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# Some $\phi$-analogues of Hermite-Hadamard Inequality for $s$-convex Functions in the Second Sense and Related Estimates

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**Abstract.** In this paper, the right hand side for quantum analogue of the famous Hermite-Hadamard's inequality for $s$-convex functions is presented. Some quantum estimates for the right hand side of the $\phi$-analogue of the well known Hermite-Hadamard inequality by using the $s$-convexity of the absolute value of the $\phi$-derivatives are obtained. Some inequalities similar to Hermite-Hadamard for the products of the functions which belong to the class of convex and $s$-convex function are proved by using quantum calculus.

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## 1. Introduction

The study of calculus without limits is known as quantum calculus or $\phi$-calculus. The famous mathematician Euler initiated the study $\phi$-calculus in the eighteenth century by introducing the parameter $\phi$ in Newton's work of infinite series. In the nineteenth century, many outstanding results such as Jacobi's triple product identity and the theory of $\phi$-hypergeometric functions were obtained. In early twentieth century, Jackson [7] has started a symmetric study of $\phi$-calculus and introduced $\phi$-definite integrals. The subject of quantum calculus has numerous applications in various areas of mathematics and physics such as number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, quantum theory, mechanics and in theory of relativity. This subject has received outstanding attention by many researchers and hence it is considered as an incorporative subject between mathematics and physics. Interested readers are referred to [4, 5, 8] for some current advances in the theory of quantum calculus and theory of inequalities in quantum calculus.

Recall that a function $\varphi : [\alpha_1, \alpha_2] \subseteq (-\infty, \infty) \to (-\infty, \infty)$ is a convex function if the inequality

$$\varphi\left(\mu\nu + (1-\mu)\xi\right) \leq \mu\varphi(\nu) + (1-\mu)\varphi(\xi)$$

holds for all $\nu, \xi \in [\alpha_1, \alpha_2]$ and $\mu \in [0,1]$.

The following remarkable result is considered a necessary and sufficient condition for a function $\varphi : [\alpha_1, \alpha_2] \subset (-\infty, \infty) \to (-\infty, \infty)$ to be convex on $[\alpha_1, \alpha_2]$, where $\alpha_1 < \alpha_2$

$$\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right) \leq \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\,d\nu \leq \frac{\varphi(\alpha_1)+\varphi(\alpha_2)}{2}. \tag{1. 1}$$

The inequalities that appear in ( 1. 1 ) are acknowledged in the literature as Hermite-Hadamard inequalities. Theory of inequalities and theory of convex functions have been observed to be profoundly dependent on each other and consequently a vast literature on inequalities has been produced by a number of researchers by using convex functions, see [1, 2].

In the paper [6], Hudzik and Maligranda have formulated a new class of $s$-convex functions in the second sense. This class is defined as follows:

**Definition 1.1.** [6] *A function $\varphi : [0, \infty) \to (-\infty, \infty)$ is said to be an $s$-convex function in the second sense if*

$$\varphi\left(\mu\nu + (1-\mu)\xi\right) \leq \mu^s\varphi(\nu) + (1-\mu)^s\varphi(\xi)$$

*holds valid for all $\nu$, $\xi \in [0, \infty)$, $\mu \in [0,1]$, where $s \in (0,1]$ is fixed. The class of all $s$-convex functions in the second sense is denoted by $K_s^2$.*

It has be shown in [6] that all functions in the class $K_s^2$ are non-negative for $s \in (0,1)$.

Dragomir and Fitzpatrick [3] proved the following result as a variant of ( 1. 1 ) for $s$-convex functions in the second sense.

**Theorem 1.2.** [3] *Let $\varphi : [0, \infty) \to [0, \infty)$ be a function which belongs to the class $K_s^2$ and let $\alpha_1, \alpha_2 \in [0, \infty)$, $\alpha_1 < \alpha_2$. If $\varphi \in L([\alpha_1, \alpha_2])$, the following inequalities hold*

$$2^{s-1}\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right) \leq \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\,d\nu \leq \frac{\varphi(\alpha_1)+\varphi(\alpha_2)}{s+1}. \tag{1. 2}$$

*The second inequality in ( 1. 2 ) is sharp.*

In a very fresh article, Tariboon et al. [15, 16] studied the concept of quantum derivatives and quantum integrals over the intervals of the form $[\alpha_1, \alpha_2]$, $\alpha_1$, $\alpha_2 \in (-\infty, \infty)$ and settled a number of quantum analogues of some well-known results such as Hölder inequality, Hermite-Hadamard inequality and Ostrowski inequality, Cauchy-Bunyakovsky-Schwarz, Grüss, Grüss-Čebyšev and other integral inequalities using classical convexity.

Most recently, Noor et al. [13, 14] and Zhuang et al. [18] have contributed to the ongoing research and have developed some integral inequalities which provide quantum estimates for the right part of the quantum analogue of Hermite-Hadamard inequality through $\phi$-differentiable convex and $\phi$-differentiable quasi-convex functions.

Inspired by the recent progress in the field quantum calculus, our aim is to establish a variant of ( 1. 2 ) in quantum calculus. Furthermore, we will also prove some new quantum estimates by using the $s$-convexity of the absolute value of the $\phi$-derivatives.

## 2. BASICS OF $\phi$-CALCULUS

In this part of the manuscript, some basics of $\phi$-calculus over finite intervals are discussed.

Let $[\alpha_1, \alpha_2] \subseteq (-\infty, \infty)$ be an interval and $0 < \phi < 1$, $\phi$-derivative of a function $\varphi : [\alpha_1, \alpha_2] \to (-\infty, \infty)$ at $\nu \in [\alpha_1, \alpha_2]$ is given in the following definition.

**Definition 2.1.** [15] *For a continuous function $\varphi : [\alpha_1, \alpha_2] \to (-\infty, \infty)$ then $\phi$-derivative of $\varphi$ at $\nu \in [\alpha_1, \alpha_2]$ is characterized by the expression*

$$ {}_{\alpha_1}D_\phi \varphi(\nu) = \frac{\varphi(\nu) - \varphi(\phi\nu + (1-\phi)\alpha_1)}{(1-\phi)(\nu - \alpha_1)}, \nu \neq \alpha_1. \tag{2. 3} $$

*Since $\varphi : [\alpha_1, \alpha_2] \to (-\infty, \infty)$ is a continuous function, thus we have ${}_{\alpha_1}D_\phi \varphi(\alpha_1) = \lim_{\nu \to \alpha_1} {}_{\alpha_1}D_\phi \varphi(\nu)$. The function $\varphi$ is said to be $\phi$-differentiable on $[\alpha_1, \alpha_2]$ if ${}_{\alpha_1}D_\phi \varphi(\nu)$ exists for all $\nu \in [\alpha_1, \alpha_2]$. If $\alpha_1 = 0$ in ( 2. 3 ), then ${}_0D_\phi \varphi(\nu) = D_\phi \varphi(\nu)$, where $D_\phi \varphi(\nu)$ is the familiar $\phi$-derivative of $\varphi$ defined by the expression*

$$ D_\phi \varphi(\nu) = \frac{\varphi(\nu) - \varphi(\phi\nu)}{(1-\phi)\nu}, \nu \neq 0. \tag{2. 4} $$

*For more details on $\phi$-derivative given above by ( 2. 4 ), we refer the reader to* [8].

**Definition 2.2.** [16] *For a continuous function $\varphi : [\alpha_1, \alpha_2] \to (-\infty, \infty)$, a second-order $\phi$-derivative on $[\alpha_1, \alpha_2]$ is symbolized as ${}_{\alpha_1}D_\phi^2 \varphi$, as long as ${}_{\alpha_1}D_\phi \varphi$ is $\phi$-differentiable on $[\alpha_1, \alpha_2]$, is defined as ${}_{\alpha_1}D_\phi^2 \varphi = {}_{\alpha_1}D_\phi ({}_{\alpha_1}D_\phi \varphi) : [\alpha_1, \alpha_2] \to (-\infty, \infty)$. Higher order $\phi$-derivative on $[\alpha_1, \alpha_2]$ can be defined as ${}_{\alpha_1}D_\phi^n \varphi = {}_{\alpha_1}D_\phi \left({}_{\alpha_1}D_\phi^{n-1} \varphi\right) : [\alpha_1, \alpha_2] \to (-\infty, \infty)$.*

The following result is very important to evaluate $\phi$-derivatives.

**Lemma 2.3.** [15] *Let $\alpha \in (-\infty, \infty)$ and $0 < \phi < 1$, we have*

$$ {}_{\alpha_1}D_\phi (\nu - \alpha_1)^\alpha = \left(\frac{1-\phi^\alpha}{1-\phi}\right)(\nu - \alpha_1)^{\alpha - 1}. $$

One can find further properties of $\phi$-derivatives in [17].

