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# Optimal Control of Air Pollution

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**Abstract.**In this article, mathematical expressions that represent the dynamics of air pollution associated with electric power generating industries ( in particular, atmospheric $Co_2$ ) is proposed using an optimal control theory approach. The model is characterized via Pontyagins maximum/minimum principles. Electric power plants efficiency improvement was introduced through applying technologies. The optimality system is established in attempt to minimize both the cost of applying technology for efficiency improvement as well as the atmospheric emission while maximizing the electric power generation output.

**AMS (MOS) Subject Classification Codes: 49-XX; 34K35; 35H05**
**Key Words:** Optimal control, air pollution, fossil fuels, electric power generation, emission free equilibrium.

## 1. Introduction

Air pollution is a mixture of solid and gaseous particles (biological particles or other harmful materials) such as car emissions, dust, factories chemicals, and so on in the air, which may results in humans/living organisms diseases, damage or death. Electric power generation is responsible for a large measure of air pollution which is caused by burning conventional fuels like coal, oil or gas which occurs in thermal electric power plants. Therefore, substantial investments should be targeted towards the development of power plants efficiency and improvement through variety of adjustments. Electric power plants can be categorized generally into two groups. Power plants are powered by fossil fuel and by non- fossils fuel . The fossil fuel power plants are those that use coal, natural gas, and oil as their source of electrical energy. The chemical energy stored in the fossil fuels is transformed into electric energy in steam-electric power plants by burning the fuel in the combustion chamber. Heat energy is released and is used to produce steam in the boiler, which is passed through the steam turbines and drives the electric generator. The overall process is associated with air and thermal pollutant problems. Air pollutants are emitted via the exhaust gasses and the thermal pollutant is associated with the vast amounts of heat losses in

the condenser cooling water which caused severe problem in aquatic life and great environmental threat. The non- fossil fuel power plants are those plants that use nuclear energy, hydro energy and alternatives(renewable energy like solar, wind, geothermal e.t.c.) as their source of electrical energy. This is perfectly air pollutionfree but very scarce resources.
However, electric power plant pollutants can be reduce through a variety of adjustment in the plants. Such as fuel balancing, fuel switching, implementation of improvement technologies to existing power plants. As well as using non- fossil fuel as energy source or using energy source that are renewable such as solar, wind turbines e.t.c., Bamigbola and Aderinto (2010), Charles (1976), Harry et al (1996), Olle (1970), Rayput (2003),Shammakh et al (2006). Many researchers have worked on the electric power generation as well as on air pollution and the control of air pollution with application to resource management control. Genchi, et al (2002) used optimization model to assess the reduction of Co2 emissions. Hashim et al (2005) studied Co2 emission and electric energy planning. Marco, et al (2007) worked on Co2 emission management in attempt to reduce its greenhouse effect. Shammakh, et al (2006) looked at some strategies to control air pollution optimally in the power generating system sector. Dymtro, et al (2007) used variational inequalities to model fuel supply in electric power networks, to mention a few.
However, the present effort is to apply the Pontraygins maximum/minimum Principles of optimal control theory to air pollution associated with electric power generation. The main aim is to propose mathematical expressions in form of optimal control model that describes the way, effect, and relation of Co2 emission with respect to electric power generating systems, and application of the efficiency technology towards removal / reduction of Co2 emission during power generation via optimal control approach. And for better understanding the model is characterized and the optimality system is established in an attempt to reduce the Co2 emission with respect to electric power generation.

## 2. Formulation of Model

Let $X_{ij}(t)$ be the concentration of atmospheric $\text{Co}_2$ emission from electricity generating plants. $E_{1j}(t)$ be the electric energy generated from the power plant 1 using fuel type j and $E_{2j}(t)$ be the electric energy generated from the power plant 2 using j source of energy $(j = 1, 2)$. Electric power plants with fuel type 1 are those plants that use fossil fuels as their source of energy and power plants with fuel type 2 are those plants that use non-fossil fuel and alternatives (renewable energy such as solar, wind, geothermal e.t.c.) as their source of energy.
The model is given by

