r(y)) < \mu)\,.$$

Using Remark 4, $\sigma_\varphi : \{1,\dots,k\}^{\mathbb{N}} \to \{1,\dots,k\}^{\mathbb{N}}$ is not expansive. □

THEOREM 2.1. *For bijection $\varphi : \mathbb{N} \to \mathbb{N}$ and discrete set $\{1,\dots,k\}$ with $k \geq 2$, the generalized shift dynamical system $(\{1,\dots,k\}^{\mathbb{N}}, \sigma_\varphi)$ is expansive if and only if $\frac{\mathbb{N}}{\sim_\varphi}$ is finite (i.e., there exists $n_1,\dots,n_s \in \mathbb{N}$ with $\mathbb{N} = \{\varphi^i(n_j) : j \in \{1,\dots,s\}, i \in \mathbb{Z}\}$).*

*Proof.* Use Corollary 2.1 and Lemma 2.2. □

EXAMPLE 1. *Define $\varphi_1, \varphi_2 : \mathbb{N} \to \mathbb{N}$ with:*

$$\varphi_1(n) = \begin{cases} n+2 & n \text{ is odd} \\ n-2 & n > 2 \text{ is even} \\ 1 & n = 2 \end{cases} \quad \text{and} \quad \varphi_2(n) = \begin{cases} n+1 & n \text{ is odd} \\ n-1 & n \text{ is even} \end{cases}$$

*then $(\{1,\dots,k\}^{\mathbb{N}}, \sigma_{\varphi_1})$ is expansive, and $(\{1,\dots,k\}^{\mathbb{N}}, \sigma_{\varphi_2})$ is not expansive, since $\frac{\mathbb{N}}{\sim_{\varphi_1}} = \{\mathbb{N}\}$ and $\frac{\mathbb{N}}{\sim_{\varphi_2}} = \{\{2n-1, 2n\} : n \in \mathbb{N}\}$.*

## 3. $e-$CHAOTIC GENERALIZED SHIFT DYNAMICAL SYSTEM $(\{1,\dots,k\}^{\mathbb{N}}, \sigma_\varphi)$

We call the dynamical system $(Y, f)$, $e-$chaotic, if it is expansive and the set of all periodic points (of $f : Y \to Y$) is dense in $Y$ [9], we recall that $a \in Y$ is a periodic point of $f : Y \to Y$ if there exists $n \geq 1$ with $f^n(a) = a$.

REMARK 5. *If $Y$ is a discrete topological space with at least two elements, $\Lambda$ is a nonempty set and $\eta : \Lambda \to \Lambda$ is arbitrary, then the set of periodic points of $\sigma_\eta : Y^\Lambda \to Y^\Lambda$ ($\sigma_\eta((x_\alpha)_{\alpha\in\Lambda}) = (x_{\eta(\alpha)})_{\alpha\in\Lambda}$) is dense in $Y^\Lambda$ if and only if $\eta : \Lambda \to \Lambda$ is one to one* [4, Theorem 2.6].

THEOREM 3.1 (main). *For bijection $\varphi : \mathbb{N} \to \mathbb{N}$ discrete set $\{1,\dots,k\}$ with $k \geq 2$, in the generalized shift dynamical system $(\{1,\dots,k\}^{\mathbb{N}}, \sigma_\varphi)$, the following statements are equivalent:*

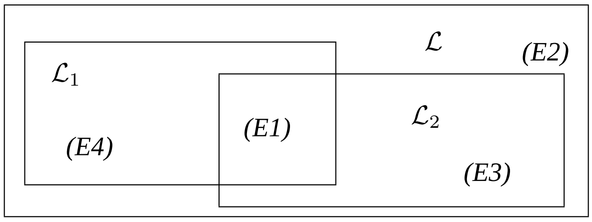
- *$(\{1,\dots,k\}^{\mathbb{N}}, \sigma_\varphi)$ is $e-$chaotic;*
- *$(\{1,\dots,k\}^{\mathbb{N}}, \sigma_\varphi)$ is expansive;*
- *$\frac{\mathbb{N}}{\sim_\varphi}$ is finite (i.e., there exists $n_1,\dots,n_s \in \mathbb{N}$ with $\mathbb{N} = \{\varphi^i(n_j) : j \in \{1,\dots,s\}, i \in \mathbb{Z}\}$, or equivalently $\{\{\varphi^i(n) : i \in \mathbb{Z}\} : n \in \mathbb{N}\}$ is a finite partition of $\mathbb{N}$).*