**Definition 2.4.** [15] *Let $\varphi : [\alpha_1, \alpha_2] \to (-\infty, \infty)$ be a continuous function. The definite $\phi$-integral on $[\alpha_1, \alpha_2]$ is delineated as*

$$ \int_{\alpha_1}^{\nu} \varphi(\gamma) \ {}_{\alpha_1}d_\phi \gamma \ = (\nu - \alpha_1)(1-\phi) \sum_{n=0}^{\infty} \phi^n \varphi(\phi^n \nu + (1-\phi^n)\alpha_1) \tag{2. 5} $$

*for $\nu \in [\alpha_1, \alpha_2]$. If $c \in (\alpha_1, \nu)$, then the definite $\phi$-integral on $[\alpha_1, \alpha_2]$ is expressed as*

$$\begin{aligned}\int_c^{\nu} \varphi(\gamma)\ {}_{\alpha_1}d_\phi\gamma &= \int_{\alpha_1}^{\nu} \varphi(\gamma)\ {}_{\alpha_1}d_\phi\gamma - \int_{\alpha_1}^{c} \varphi(\gamma)\ {}_{\alpha_1}d_\phi\gamma \\ &= (\nu - \alpha_1)(1-\phi)\sum_{n=0}^{\infty}\phi^n\varphi(\phi^n\nu + (1-\phi^n)\alpha_1) \\ &\quad - (c-\alpha_1)(1-\phi)\sum_{n=0}^{\infty}\phi^n\varphi(\phi^n c + (1-\phi^n)\alpha_1).\end{aligned}$$

*If $\alpha_1 = 0$ in ( 2. 5 ), then one can get the classical $\phi$-definite integral defined by (see* [4, 8]*)*

$$\int_0^{\nu} \varphi(\gamma)\ {}_0d_\phi\gamma = (1-\phi)\,\nu\sum_{n=0}^{\infty}\phi^n\varphi(\phi^n\nu),\ \nu \in [0,\infty).$$

**Definition 2.5.** [8] *Let $\alpha$ be a real number. Then*

$$[\alpha] = \frac{1-\phi^\alpha}{1-\phi}.$$

*It is clear that if $n \in \mathbb{N}$, then $[n] = 1 + \phi + \cdots + \phi^{n-1}$.*

The following results hold about definite $\phi$-integrals.

**Theorem 2.6.** [17] *Let $\varphi : [\alpha_1, \alpha_2] \to (-\infty, \infty)$ be a continuous function. Then*

(1) ${}_{\alpha_1}D_\phi \int_{\alpha_1}^{\nu} \varphi(\gamma)\ {}_{\alpha_1}d_\phi\gamma = \varphi(\nu)$
(2) $\int_c^{\nu}\ {}_{\alpha_1}D_\phi\varphi(\gamma)\ {}_{\alpha_1}d_\phi\gamma = \varphi(\nu) - \varphi(c)$, $c \in (\alpha_1, \nu)$.

**Theorem 2.7.** [17] *Suppose that $\varphi, \psi : [\alpha_1, \alpha_2] \to (-\infty, \infty)$ are continuous functions, $\alpha \in (-\infty, \infty)$. Then, for $\nu \in [\alpha_1, \alpha_2]$,*

(1) $\int_{\alpha_1}^{\nu} [\varphi(\gamma) + \psi(\gamma)]\ {}_{\alpha_1}d_\phi\gamma = \int_{\alpha_1}^{\nu} \varphi(\gamma)\ {}_{\alpha_1}d_\phi\gamma + \int_{\alpha_1}^{\nu} \psi(\gamma)\ {}_{\alpha_1}d_\phi\gamma$;
(2) $\int_{\alpha_1}^{\nu} \alpha\varphi(\gamma)\ {}_{\alpha_1}d_\phi\gamma = \alpha\int_{\alpha_1}^{\nu} \varphi(\gamma)\ {}_{\alpha_1}d_\phi\gamma$;
(3) $\int_c^{\nu} \varphi(\gamma)_{\alpha_1} D_\phi\psi(\gamma)\ {}_{\alpha_1}d_\phi\gamma = \varphi(\gamma)\psi(\gamma)|_c^{\nu} - \int_c^{\nu} \psi(\phi\gamma + (1-\phi)\alpha_1)_{\alpha_1} D_\phi\varphi(\gamma)_{\alpha_1} d_\phi\gamma$, $c \in (\alpha_1, \nu)$.

The following is a valuable results to evaluate definite $\phi$-integrals.

**Lemma 2.8.** [15] *For $\alpha \in (-\infty, \infty) \setminus \{-1\}$ and $0 < \phi < 1$, the following formula holds:*

$$\int_{\alpha_1}^{\nu} (\gamma - \alpha_1)^\alpha\ {}_{\alpha_1}d_\phi\gamma = \left(\frac{1-\phi}{1-\phi^{\alpha+1}}\right)(\nu - \alpha_1)^{\alpha+1}.$$

## 3. Main Results

The following result provides the right hand side of a $\phi$-analogue of the Hermite-Hadamard type inequality for $s$-convex functions.

**Theorem 3.1.** *Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous function such that $f \in K_s^2$, where $s \in (0, \infty)$, $0 < \phi < 1$. If $\alpha_1, \alpha_2 \in [0, \infty)$, $\alpha_1 < \alpha_2$ and if $\varphi$ is $\phi$-integrable on $[\alpha_1, \alpha_2]$, then the following inequality holds:*

$$\frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2} \varphi(\nu)\ {}_{\alpha_1}d_\phi\nu \le I_1(\phi, s)\,\varphi(\alpha_1) + I_2(\phi, s)\,\varphi(\alpha_2). \tag{3. 6}$$

*where*

$$I_1(\phi,s) = (1-\phi)\sum_{n=0}^{\infty}\phi^n(1-\phi^n)^s,$$
$$I_2(\phi,s) = (1-\phi)\sum_{n=0}^{\infty}\left(\phi^{s+1}\right)^n.$$

*Proof.* A ussage of the $s$-convexity $\varphi$, gives

$$\varphi((1-\gamma)\alpha_1+\gamma\alpha_2) \le (1-\gamma)^s\varphi(\alpha_1)+\gamma^s\varphi(\alpha_2)$$

for all $\gamma\in[0,1]$. By $\phi$-integration of the above inequality on $[0,1]$, one gets

$$\int_0^1 \varphi((1-\gamma)\alpha_1+\gamma\alpha_2)_0\, d_\phi\gamma \le \varphi(\alpha_1)\int_0^1(1-\gamma)^s\ {}_0d_\phi\gamma+\varphi(\alpha_2)\int_0^1\gamma^s\ {}_0d_\phi\gamma.$$

Using Definition 2.4, we have

$$\int_0^1(1-\gamma)^s\ {}_0d_\phi\gamma = (1-\phi)\sum_{n=0}^{\infty}\phi^n(1-\phi^n)^s = I_1(\phi,s), \tag{3. 7}$$

$$\int_0^1\gamma^s\ {}_0d_\phi\gamma = (1-\phi)\sum_{n=0}^{\infty}\left(\phi^{s+1}\right)^n = I_2(\phi,s). \tag{3. 8}$$

and

$$\begin{aligned}
&\int_0^1\varphi((1-\gamma)\alpha_1+\gamma\alpha_2)\ {}_0d_\phi\gamma \\
&= \frac{1}{\alpha_2-\alpha_1}\left[(1-\phi)(\alpha_2-\alpha_1)\sum_{n=0}^{\infty}\phi^n\varphi((1-\phi^n)\alpha_1+\phi^n\alpha_2)\right] \\
&= \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\ {}_{\alpha_1}d_\phi\nu.
\end{aligned}$$

Hence

$$\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\ {}_{\alpha_1}d_\phi\nu \le I_1(\phi,s)\varphi(\alpha_1)+I_2(\phi,s)\varphi(\alpha_2).$$

This completes the proof. □

**Corollary 3.2.** *If one takes $s=1$ in ( 3. 6 ), than one has the following inequality*

$$\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\ {}_{\alpha_1}d_\phi\nu \le \frac{\phi\varphi(\alpha_1)+\varphi(\alpha_2)}{1+\phi}. \tag{3. 9}$$

*Proof.* It is clear, since

$$I_1(\phi,1) = (1-\phi)\sum_{n=0}^{\infty}\phi^n(1-\phi^n) = \frac{\phi}{1+\phi},$$
$$I_2(\phi,1) = (1-\phi)\sum_{n=0}^{\infty}\left(\phi^2\right)^n = \frac{1}{1+\phi}.$$

□

**Corollary 3.3.** *If one takes $\phi \to 1^-$ in ( 3. 6 ), than one has the following inequality*

$$\frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(\nu) \ {}_{\alpha_1}d_\phi \nu \le \frac{\varphi(\alpha_1) + \varphi(\alpha_2)}{s+1}. \tag{3. 10}$$

*Proof.* It is clear, since

$$\lim_{\phi \to 1^-} \int_{\alpha_1}^{\alpha_2} \varphi(\nu) \ {}_{\alpha_1}d_\phi \nu = \int_{\alpha_1}^{\alpha_2} \varphi(\nu) \, dv,$$

$$\lim_{\phi \to 1^-} I_1(\phi, s) = \lim_{\phi \to 1^-} \int_0^1 (1-\gamma)^s \ {}_0d_\phi \gamma = \int_0^1 (1-\gamma)^s \ d\gamma = \frac{1}{s+1},$$

$$\lim_{\phi \to 1^-} I_2(\phi, s) = \lim_{\phi \to 1^-} \int_0^1 \gamma^s \ {}_0d_\phi \gamma = \int_0^1 \gamma^s \ d\gamma = \frac{1}{s+1}.$$

□

**Remark 3.4.** *In inequality ( 3. 9 ), we recapture the right hand side of the inequality which is prowed in* [16, Theorem 3.2, page 5] *(see also* [10]*), in inequality ( 3. 10 ), we recapture the right hand side of the inequality ( 1. 2 ).*

Although the following Lemma has been proved in [13] and [15], we will prove it by using (3) of Theorem 2.7.

**Lemma 3.5.** *Let $\varphi : [\beta_1, \beta_2] \subset (-\infty, \infty) \to (-\infty, \infty)$ satisfy the $\phi$-differentiability condition on $(\beta_1, \beta_2)$. If ${}_{\alpha_1}D_\phi \varphi$ is continuous and $\phi$-integrable on $[\alpha_1, \alpha_2]$, $\alpha_1$, $\alpha_2 \in (\beta_1, \beta_2)$, where $0 < \phi < 1$, then*

$$\begin{aligned} \Upsilon_\phi(\alpha_1, \alpha_2)(\varphi) &:= \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \varphi(\nu) \ {}_{\alpha_1}d_\phi \nu - \frac{\phi\varphi(\alpha_1) + \varphi(\alpha_2)}{\phi + 1} \\ &= \frac{\phi(\alpha_2 - \alpha_1)}{1+\phi} \int_0^1 (1 - (1+\phi)\gamma)_{\alpha_1} D_\phi \varphi((1-\gamma)\alpha_1 + \gamma\alpha_2) \ {}_0d_\phi \gamma. \end{aligned} \tag{3. 11}$$

*Proof.* By making use of the change of variables $(1-\gamma)\alpha_1 + \gamma\alpha_2 = \nu$ and using (3) of Theorem 2.7, we have

$$\begin{aligned} &\int_0^1 (1 - (1+\phi)\gamma) \ {}_{\alpha_1}D_\phi \varphi((1-\gamma)\alpha_1 + \gamma\alpha_2) \ {}_0d_\phi \gamma \\ &= \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \left(1 - (1+\phi)\left(\frac{\nu - \alpha_1}{\alpha_2 - \alpha_1}\right)\right)_{\alpha_1} D_\phi \varphi(\nu) \ {}_{\alpha_1}d_\phi \nu \\ &= \frac{1}{\alpha_2 - \alpha_1} \left[ \left(1 - (1+\phi)\left(\frac{\nu - \alpha_1}{\alpha_2 - \alpha_1}\right)\right) \varphi(\nu) \Big|_{\alpha_1}^{\alpha_2} \right. \\ &\left. - \int_{\alpha_1}^{\alpha_2} \varphi(\phi\nu + (1-\phi)\alpha_1)_{\alpha_1} D_\phi \left(1 - (1+\phi)\left(\frac{\nu - \alpha_1}{\alpha_2 - \alpha_1}\right)\right) \ {}_{\alpha_1}d_\phi \nu \right] \\ &= -\frac{\phi\varphi(\alpha_2) + \varphi(\alpha_1)}{\alpha_2 - \alpha_1} + \frac{1+\phi}{(\alpha_2 - \alpha_1)^2} \int_{\alpha_1}^{\alpha_2} \varphi(\phi\nu + (1-\phi)\alpha_1) \ {}_{\alpha_1}d_\phi \nu. \end{aligned} \tag{3. 12}$$

Now by Definition 2.4, we have

$$\int_{\alpha_1}^{\alpha_2} \varphi\left(\phi\nu+(1-\phi)\alpha_1\right)\ {}_{\alpha_1}d_\phi\nu \tag{3. 13}$$
$$=(\alpha_2-\alpha_1)(1-\phi)\sum_{n=0}^{\infty}\phi^n\varphi\left(\phi\left(\phi^n\alpha_2+(1-\phi^n)\alpha_1\right)+(1-\phi)\alpha_1\right)$$
$$=(\alpha_2-\alpha_1)(1-\phi)\sum_{n=0}^{\infty}\phi^n\varphi\left(\phi^{n+1}\alpha_2+\left(1-\phi^{n+1}\right)\alpha_1\right)$$
$$=\frac{(\alpha_2-\alpha_1)(1-\phi)}{\phi}\sum_{n=1}^{\infty}\phi^n\varphi\left(\phi^n\alpha_2+(1-\phi^n)\alpha_1\right)$$
$$=\frac{(\alpha_2-\alpha_1)(1-\phi)}{\phi}\sum_{n=0}^{\infty}\phi^n\varphi\left(\phi^n\alpha_2+(1-\phi^n)\alpha_1\right)-\frac{(\alpha_2-\alpha_1)(1-\phi)\varphi(\alpha_2)}{\phi}$$
$$=\frac{1}{\phi}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)_{\alpha_1}d_\phi\nu-\frac{(\alpha_2-\alpha_1)(1-\phi)\varphi(\alpha_2)}{\phi}.$$

Using ( 3. 13 ) in ( 3. 12 ), we get

$$\int_0^1 \left(1-(1+\phi)\gamma\right)_{\alpha_1}D_\phi\varphi\left((1-\gamma)\alpha_1+\gamma\alpha_2\right)\ {}_0d_\phi\gamma \tag{3. 14}$$
$$=-\frac{\phi\varphi(\alpha_2)+\varphi(\alpha_1)}{\alpha_2-\alpha_1}+\frac{1+\phi}{\phi(\alpha_2-\alpha_1)^2}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\ {}_{\alpha_1}d_\phi\nu-\frac{\left(1-\phi^2\right)\varphi(\alpha_2)}{\phi(\alpha_2-\alpha_1)}$$
$$=-\frac{\phi\varphi(\alpha_1)+\varphi(\alpha_2)}{\phi(\alpha_2-\alpha_1)}+\frac{1+\phi}{\phi(\alpha_2-\alpha_1)^2}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\ {}_{\alpha_1}d_\phi\nu.$$

Multiplication on both sides of ( 3. 14 ) with $\frac{\phi(\alpha_2-\alpha_1)}{1+\phi}$, produces ( 3. 11 ). □

We are now able to present some new estimates for ( 1. 1 ) in $\phi$-calculus by using $s$-convexity of functions.

**Theorem 3.6.** *Suppose that for a function $\varphi:[\beta_1,\beta_2]\to(-\infty,\infty)$, the $\phi$-derivative exist on $(\beta_1,\beta_2)$ with $[0,\infty)\subset(\beta_1,\beta_2)$. If ${}_{\alpha_1}D_\phi\varphi$ is continuous and $\phi$-integrable on $[\alpha_1,\alpha_2]$, where $\alpha_1,\ \alpha_2\in[0,\infty)$, $\alpha_1<\alpha_2$, $0<\phi<1$ and $\left|{}_{\alpha_1}D_\phi\varphi\right|^{r_1}\in K_s^2$, $s\in(0,1]$ with $r_1\geq 1$, then*

$$\left|\Upsilon_\phi(\alpha_1,\alpha_2)(\varphi)\right|$$
$$\leq\frac{\phi(\alpha_2-\alpha_1)}{1+\phi}\left[\frac{2\phi}{(1+\phi)^2}\right]^{1-\frac{1}{r_1}}\left(I_3(\phi,s)\left|{}_{\alpha_1}D_\phi\varphi(\alpha_1)\right|^{r_1}+I_4(\phi,s)\left|{}_{\alpha_1}D_\phi\varphi(\alpha_2)\right|^{r_1}\right)^{\frac{1}{r_1}},$$

*where*

$$I_3(\phi,s)=2\frac{1-\phi}{1+\phi}\sum_{n=0}^{\infty}\phi^n(1-\phi^n)\left(1-\frac{\phi^n}{1+\phi}\right)^s$$
$$+(1-\phi)\sum_{n=0}^{\infty}\phi^n\left((1+\phi)\phi^n-1\right)(1-\phi^n)^s,$$
$$I_4(\phi,s)=2\frac{1-\phi}{1+\phi}\sum_{n=0}^{\infty}\phi^n(1-\phi^n)\left(\frac{\phi^n}{1+\phi}\right)^s+(1-\phi)\sum_{n=0}^{\infty}\phi^n\left[\left((1+\phi)\phi^n-1\right)\phi^{ns}\right].$$

*Proof.* Considering ( 3. 11 ), taking the absolute value on both sides and by the application of the Hölder's inequality, gives

$$\left|\Upsilon_{\phi}\left(\alpha_1,\alpha_2\right)(\varphi)\right| \le \frac{\phi\left(\alpha_2-\alpha_1\right)}{1+\phi}\left(\int_0^1 \left|1-(1+\phi)\gamma\right| \ {}_0d_{\phi}\gamma\right)^{1-\frac{1}{r_1}} \tag{3. 15}$$
$$\times\left(\int_0^1 \left|1-(1+\phi)\gamma\right|\left|{}_{\alpha_1}D_{\phi}\varphi\left((1-\gamma)\alpha_1+\gamma\alpha_2\right)\right|^{r_1} \ {}_0d_{\phi}\gamma\right)^{\frac{1}{r_1}}.$$

Since $\left|{}_{\alpha_1}D_{\phi}\varphi\right|^{r_1}$ is $s$-convex, $s\in(0,1]$, we have

$$\int_0^1 \left|1-(1+\phi)\gamma\right|\left|{}_{\alpha_1}D_{\phi}\varphi\left((1-\gamma)\alpha_1+\gamma\alpha_2\right)\right|^{r_1} \ {}_0d_{\phi}\gamma \tag{3. 16}$$
$$\le \left|{}_{\alpha_1}D_{\phi}\varphi\left(\alpha_1\right)\right|^{r_1}\int_0^1 \left|1-(1+\phi)\gamma\right|(1-\gamma)^s \ {}_0d_{\phi}\gamma$$
$$+\left|{}_{\alpha_1}D_{\phi}\varphi\left(\alpha_2\right)\right|^{r_1}\int_0^1 \left|1-(1+\phi)\gamma\right|\gamma^s \ {}_0d_{\phi}\gamma.$$

We also have

$$\int_0^1 \left|1-(1+\phi)\gamma\right| \ {}_0d_{\phi}\gamma \tag{3. 17}$$
$$=\int_0^{\frac{1}{1+\phi}}\left(1-(1+\phi)\gamma\right) \ {}_0d_{\phi}\gamma+\int_{\frac{1}{1+\phi}}^1\left((1+\phi)\gamma-1\right) \ {}_0d_{\phi}\gamma=\frac{2\phi}{(1+\phi)^2}.$$