*Proof.* Use Remark 5 and Theorem 2.1. □

EXAMPLE 2. *Using* [4, Theorem 2.13], *for discrete topological space $Y$ with at least two elements and $\eta : \mathbb{N} \to \mathbb{N}$, the generalized shift dynamical system $(Y^{\mathbb{N}}, \sigma_\eta)$ is Devaney chaotic if and only if $\eta : \mathbb{N} \to \mathbb{N}$ is one to one without any periodic point. Let:*

- *$\mathcal{L}$ = the class of all generalized shift dynamical systems $(Y^{\mathbb{N}}, \sigma_\eta)$, where $\eta : \mathbb{N} \to \mathbb{N}$ is bijective.*
- *$\mathcal{L}_1$ = the class of all Devaney chaotic generalized shift dynamical systems $(Y^{\mathbb{N}}, \sigma_\eta)$, where $\eta : \mathbb{N} \to \mathbb{N}$ is bijective.*
- *$\mathcal{L}_2$ = the class of all $e-$chaotic generalized shift dynamical systems $(y^{\mathbb{N}}, \sigma_\eta)$, where $\eta : \mathbb{N} \to \mathbb{N}$ is bijective.*

*We have the following diagram:*




*where:*

- *E1 is* $(\{1,\ldots,k\}^{\mathbb{N}},\sigma_{\varphi_1})$ *as in Example 1;*
- *E2 is* $(\{1,\ldots,k\}^{\mathbb{N}},\sigma_{\varphi_2})$ *as in Example 1;*
- *E3 is* $(\{1,\ldots,k\}^{\mathbb{N}},\sigma_{\varphi_3})$ *for* $\varphi_3:\mathbb{N}\to\mathbb{N}$ *with* $(k\geq 2)$*:*

$$\varphi_3(n)=\begin{cases}1 & n=1\,,\\ \varphi_1(n-1)+1 & n>1\,;\end{cases}$$

- *E4 is* $(\{1,\ldots,k\}^{\mathbb{N}},\sigma_{\varphi_4})$ *for* $\varphi_4:\mathbb{N}\to\mathbb{N}$ *with* $(k\geq 2)$*(suppose* $p_m$ *is the* $m$*th prime number and* $\mathbb{N}\setminus\{p_m^k:m,k\geq 1\}=\{w_1,w_2,\ldots\}$ *for* $w_1<w_2<\cdots$*):*

$$\varphi_4(n)=\begin{cases}p_m^{\varphi_1(k)} & n=p_m^k\,,\\ w_{\varphi_1(k)} & n=w_k\,.\end{cases}$$

## 4. More details on expansive generalized shift dynamical systems

Regarding previous sections, let's call the dynamical system $((Z,\mathcal{F}),f)$ with uniform phase space $(Z,\mathcal{F})$ and homeomorphism $f:Z\to Z$ *expansive* if there exists $\mu\in\mathcal{F}$ such that for all distinct $x,y\in Z$ there exists $n\in\mathbb{Z}$ with $(f^n(x),f^n(y))\notin\mu$. In this section suppose $(Y,\mathcal{K})$ is a uniform Hausdorff space with at least two elements, $\Lambda$ is a nonempty set and $\lambda:\Lambda\to\Lambda$ is an arbitrary map. It is well-known that product and subspaces of uniform spaces are uniformzable.
In this section we prove that if the generalized shift dynamical system $(Y^\Lambda,\sigma_\lambda)$ with bijection $\lambda:\Lambda\to\Lambda$ is expansive (with any compatible uniformity on $Y^\Lambda$, where $Y^\Lambda$ equipped with product topology), then $\Lambda$ is countable and $\{\{\lambda^n(\alpha):n\in\mathbb{Z}\}:\alpha\in\Lambda\}$ is a finite partition of $\Lambda$.