Now we calculate the other $\phi$-integrals involved in ( 3. 16 ) as follows

$$\int_0^1 \left|1-(1+\phi)\gamma\right|(1-\gamma)^s \ {}_0d_{\phi}\gamma \tag{3. 18}$$
$$=\int_0^{\frac{1}{1+\phi}}\left(1-(1+\phi)\gamma\right)(1-\gamma)^s \ {}_0d_{\phi}\gamma+\int_{\frac{1}{1+\phi}}^1\left((1+\phi)\gamma-1\right)(1-\gamma)^s \ {}_0d_{\phi}\gamma$$
$$=\int_0^{\frac{1}{1+\phi}}\left(1-(1+\phi)\gamma\right)(1-\gamma)^s \ {}_0d_{\phi}\gamma$$
$$+\int_0^1\left((1+\phi)\gamma-1\right)(1-\gamma)^s \ {}_0d_{\phi}\gamma-\int_0^{\frac{1}{1+\phi}}\left((1+\phi)\gamma-1\right)(1-\gamma)^s \ {}_0d_{\phi}\gamma$$
$$=2\int_0^{\frac{1}{1+\phi}}\left(1-(1+\phi)\gamma\right)(1-\gamma)^s \ {}_0d_{\phi}\gamma+\int_0^1\left((1+\phi)\gamma-1\right)(1-\gamma)^s \ {}_0d_{\phi}\gamma$$
$$=2\frac{1-\phi}{1+\phi}\sum_{n=0}^{\infty}\phi^n\left(1-\phi^n\right)\left(1-\frac{\phi^n}{1+\phi}\right)^s+(1-\phi)\sum_{n=0}^{\infty}\phi^n\left((1+\phi)\phi^n-1\right)\left(1-\phi^n\right)^s$$
$$=I_3(\phi,s),$$

and

$$\begin{aligned}
&\int_0^1 \left|1-(1+\phi)\gamma\right|\gamma^s\ {}_0d_\phi\gamma \qquad (3.\ 19)\\
&=\int_0^{\frac{1}{1+\phi}}\left(1-(1+\phi)\gamma\right)\gamma^s\ {}_0d_\phi\gamma+\int_{\frac{1}{1+\phi}}^1\left((1+\phi)\gamma-1\right)\gamma^s\ {}_0d_\phi\gamma\\
&=\int_0^{\frac{1}{1+\phi}}\left(1-(1+\phi)\gamma\right)\gamma^s\ {}_0d_\phi\gamma\\
&+\int_0^1\left((1+\phi)\gamma-1\right)\gamma^s\ {}_0d_\phi\gamma-\int_0^{\frac{1}{1+\phi}}\left((1+\phi)\gamma-1\right)\gamma^s\ {}_0d_\phi\gamma\\
&=2\int_0^{\frac{1}{1+\phi}}\left(1-(1+\phi)\gamma\right)\gamma^s\ {}_0d_\phi\gamma+\int_0^1\left((1+\phi)\gamma-1\right)\gamma^s\ {}_0d_\phi\gamma\\
&=2\frac{1-\phi}{1+\phi}\sum_{n=0}^{\infty}\phi^n\left(1-\phi^n\right)\left(\frac{\phi^n}{1+\phi}\right)^s+(1-\phi)\sum_{n=0}^{\infty}\phi^n\left[\left((1+\phi)\phi^n-1\right)\phi^{ns}\right]\\
&=I_4(\phi,s).
\end{aligned}$$

Applying ( 3. 16 )-( 3. 19 ) in ( 3. 15 ), we get the required inequality. □

**Corollary 3.7.** *If one takes $s=1$ in Theorem 3.6, than one has the following inequality*

$$\begin{aligned}
&\left|\Upsilon_\phi(\alpha_1,\alpha_2)(\varphi)\right| \qquad (3.\ 20)\\
&\le\frac{\phi(\alpha_2-\alpha_1)}{1+\phi}\left[\frac{2\phi}{(1+\phi)^2}\right]^{1-\frac{1}{r_1}}\left(\frac{\phi\left(1+3\phi^2+2\phi^3\right)}{(1+\phi)^3\left(1+\phi+\phi^2\right)}\left|{}_{\alpha_1}D_\phi\varphi(\alpha_1)\right|^{r_1}\right.\\
&\left.+\frac{\phi\left(1+4\phi+\phi^2\right)}{(1+\phi)^3\left(1+\phi+\phi^2\right)}\left|{}_{\alpha_1}D_\phi\varphi(\alpha_2)\right|^{r_1}\right)^{\frac{1}{r_1}},
\end{aligned}$$

*Proof.* It is clear, since

$$\begin{aligned}
I_3(\phi,1)&=2\frac{1-\phi}{1+\phi}\sum_{n=0}^{\infty}\phi^n\left(1-\phi^n\right)\left(1-\frac{\phi^n}{1+\phi}\right)+(1-\phi)\sum_{n=0}^{\infty}\phi^n\left((1+\phi)\phi^n-1\right)\left(1-\phi^n\right)\\
&=\frac{\phi\left(1+3\phi^2+2\phi^3\right)}{(1+\phi)^3\left(1+\phi+\phi^2\right)},
\end{aligned}$$

$$\begin{aligned}
I_4(\phi,1)&=2\frac{1-\phi}{1+\phi}\sum_{n=0}^{\infty}\phi^n\left(1-\phi^n\right)\left(\frac{\phi^n}{1+\phi}\right)+(1-\phi)\sum_{n=0}^{\infty}\phi^n\left[\left((1+\phi)\phi^n-1\right)\phi^n\right]\\
&=\frac{\phi\left(1+4\phi+\phi^2\right)}{(1+\phi)^3\left(1+\phi+\phi^2\right)}.
\end{aligned}$$

□

**Corollary 3.8.** *If one takes $\phi \to 1^-$ in Theorem 3.6, than one has the following inequality*

$$\left|\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\,d\nu-\frac{\varphi(\alpha_1)+\varphi(\alpha_2)}{2}\right| \tag{3. 21}$$
$$\le\frac{(\alpha_2-\alpha_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{r_1}}\left[\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^{\frac{1}{r_1}}\left(\left|\varphi'(\alpha_1)\right|^{r_1}+\left|\varphi'(\alpha_2)\right|^{r_1}\right)^{\frac{1}{r_1}}.$$

*Proof.* It is clear, since

$$\lim_{\phi\to1^-}\left|\Upsilon_\phi(\alpha_1,\alpha_2)(\varphi)\right|=\left|\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\,d\nu-\frac{\varphi(\alpha_1)+\varphi(\alpha_2)}{2}\right|,$$

$$\lim_{\phi\to1^-}I_3(\phi,s)=\lim_{\phi\to1^-}\int_0^1\left|1-(1+\phi)\gamma\right|(1-\gamma)^s\ {}_0d_\phi\gamma$$
$$=\int_0^1\left|1-2\gamma\right|(1-\gamma)^s\ d\gamma=\left[\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^{\frac{1}{r_1}},$$

$$\lim_{\phi\to1^-}I_4(\phi,s)=\lim_{\phi\to1^-}\int_0^1\left|1-(1+\phi)\gamma\right|\gamma^s\ {}_0d_\phi\gamma$$
$$=\int_0^1\left|1-2\gamma\right|\gamma^s\ d\gamma=\left[\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^{\frac{1}{r_1}},$$

$$\lim_{\phi\to1^-}\left|{}_{\alpha_1}D_\phi\varphi(\alpha_1)\right|^{r_1}=\left|\varphi'(\alpha_1)\right|^{r_1},$$

$$\lim_{\phi\to1^-}\left|{}_{\alpha_1}D_\phi\varphi(\alpha_2)\right|^{r_1}=\left|\varphi'(\alpha_2)\right|^{r_1}.$$

□

**Remark 3.9.** *In inequality ( 3. 20 ), we recapture the inequality proved in* [13, Theorem 3.2, page 677] *and* [15, Theorem 4.2, page 786] *(see also* [11, 12]*). In inequality ( 3. 21 ), we recapture the inequality proved in* [9, Theorem 1, page 28].

**Theorem 3.10.** *Suppose that for a function $\varphi:[\beta_1,\beta_2]\to(-\infty,\infty)$, the $\phi$-derivative exist on $(\beta_1,\beta_2)$ with $[0,\infty)\subset(\beta_1,\beta_2)$. If ${}_{\alpha_1}D_\phi\varphi$ is continuous and $\phi$-integrable on $[\alpha_1,\alpha_2]$, where $\alpha_1$, $\alpha_2\in[0,\infty)$, $\alpha_1<\alpha_2$, $0<\phi<1$ and $\left|{}_{\alpha_1}D_\phi\varphi\right|^{r_1}\in K_s^2$ for some fixed $s\in(0,1]$ with $r_1,r_2>1$, then*

$$\left|\Upsilon_\phi(\alpha_1,\alpha_2)(\varphi)\right| \tag{3. 22}$$
$$\le\frac{\phi(\alpha_2-\alpha_1)}{1+\phi}\left(I_5(\phi)\right)^{\frac{1}{r_2}}\left(\left|{}_{\alpha_1}D_\phi\varphi(\alpha_1)\right|^{r_1}I_1(\phi,s)+\left|{}_{\alpha_1}D_\phi\varphi(\alpha_2)\right|^{r_1}I_2(\phi,s)\right)^{\frac{1}{r_1}},$$

*where $r_1$ and $r_2$ are Hölder conjugates of each other, and*

$$I_5(\phi)=\frac{1-\phi}{1+\phi}\left[\sum_{n=0}^{\infty}\phi^n\left(1-\phi^n\right)^{r_2}+(1+\phi)\sum_{n=0}^{\infty}\phi^n\left((1+\phi)\phi^n-1\right)^{r_2}-\sum_{n=0}^{\infty}\phi^n\left(\phi^n-1\right)^{r_2}\right].$$

*Proof.* Taking absolute value on both sides of ( 3. 11 ) and using Hölder inequality, we have

$$|\Upsilon_\phi(\alpha_1,\alpha_2)(\varphi)| \leq \frac{\phi(\alpha_2-\alpha_1)}{1+\phi}\left(\int_0^1 |1-(1+\phi)\gamma|^{r_2}\ {}_0d_\phi\gamma\right)^{\frac{1}{r_2}} \times\left(\int_0^1 |{}_{\alpha_1}D_\phi\varphi((1-\gamma)\alpha_1+\gamma\alpha_2)|^{r_1}\ {}_0d_\phi\gamma\right)^{\frac{1}{r_1}}. \tag{3. 23}$$

We now evaluate the integrals involved in ( 3. 23 ), we get

$$\begin{aligned}
&\int_0^1 |1-(1+\phi)\gamma|^{r_2}\ {}_0d_\phi\gamma \\
&= \int_0^{\frac{1}{1+\phi}} (1-(1+\phi)\gamma)^{r_2}\ {}_0d_\phi\gamma + \int_{\frac{1}{1+\phi}}^1 ((1+\phi)\gamma-1)^{r_2}\ {}_0d_\phi\gamma \\
&= \int_0^{\frac{1}{1+\phi}} (1-(1+\phi)\gamma)^{r_2}\ {}_0d_\phi\gamma + \int_0^1 ((1+\phi)\gamma-1)^{r_2}\ {}_0d_\phi\gamma \\
&- \int_0^{\frac{1}{1+\phi}} ((1+\phi)\gamma-1)^{r_2}\ {}_0d_\phi\gamma \\
&= \frac{1-\phi}{1+\phi}\left[\sum_{n=0}^\infty \phi^n(1-\phi^n)^{r_2} + (1+\phi)\sum_{n=0}^\infty \phi^n((1+\phi)\phi^n-1)^{r_2} - \sum_{n=0}^\infty \phi^n(\phi^n-1)^{r_2}\right] \\
&= I_5(\phi).
\end{aligned} \tag{3. 24}$$

Using ( 3. 7 ), ( 3. 8 ), and the $s$-convexity of $|{}_{\alpha_1}D_\phi\varphi|^{r_1}$ for some fixed $s\in(0,1]$, we have

$$\begin{aligned}
&\int_0^1 |{}_{\alpha_1}D_\phi\varphi((1-\gamma)\alpha_1+\gamma\alpha_2)|^{r_1}\ {}_0d_\phi\gamma \\
&\leq |{}_{\alpha_1}D_\phi\varphi(\alpha_1)|^{r_1}\int_0^1 (1-\gamma)^s\ {}_0d_\phi\gamma + |{}_{\alpha_1}D_\phi\varphi(\alpha_2)|^{r_1}\int_0^1 \gamma^s\ {}_0d_\phi\gamma \\
&= |{}_{\alpha_1}D_\phi\varphi(\alpha_1)|^{r_1} I_1(\phi,s) + |{}_{\alpha_1}D_\phi\varphi(\alpha_2)|^{r_1} I_2(\phi,s).
\end{aligned} \tag{3. 25}$$

Making use of ( 3. 24 ) and ( 3. 25 ) in ( 3. 23 ), we get the required result. □

**Corollary 3.11.** *If one takes $s=1$ in Theorem 3.10, than one has the following inequality*

$$\begin{aligned}
&|\Upsilon_\phi(\alpha_1,\alpha_2)(\varphi)| \\
&\leq \frac{\phi(\alpha_2-\alpha_1)}{1+\phi}(I_5(\phi))^{\frac{1}{r_2}}\left(\frac{\phi|{}_{\alpha_1}D_\phi\varphi(\alpha_1)|^{r_1}+|{}_{\alpha_1}D_\phi\varphi(\alpha_2)|^{r_1}}{1+\phi}\right)^{\frac{1}{r_1}},
\end{aligned} \tag{3. 26}$$

*Proof.* It is clear, since

$$I_1(\phi,1) = (1-\phi)\sum_{n=0}^\infty \phi^n(1-\phi^n) = \frac{\phi}{1+\phi},$$

$$I_2(\phi,1) = (1-\phi)\sum_{n=0}^\infty (\phi^2)^n = \frac{1}{1+\phi}.$$

□

**Corollary 3.12.** *If one takes $\phi \to 1^-$ in Theorem 3.10, than one has the following inequality*

$$\left|\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\,d\nu-\frac{\varphi(\alpha_1)+\varphi(\alpha_2)}{2}\right| \tag{3. 27}$$
$$\leq \frac{(\alpha_2-\alpha_1)}{2}\left(\frac{1}{r_2+1}\right)^{\frac{1}{r_2}}\left(\frac{|\varphi'(\alpha_1)|^{r_1}+|\varphi'(\alpha_2)|^{r_1}}{s+1}\right)^{\frac{1}{r_1}}.$$

*Proof.* It is clear, since

$$\lim_{\phi\to 1^-}|\Upsilon_\phi(\alpha_1,\alpha_2)(\varphi)|=\left|\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\,d\nu-\frac{\varphi(\alpha_1)+\varphi(\alpha_2)}{2}\right|,$$

$$\begin{aligned}\lim_{\phi\to 1^-}I_5(\phi)&=\lim_{\phi\to 1^-}\int_0^1|1-(1+\phi)\gamma|^{r_2}\ {}_0d_\phi\gamma\\&=\int_0^1|1-2\gamma|^{r_2}\ d\gamma=\int_0^{\frac{1}{2}}(1-2\gamma)^{r_2}\ d\gamma+\int_{\frac{1}{2}}^1(2\gamma-1)^{r_2}\ d\gamma\\&=\frac{1}{2(r_2+1)}+\frac{1}{2(r_2+1)}=\frac{1}{r_2+1},\end{aligned}$$

$$\lim_{\phi\to 1^-}I_1(\phi,s)=\lim_{\phi\to 1^-}\int_0^1(1-\gamma)^s\ {}_0d_\phi\gamma=\int_0^1(1-\gamma)^s\ d\gamma=\frac{1}{s+1},$$

$$\lim_{\phi\to 1^-}I_2(\phi,s)=\lim_{\phi\to 1^-}\int_0^1\gamma^s\ {}_0d_\phi\gamma=\int_0^1\gamma^s\ d\gamma=\frac{1}{s+1},$$

$$\lim_{\phi\to 1^-}|{}_{\alpha_1}D_\phi\varphi(\alpha_1)|^{r_1}=|\varphi'(\alpha_1)|^{r_1},$$

$$\lim_{\phi\to 1^-}|{}_{\alpha_1}D_\phi\varphi(\alpha_2)|^{r_1}=|\varphi'(\alpha_2)|^{r_1}.$$

□

**Theorem 3.13.** *Suppose that for a function $\varphi:[\beta_1,\beta_2]\to(-\infty,\infty)$, the $\phi$-derivative exist on $(\beta_1,\beta_2)$ with $[0,\infty)\subset(\beta_1,\beta_2)$. If ${}_{\alpha_1}D_\phi\varphi$ is continuous and $\phi$-integrable on $[\alpha_1,\alpha_2]$, where $\alpha_1$, $\alpha_2\in[0,\infty)$, $\alpha_1<\alpha_2$, $0<\phi<1$ and $|{}_{\alpha_1}D_\phi\varphi|^{r_1}\in K_s^2$ for some fixed $s\in(0,1]$ with $r_1,r_2>1$, then*

$$|\Upsilon_\phi(\alpha_1,\alpha_2)(\varphi)| \tag{3. 28}$$
$$\leq\frac{\phi(\alpha_2-\alpha_1)}{1+\phi}(I_6(\phi)\ )^{\frac{1}{r_2}}\left(|{}_{\alpha_1}D_\phi\varphi(\alpha_1)|^{r_1}I_8(\phi,s)+|{}_{\alpha_1}D_\phi\varphi(\alpha_2)|^{r_1}I_9(\phi,s)\right)^{\frac{1}{r_1}}$$
$$+\frac{\phi(\alpha_2-\alpha_1)}{1+\phi}(I_7(\phi)\ )^{\frac{1}{r_2}}\left(|{}_{\alpha_1}D_\phi\varphi(\alpha_1)|^{r_1}I_{10}(\phi,s)+|{}_{\alpha_1}D_\phi\varphi(\alpha_2)|^{r_1}I_{11}(\phi,s)\right)^{\frac{1}{r_1}},$$

*where $r_1$ and $r_2$ are Hölder conjugates of each other, and*

$$I_6(\phi)=\frac{1-\phi}{1+\phi}\sum_{n=0}^{\infty}\phi^n(1-\phi^n)^{r_2},$$

$$I_7(\phi)=(1-\phi)\sum_{n=0}^{\infty}\phi^n((1+\phi)\phi^n-1)^{r_2}-\frac{1-\phi}{1+\phi}\sum_{n=0}^{\infty}\phi^n(\phi^n-1)^{r_2},$$

$$I_8(\phi,s)=\int_0^{\frac{1}{1+\phi}}(1-\gamma)^s\ {}_0d_\phi\gamma,$$

$$I_9(\phi,s)=\int_0^{\frac{1}{1+\phi}}\gamma^s\ {}_0d_\phi\gamma,$$

$$I_{10}(\phi,s)=\int_{\frac{1}{1+\phi}}^1(1-\gamma)^s\ {}_0d_\phi\gamma,$$

$$I_{11}(\phi,s)=\int_{\frac{1}{1+\phi}}^1\gamma_0^s d_\phi\gamma.$$

*Proof.* Taking absolute value on both sides of ( 3. 11 ) and using Hölder inequality, we have

$$|\Upsilon_\phi(\alpha_1,\alpha_2)(\varphi)|\le\frac{\phi(\alpha_2-\alpha_1)}{1+\phi}\int_0^1|1-(1+\phi)\gamma|\,|_{\alpha_1}D_\phi\varphi((1-\gamma)\alpha_1+\gamma\alpha_2)|\ {}_0d_\phi\gamma \tag{3. 29}$$