COROLLARY 4.1. *Using Theorem 3.1 if* $M$ *is countable,* $W$ *is finite discrete with at least two elements and* $\psi:M\to M$ *is bijective, then the following statements are equivalent (note that* $W^M$ *with product topology is a compact metrizable space):*

- $(W^M,\sigma_\psi)$ *is* $e-$*chaotic,*
- $(W^M,\sigma_\psi)$ *is expansive,*
- $\frac{M}{\sim_\psi}$ *is finite.*

THEOREM 4.1. *For bijection* $\lambda:\Lambda\to\Lambda$ *if the generalized shift dynamical system* $(Y^\Lambda,\sigma_\lambda)$ *is expansive, then* $\Lambda$ *is countable and and* $\frac{\Lambda}{\sim_\lambda}$ *is finite.*

*Proof.* First of all note that for all $\alpha\in\Lambda$, $\frac{\alpha}{\sim_\lambda}=\{\lambda^n(\alpha):n\in\mathbb{Z}\}$ is countable. Suppose $\{\alpha_n\}_{n\geq 1}$ is a sequence in $\Lambda$ and $p,q$ are two distinct elements of $Y$. Let $M=\bigcup\{\frac{\alpha_n}{\sim_\lambda}:n\geq 1\}$. Since $(Y^\Lambda,\sigma_\lambda)$ is expansive, $(\{p,q\}^M,\sigma_{\lambda\restriction_M})$ is expansive too. Since $\{p,q\}$ is a discrete set with two elements and $M$ is countable, using Corollary 4.1, for $\psi=\lambda\restriction_M$ the set $\frac{M}{\sim_\psi}(=\{\frac{\alpha_n}{\sim_\lambda}:n\geq 1\})$ is finite. Thus we don't have any infinite sequence in $\frac{\Lambda}{\sim_\lambda}$ and $\frac{\Lambda}{\sim_\lambda}$ is finite. Since $\frac{\Lambda}{\sim_\lambda}$ is finite and all $\frac{\alpha}{\sim_\lambda}(\in\frac{\Lambda}{\sim_\lambda})$ is countable, the set $\bigcup\frac{\Lambda}{\sim_\lambda}=\Lambda$ is countable too. $\square$

**Positively expansive dynamical system.** We call the dynamical system $(Y, f)$ (or briefly $f : Y \to Y$) with compact metric space $(Y, \rho)$, *positively expansive* if there exists $\mu > 0$ such that for all distinct $x, y \in Y$ there exists $n \geq 0$ with $\rho(f^n(x), f^n(y)) > \mu$ [8]. It's evident that for homeomorphism $f : Y \to Y$, if $(Y, f)$ is positively expansive, then it is expansive. Using the same method described in Remark 4 positively expansivity of continuous map $f : Y \to Y$ does not depends on $\rho$ and we may choose any compatible metric on $Y$. Using the same proof as in Lemma 2.2, for arbitrary self-map $\mu : \mathbb{N} \to \mathbb{N}$, if the generalized shift dynamical system $(\{1, \ldots, k\}^{\mathbb{N}}, \sigma_\mu)$ is positively expansive, then $\frac{\mathbb{N}}{\sim_\mu}$ is finite. However for constant map $\mu : \mathbb{N} \to \mathbb{N}$ with $\mu(n) = 1$, the dynamical system $(\{1, \ldots, k\}^{\mathbb{N}}, \sigma_\mu)$ is not positively expansive, although $\frac{\mathbb{N}}{\sim_\mu}$ is finite. Moreover using Lemma 2.1 and Theorem 3.1 we have the following corollary.

COROLLARY 4.2. *For bijection $\varphi : \mathbb{N} \to \mathbb{N}$ and discrete set $\{1, \ldots, k\}$ with $k \geq 2$, the generalized shift dynamical system $(\{1, \ldots, k\}^{\mathbb{N}}, \sigma_\varphi)$ is expansive if and only if it is positively expansive.*

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