$$\begin{aligned}&=\frac{\phi(\alpha_2-\alpha_1)}{1+\phi}\int_0^{\frac{1}{1+\phi}}(1-(1+\phi)\gamma)\,|_{\alpha_1}D_\phi\varphi((1-\gamma)\alpha_1+\gamma\alpha_2)|\ {}_0d_\phi\gamma\\&+\frac{\phi(\alpha_2-\alpha_1)}{1+\phi}\int_{\frac{1}{1+\phi}}^1((1+\phi)\gamma-1)\,|_{\alpha_1}D_\phi\varphi((1-\gamma)\alpha_1+\gamma\alpha_2)|\ {}_0d_\phi\gamma\\&\le\frac{\phi(\alpha_2-\alpha_1)}{1+\phi}\left(\int_0^{\frac{1}{1+\phi}}(1-(1+\phi)\gamma)^{r_2}\ {}_0d_\phi\gamma\right)^{\frac{1}{r_2}}\\&\times\left(\int_0^{\frac{1}{1+\phi}}|_{\alpha_1}D_\phi\varphi((1-\gamma)\alpha_1+\gamma\alpha_2)|^{r_1}\ {}_0d_\phi\gamma\right)^{\frac{1}{r_1}}\\&+\frac{\phi(\alpha_2-\alpha_1)}{1+\phi}\left(\int_{\frac{1}{1+\phi}}^1((1+\phi)\gamma-1)^{r_2}\ {}_0d_\phi\gamma\right)^{\frac{1}{r_2}}\\&\times\left(\int_{\frac{1}{1+\phi}}^1|_{\alpha_1}D_\phi\varphi((1-\gamma)\alpha_1+\gamma\alpha_2)|^{r_1}\ {}_0d_\phi\gamma\right)^{\frac{1}{r_1}}.\end{aligned}$$

We now evaluate the integrals involved in ( 3. 29 ), we get

$$\int_0^{\frac{1}{1+\phi}}(1-(1+\phi)\gamma)^{r_2}\ {}_0d_\phi\gamma=\frac{1-\phi}{1+\phi}\sum_{n=0}^{\infty}\phi^n(1-\phi^n)^{r_2}=I_6(\phi) \tag{3. 30}$$

and

$$\begin{aligned}&\int_{\frac{1}{1+\phi}}^1((1+\phi)\gamma-1)^{r_2}\ {}_0d_\phi\gamma \qquad (3.\ 31)\\&=(1-\phi)\sum_{n=0}^{\infty}\phi^n((1+\phi)\phi^n-1)^{r_2}-\frac{1-\phi}{1+\phi}\sum_{n=0}^{\infty}\phi^n(\phi^n-1)^{r_2}=I_7(\phi).\end{aligned}$$

Moreover, since $\left|{}_{\alpha_1}D_\phi\varphi\right|^{r_1} \in K_s^2$, we get

$$\int_0^{\frac{1}{1+\phi}} \left|{}_{\alpha_1}D_\phi\varphi\left((1-\gamma)\alpha_1+\gamma\alpha_2\right)\right|^{r_1} \ {}_0d_\phi\gamma \tag{3. 32}$$
$$\leq \left|{}_{\alpha_1}D_\phi\varphi(\alpha_1)\right|^{r_1} \int_0^{\frac{1}{1+\phi}} (1-\gamma)^s \ {}_0d_\phi\gamma + \left|{}_{\alpha_1}D_\phi\varphi(\alpha_2)\right|^{r_1} \int_0^{\frac{1}{1+\phi}} \gamma^s \ {}_0d_\phi\gamma$$
$$= \left|{}_{\alpha_1}D_\phi\varphi(\alpha_1)\right|^{r_1} I_8(\phi,s) + \left|{}_{\alpha_1}D_\phi\varphi(\alpha_2)\right|^{r_1} I_9(\phi,s),$$

and

$$\int_{\frac{1}{1+\phi}}^1 \left|{}_{\alpha_1}D_\phi\varphi\left((1-\gamma)\alpha_1+\gamma\alpha_2\right)\right|^{r_1} \ {}_0d_\phi\gamma \tag{3. 33}$$
$$\leq \left|{}_{\alpha_1}D_\phi\varphi(\alpha_1)\right|^{r_1} \int_{\frac{1}{1+\phi}}^1 (1-\gamma)^s \ {}_0d_\phi\gamma + \left|{}_{\alpha_1}D_\phi\varphi(\alpha_2)\right|^{r_1} \int_{\frac{1}{1+\phi}}^1 \gamma^s \ {}_0d_\phi\gamma$$
$$= \left|{}_{\alpha_1}D_\phi\varphi(\alpha_1)\right|^{r_1} I_{10}(\phi,s) + \left|{}_{\alpha_1}D_\phi\varphi(\alpha_2)\right|^{r_1} I_{11}(\phi,s).$$

Using ( 3. 30 )-( 3. 33 ) in ( 3. 29 ), we get the desired result. □

**Corollary 3.14.** *If one takes $s = 1$ in Theorem 3.13, than one has the following inequality*

$$\left|\Upsilon_\phi(\alpha_1,\alpha_2)(\varphi)\right|$$
$$\leq \frac{\phi(\alpha_2-\alpha_1)}{1+\phi}\left(I_6(\phi)\right)^{\frac{1}{r_2}}\left(\left|{}_{\alpha_1}D_\phi\varphi(\alpha_1)\right|^{r_1}\frac{2\phi+\phi^2}{(1+\phi)^3}+\left|{}_{\alpha_1}D_\phi\varphi(\alpha_2)\right|^{r_1}\frac{1}{(1+\phi)^3}\right)^{\frac{1}{r_1}}$$
$$+\frac{\phi(\alpha_2-\alpha_1)}{1+\phi}\left(I_7(\phi)\right)^{\frac{1}{r_2}}\left(\left|{}_{\alpha_1}D_\phi\varphi(\alpha_1)\right|^{r_1}\frac{-\phi+\phi^2+\phi^3}{(1+\phi)^3}+\left|{}_{\alpha_1}D_\phi\varphi(\alpha_2)\right|^{r_1}\frac{2\phi+\phi^2}{(1+\phi)^3}\right)^{\frac{1}{r_1}}$$

*Proof.* It is clear, since

$$I_8(\phi,1) = \int_0^{\frac{1}{1+\phi}} (1-\gamma) \ {}_0d_\phi\gamma = \frac{2\phi+\phi^2}{(1+\phi)^3},$$

$$I_9(\phi,1) = \int_0^{\frac{1}{1+\phi}} \gamma \ {}_0d_\phi\gamma = \frac{1}{(1+\phi)^3},$$

$$I_{10}(\phi,1) = \int_{\frac{1}{1+\phi}}^1 (1-\gamma) \ {}_0d_\phi\gamma = \frac{-\phi+\phi^2+\phi^3}{(1+\phi)^3},$$

$$I_{11}(\phi,1) = \int_{\frac{1}{1+\phi}}^1 \gamma \ {}_0d_\phi\gamma = \frac{2\phi+\phi^2}{(1+\phi)^3}.$$

□

**Corollary 3.15.** *If one takes $\phi \to 1^-$ in Theorem 3.13, than one has the following inequality*

$$\left|\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\,d\nu-\frac{\varphi(\alpha_1)+\varphi(\alpha_2)}{2}\right|$$
$$\leq \frac{\phi(\alpha_2-\alpha_1)}{1+\phi}\left(\frac{1}{2(r_2+1)}\right)^{\frac{1}{r_2}}\left(|\varphi'(\alpha_1)|^{r_1}\frac{1-\left(\frac{1}{2}\right)^{s+1}}{s+1}+|\varphi'(\alpha_2)|^{r_1}\frac{\left(\frac{1}{2}\right)^{s+1}}{s+1}\right)^{\frac{1}{r_1}}$$
$$+\frac{\phi(\alpha_2-\alpha_1)}{1+\phi}\left(\frac{1}{2(r_2+1)}\right)^{\frac{1}{r_2}}\left(|\varphi'(\alpha_1)|^{r_1}\frac{\left(\frac{1}{2}\right)^{s+1}}{s+1}+|\varphi'(\alpha_2)|^{r_1}\frac{1-\left(\frac{1}{2}\right)^{s+1}}{s+1}\right)^{\frac{1}{r_1}}.$$

*Proof.* It is clear, since

$$\lim_{\phi\to1^-}|\Upsilon_\phi(\alpha_1,\alpha_2)(\varphi)|=\left|\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\,d\nu-\frac{\varphi(\alpha_1)+\varphi(\alpha_2)}{2}\right|,$$

$$\lim_{\phi\to1^-}I_6(\phi)=\lim_{\phi\to1^-}\int_0^{\frac{1}{1+\phi}}(1-(1+\phi)\gamma)^{r_2}\ {}_0d_\phi\gamma=\int_0^{\frac{1}{2}}(1-2\gamma)^{r_2}\,d\gamma=\frac{1}{2(r_2+1)},$$
$$\lim_{\phi\to1^-}I_7(\phi)=\lim_{\phi\to1^-}\int_{\frac{1}{1+\phi}}^1((1+\phi)\gamma-1)^{r_2}\ {}_0d_\phi\gamma=\int_{\frac{1}{2}}^1(2\gamma-1)^{r_2}\,d\gamma=\frac{1}{2(r_2+1)},$$
$$\lim_{\phi\to1^-}I_8(\phi,s)=\lim_{\phi\to1^-}\int_0^{\frac{1}{1+\phi}}(1-\gamma)^s\ {}_0d_\phi\gamma=\int_0^{\frac{1}{2}}(1-\gamma)^s\,d\gamma=\frac{1-\left(\frac{1}{2}\right)^{s+1}}{s+1},$$
$$\lim_{\phi\to1^-}I_9(\phi,s)=\lim_{\phi\to1^-}\int_0^{\frac{1}{1+\phi}}\gamma^s\ {}_0d_\phi\gamma=\int_0^{\frac{1}{2}}\gamma^s d\gamma=\frac{\left(\frac{1}{2}\right)^{s+1}}{s+1},$$
$$\lim_{\phi\to1^-}I_{10}(\phi,s)=\lim_{\phi\to1^-}\int_{\frac{1}{1+\phi}}^1(1-\gamma)^s\ {}_0d_\phi\gamma=\int_{\frac{1}{2}}^1(1-\gamma)^s\,d\gamma=\frac{\left(\frac{1}{2}\right)^{s+1}}{s+1},$$
$$\lim_{\phi\to1^-}I_{11}(\phi,s)=\lim_{\phi\to1^-}\int_{\frac{1}{1+\phi}}^1\gamma^s\ {}_0d_\phi\gamma=\int_{\frac{1}{2}}^1\gamma^s\ {}_0d_\phi\gamma=\frac{1-\left(\frac{1}{2}\right)^{s+1}}{s+1}.$$
$$\lim_{\phi\to1^-}|{}_{\alpha_1}D_\phi\varphi(\alpha_1)|^{r_1}=|\varphi'(\alpha_1)|^{r_1},$$
$$\lim_{\phi\to1^-}|{}_{\alpha_1}D_\phi\varphi(\alpha_2)|^{r_1}=|\varphi'(\alpha_2)|^{r_1}.$$

□

## 4. Inequalities for products of two $\phi$-integrable functions

**Theorem 4.1.** *Let $\varphi,\psi:[0,\infty)\to(-\infty,\infty)$, $\alpha_1,\alpha_2\in[0,\infty)$, $\alpha_1<\alpha_2$, be functions such that $\varphi$, $\psi$ and $\varphi\psi$ are $\phi$-integrable over $[\alpha_1,\alpha_2]$, $0<\phi<1$. If $\varphi$ is non-negative and convex on $[\alpha_1,\alpha_2]$, if $\psi$ is non-negative and $\psi\in K_s^2$ for some fixed $s\in(0,1]$, then*

$$\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\psi(\nu)\ {}_{\alpha_1}d_\phi\nu \tag{4. 34}$$
$$\leq\varphi(\alpha_2)\psi(\alpha_2)\int_0^1\gamma^{s+1}\ {}_0d_\phi\gamma+\varphi(\alpha_1)\psi(\alpha_1)\int_0^1(1-\gamma)^{s+1}\ {}_0d_\phi\gamma$$
$$+\varphi(\alpha_2)\psi(\alpha_1)\int_0^1\gamma(1-\gamma)^s\ {}_0d_\phi\gamma+\varphi(\alpha_1)\psi(\alpha_2)\int_0^1\gamma^s(1-\gamma)\ {}_0d_\phi\gamma.$$

*Proof.* By the convexity of $\varphi$ on $[\alpha_1, \alpha_2]$ and $s$-convexity of $\psi$, we have

$$\varphi\left(\gamma\alpha_2 + (1-\gamma)\,\alpha_1\right) \leq \gamma\varphi\left(\alpha_2\right) + (1-\gamma)\,\varphi\left(\alpha_1\right)$$

and

$$\psi\left(\gamma\alpha_2 + (1-\gamma)\,\alpha_1\right) \leq \gamma^s\psi\left(\alpha_2\right) + (1-\gamma)^s\,\psi\left(\alpha_1\right)$$

for all $\gamma \in [0,1]$. Since $\varphi$ and $\psi$ are non-negative, we have

$$\begin{aligned}&\varphi\left(\gamma\alpha_2 + (1-\gamma)\,\alpha_1\right)\psi\left(\gamma\alpha_2 + (1-\gamma)\,\alpha_1\right)\\ &\leq \gamma^{s+1}\varphi\left(\alpha_2\right)\psi\left(\alpha_2\right) + \gamma\left(1-\gamma\right)^s\varphi\left(\alpha_2\right)\psi\left(\alpha_1\right)\\ &+\gamma^s\left(1-\gamma\right)\varphi\left(\alpha_1\right)\psi\left(\alpha_2\right) + \left(1-\gamma\right)^{s+1}\varphi\left(\alpha_1\right)\psi\left(\alpha_1\right).\end{aligned}$$

By $\phi$-integration on both sides of the above inequality on $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 \varphi\left(\gamma\alpha_2 + (1-\gamma)\,\alpha_1\right)\psi\left(\gamma\alpha_2 + (1-\gamma)\,\alpha_1\right)\ {}_0d_\phi\gamma \\ &\leq \varphi\left(\alpha_2\right)\psi\left(\alpha_2\right)\int_0^1 \gamma^{s+1}\ {}_0d_\phi\gamma + \varphi\left(\alpha_2\right)\psi\left(\alpha_1\right)\int_0^1 \gamma\left(1-\gamma\right)^s\ {}_0d_\phi\gamma \\ &+\varphi\left(\alpha_1\right)\psi\left(\alpha_2\right)\int_0^1 \gamma^s\left(1-\gamma\right)\ {}_0d_\phi\gamma + \varphi\left(\alpha_1\right)\psi\left(\alpha_1\right)\int_0^1 \left(1-\gamma\right)^{s+1}\ {}_0d_\phi\gamma.\end{aligned} \tag{4. 35}$$

By the definition of $\phi$-integral, we have

$$\int_0^1 \varphi\left(\gamma\alpha_2 + (1-\gamma)\,\alpha_1\right)\psi\left(\gamma\alpha_2 + (1-\gamma)\,\alpha_1\right)\ {}_0d_\phi\gamma = \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi\left(\nu\right)\psi\left(\nu\right)\ {}_{\alpha_1}d_\phi\nu. \tag{4. 36}$$

Applying ( 4. 36 ) in ( 4. 35 ), we get the desired inequality. □

**Corollary 4.2.** *If one takes $s = 1$ in Theorem 4.1, than one has the following inequality*

$$\begin{aligned}&\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi\left(\nu\right)\psi\left(\nu\right)\ {}_{\alpha_1}d_\phi\nu \\ &\leq \varphi\left(\alpha_2\right)\psi\left(\alpha_2\right)\frac{1}{1+\phi+\phi^2} + \varphi\left(\alpha_1\right)\psi\left(\alpha_1\right)\frac{\phi+\phi^3}{(1+\phi)\left(1+\phi+\phi^2\right)} \\ &+ N\left(\alpha_1,\alpha_2\right)\frac{\phi^2}{(1+\phi)\left(1+\phi+\phi^2\right)}\end{aligned} \tag{4. 37}$$

*where* $N\left(\alpha_1,\alpha_2\right) = \varphi\left(\alpha_1\right)\psi\left(\alpha_2\right) + \varphi\left(\alpha_2\right)\psi\left(\alpha_1\right).$

*Proof.* It is clear, since

$$\int_0^1 \gamma^2\ {}_0d_\phi\gamma = \frac{1}{1+\phi+\phi^2},$$

$$\int_0^1 \left(1-\gamma\right)^2\ {}_0d_\phi\gamma = \frac{\phi+\phi^3}{(1+\phi)\left(1+\phi+\phi^2\right)},$$

$$\int_0^1 \gamma\left(1-\gamma\right)\ {}_0d_\phi\gamma = \frac{\phi^2}{(1+\phi)\left(1+\phi+\phi^2\right)}.$$

□

**Corollary 4.3.** *If one takes $\phi \to 1^-$ in Theorem 4.1, than one has the following inequality*

$$\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\psi(\nu)\ dv \leq \frac{M(\alpha_1,\alpha_2)}{s+2}+\frac{N(\alpha_1,\alpha_2)}{(s+1)(s+2)} \tag{4. 38}$$

*where* $M(\alpha_1,\alpha_2) = \varphi(\alpha_2)\psi(\alpha_2)+\varphi(\alpha_1)\psi(\alpha_1)$, $N(\alpha_1,\alpha_2) = \varphi(\alpha_1)\psi(\alpha_2)+\varphi(\alpha_2)\psi(\alpha_1)$

*Proof.* It is clear, since

$$\lim_{\phi\to 1^-}\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\psi(\nu)\ {}_{\alpha_1}d_\phi\nu = \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\psi(\nu)\ dv,$$

$$\lim_{\phi\to 1^-}\int_0^1\gamma^{s+1}\ {}_0d_\phi\gamma = \int_0^1\gamma^{s+1}d\gamma = \frac{1}{s+2},$$

$$\lim_{\phi\to 1^-}\int_0^1(1-\gamma)^{s+1}\ {}_0d_\phi\gamma = \int_0^1(1-\gamma)^{s+1}d\gamma = \frac{1}{s+2},$$

$$\lim_{\phi\to 1^-}\int_0^1\gamma(1-\gamma)^{s}\ {}_0d_\phi\gamma = \int_0^1\gamma(1-\gamma)^{s}d\gamma = \frac{1}{(s+1)(s+2)},$$

$$\lim_{\phi\to 1^-}\int_0^1\gamma^{s}(1-\gamma)\ {}_0d_\phi\gamma = \int_0^1\gamma^{s}(1-\gamma)\,d\gamma = \frac{1}{(s+1)(s+2)}.$$

□

**Remark 4.4.** *In ( 4. 37 ), we recapture the inequality proved in* [15, Theorem 4.3, inequality (4.6)] *(see also* [11]*), in ( 4. 38 ), we recapture the inequality proved in* [9, Theorem 5].

**Definition 4.5.** [8] *(1) For $x>0$, the $\phi$-gamma function is defined as*

$$\Gamma_\phi(x) = \int_0^{\frac{1}{1-\phi}}\gamma^{x-1}E_\phi^{-\phi\gamma}\ {}_0d_\phi\gamma,$$

*where $E_\phi^\gamma$ is one of the following $\phi$-analogues of the exponential function*

$$E_\phi^\gamma = \sum_{n=0}^{\infty}\phi^{\frac{n(n-1)}{2}}\frac{\gamma^n}{[n]!} = (1+(1-\phi)\gamma)_\phi^\infty = \prod_{j=0}^{\infty}\left(1+\phi^j(1-\phi)\gamma\right)$$

$$e_\phi^\gamma = \sum_{n=0}^{\infty}\frac{\gamma^n}{[n]!} = \frac{1}{(1-(1-\phi)\gamma)_\phi^\infty} = \frac{1}{\prod_{j=0}^{\infty}(1-\phi^j(1-\phi)\gamma)}.$$

*(2) For $x,y>0$, the $\phi$-beta function is defined as*

$$B_\phi(x,y) = \int_0^1\gamma^{x-1}(1-\phi\gamma)_\phi^{y-1}\ {}_0d_\phi\gamma,$$

*where*

$$(1-\phi\gamma)_\phi^{y-1} = \frac{(1-\phi\gamma)_\phi^\infty}{(1-\phi^y\gamma)_\phi^\infty}.$$

Some properties of $\phi$-beta and $\phi$-gamma functions are given in the following theorem.

**Theorem 4.6.** [8] *(a)* $\Gamma_\phi(x)$ *can equivalently be expressed as*

$$\Gamma_\phi(x) = \frac{(1-\phi)_\phi^{x-1}}{(1-\phi)^{x-1}}$$

*In particular one has*

$$\Gamma_\phi(1+x) = [x]\,\Gamma_\phi(x)\,,\ \textit{for all } x>0,\ \Gamma_\phi(1)=1.$$

*(b) The* $\phi$*-gamma and* $\phi$*-beta functions are related to each other by the following two equations*

$$\Gamma_\phi(x) = \frac{B_\phi(x,\infty)}{(1-\phi)^\alpha},$$

$$B_\phi(x,y) = \frac{\Gamma_\phi(x)\,\Gamma_\phi(y)}{\Gamma_\phi(x+y)}.$$

**Remark 4.7.** *It is not difficult to observe that*

$$(1-\gamma)^y \le (1-\phi\gamma)^y \le (1-\phi\gamma)_\phi^y$$

*for* $0\le\gamma\le1$*,* $y>0$ *and* $0<\phi<1$.

**Theorem 4.8.** *Let* $\varphi,\psi:[0,\infty)\to(-\infty,\infty)\,,\alpha_1,\alpha_2\in[0,\infty)\,,\alpha_1<\alpha_2$*, be functions such that* $\varphi$*,* $\psi$ *and* $\varphi\psi$ *are* $\phi$*-integrable over* $[\alpha_1,\alpha_2]$*,* $0<\phi<1$*. If* $\varphi,\psi$ *are non-negative,* $\varphi\in K_{s_1}^2$ *and* $\psi\in K_{s_2}^2$ *for some fixed* $s_1,s_2\in(0,1]$*, then*

$$\begin{aligned}
&\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\,\psi(\nu)\ {}_{\alpha_1}d_\phi\nu \\
&\le \varphi(\alpha_2)\,\psi(\alpha_2)\,\frac{1-\phi}{1-\phi^{s_1+s_2+1}} + \varphi(\alpha_1)\,\psi(\alpha_1)\,B_\phi(1,s_1+s_2+1) \\
&+ N(\alpha_1,\alpha_2)\,B_\phi(s_1+1,s_2+1)\,,
\end{aligned} \tag{4. 39}$$

*where* $B_\phi(\alpha,\beta)$*,* $\alpha,\beta>0$ *is the* $\phi$*-beta function and* $N(\alpha_1,\alpha_2)$ *are defined in Corollary 4.3.*

*Proof.* Since $\varphi$ is $s_1$-convex and $\psi$ is $s_2$-convex functions, we have

$$\varphi(\gamma\alpha_2+(1-\gamma)\alpha_1) \le \gamma^{s_1}\varphi(\alpha_2)+(1-\gamma)^{s_1}\varphi(\alpha_1)$$

and

$$\psi(\gamma\alpha_2+(1-\gamma)\alpha_1) \le \gamma^{s_2}\psi(\alpha_2)+(1-\gamma)^{s_2}\psi(\alpha_1)$$

for all $\gamma\in[0,1]$. The non-negativity of $\varphi$ and $\psi$ gives

$$\begin{aligned}
&\varphi(\gamma\alpha_2+(1-\gamma)\alpha_1)\,\psi(\gamma\alpha_2+(1-\gamma)\alpha_1) \\
&\le \gamma^{s_1+s_2}\varphi(\alpha_2)\,\psi(\alpha_2)+\gamma^{s_1}(1-\gamma)^{s_2}\varphi(\alpha_2)\,\psi(\alpha_1) \\
&+\gamma^{s_2}(1-\gamma)^{s_1}\varphi(\alpha_1)\,\psi(\alpha_2)+(1-\gamma)^{s_1}(1-\gamma)^{s_2}\varphi(\alpha_1)\,\psi(\alpha_1)\,.
\end{aligned}$$

By $\phi$-integration on both sides of the above inequality over the interval $[0,1]$, we get

$$\begin{aligned}
&\int_0^1\varphi(\gamma\alpha_2+(1-\gamma)\alpha_1)\,\psi(\gamma\alpha_2+(1-\gamma)\alpha_1)_0\,d_\phi\gamma \\
&\le \varphi(\alpha_2)\,\psi(\alpha_2)\int_0^1\gamma^{s_1+s_2}\ {}_0d_\phi\gamma+\varphi(\alpha_2)\,\psi(\alpha_1)\int_0^1\gamma^{s_1}(1-\gamma)^{s_2}\ {}_0d_\phi\gamma \\
&+\varphi(\alpha_1)\,\psi(\alpha_2)\int_0^1\gamma^{s_2}(1-\gamma)^{s_1}\ {}_0d_\phi\gamma+\varphi(\alpha_1)\,\psi(\alpha_1)\int_0^1(1-\gamma)^{s_1+s_2}\ {}_0d_\phi\gamma.
\end{aligned} \tag{4. 40}$$

By the definition of $\phi$-integral, we have

$$\int_0^1 \varphi(\gamma\alpha_2 + (1-\gamma)\alpha_1)\psi(\gamma\alpha_2 + (1-\gamma)\alpha_1)\ {}_0d_\phi\gamma = \frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2} \varphi(\nu)\psi(\nu)\ {}_{\alpha_1}d_\phi\nu.$$

We also observe that

$$\int_0^1 \gamma^{s_1+s_2}\ {}_0d_\phi\gamma = \frac{1-\phi}{1-\phi^{s_1+s_2+1}},$$

$$\int_0^1 (1-\gamma)^{s_1+s_2}\ {}_0d_\phi\gamma \le \int_0^1 (1-\phi\gamma)_\phi^{s_1+s_2}\ {}_0d_\phi\gamma = B_\phi(1, s_1+s_2+1),$$

$$\int_0^1 \gamma^{s_1}(1-\gamma)^{s_2}\ {}_0d_\phi\gamma \le \int_0^1 \gamma^{s_1}(1-\phi\gamma)_\phi^{s_2}\ {}_0d_\phi\gamma = B_\phi(s_1+1, s_2+1),$$

and

$$\begin{aligned}\int_0^1 \gamma^{s_2}(1-\gamma)^{s_1}\ {}_0d_\phi\gamma &\le \int_0^1 \gamma^{s_2}(1-\phi\gamma)_\phi^{s_1}\ {}_0d_\phi\gamma = B_\phi(s_2+1, s_1+1)\\ &= B_\phi(s_1+1, s_2+1).\end{aligned}$$

Utilizing the above observations, we get from ( 4. 40 ), the required result. □

**Corollary 4.9.** *If one takes $\phi \to 1^-$ in Theorem 4.8, than one has the following inequality*

$$\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\psi(\nu)\ dv \le \frac{M(\alpha_1,\alpha_2)}{s_1+s_2+1} + N(\alpha_1,\alpha_2)B(s_2+1, s_1+1), \quad (4.\ 41)$$

*where $M(\alpha_1,\alpha_2)$ and $N(\alpha_1,\alpha_2)$ are defined in Corollary 4.3.*

*Proof.* It is clear, since

$$\lim_{\phi\to1^-}\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\psi(\nu)\ {}_{\alpha_1}d_\phi\nu = \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(\nu)\psi(\nu)\ dv,$$

$$\lim_{\phi\to1^-}\int_0^1 \gamma^{s_1+s_2}\ {}_0d_\phi\gamma = \int_0^1 \gamma^{s_1+s_2}d\gamma = \frac{1}{s_1+s_2+1},$$

$$\lim_{\phi\to1^-}\int_0^1 (1-\gamma)^{s_1+s_2}\ {}_0d_\phi\gamma = \int_0^1 (1-\gamma)^{s_1+s_2}\,d\gamma = \frac{1}{s_1+s_2+1},$$

$$\lim_{\phi\to1^-}\int_0^1 \gamma^{s_1}(1-\gamma)^{s_2}\ {}_0d_\phi\gamma = \int_0^1 \gamma^{s_1}(1-\gamma)^{s_2}\,d\gamma = B(s_2+1, s_1+1),$$

$$\lim_{\phi\to1^-}\int_0^1 \gamma^{s_2}(1-\gamma)^{s_1}\ {}_0d_\phi\gamma = \int_0^1 \gamma^{s_2}(1-\gamma)^{s_1}\,d\gamma = B(s_2+1, s_1+1).$$

□

**Remark 4.10.** *In ( 4. 41 ), we recapture the inequality proved in* [9, Theorem 6].

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