$$\begin{aligned} X_{ij}(t) &= r_j + r_j X_{ij}\left(1 - \frac{X_{ij}}{c}\right) - \alpha_{ij}E_{1j} + (\beta + u_2)X_{ij} \\ E_{1j} &= q_F E_{1j} - l_1 E_{1j} + u_1 X_{ij} + \alpha_{ik}E_{1j} \\ E_{2j} &= q_{NF}E_{2j} - l_2 E_{2j} + hE_{1j} \\ X_{ij}(0) &= X_0,\ E_{1j}(0) = E_{1j}0,\ E_{2j}(0) = E_{1j}0; \\ X_{ij}(t) &\geq 0,\ E_{1j}(t) \geq 0,\ E_{2j}(t) \geq 0,\ i = 1, 2;\ j = 1, 2. \end{aligned} \tag{1}$$

where $r_j$ is the emission rate from power plant using fuel type $j$, $c$ is the caring capacity of the air in terms of $Co_2$ from the power plant, $\alpha_{ik}$ is the efficiency of power plant 1 if technology $k$ is applied, $\beta$ is emission rate of $Co_2$ through gross domestic product, $u_1$ is the addition of power plant efficient technique towards the removal of $Co_2$ emission through power plant, $k$ is the technology applied to increase power plant efficiency, $u_2$ is the addition of efficiency technology towards removal of $Co_2$ due to gross domestic product, $q_F$ is the generation rate for plant using fossil fuel as its source of energy. $q_{NF}$ is the generation rate for plants using non-fuel as its source of energy. $l_1, l_2$ are the power losses during generation for plants using fossil fuel and those that use non-fossil fuels as their source of energy respectively. $h$ is the rate at which plant 1 change from one type of fuel to another towards the reduction of $Co_2$ emission. The model flow diagrams are shown in figures (1,2,3).

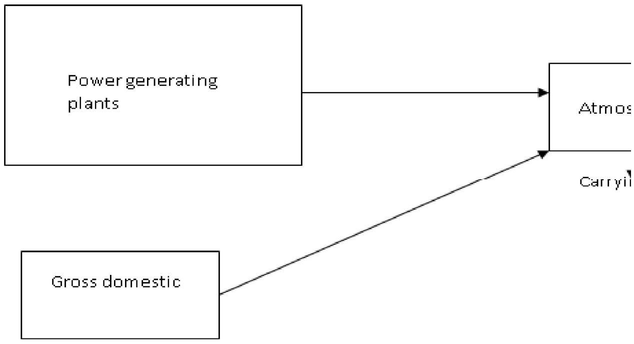


Figure.1: Scheme of the model.

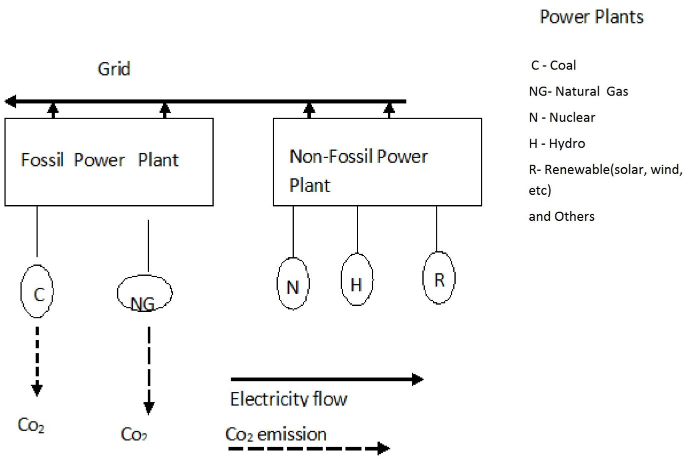


Figure. 2: Power Plant Structure.

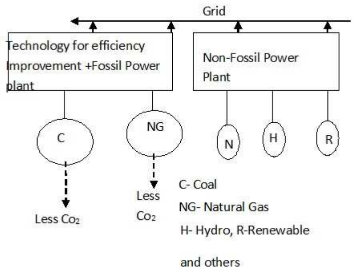


Figure.3: Power Plant with Efficiency Improvement.

## 3. RESULTS: ANALYSIS OF THE MODEL

The local stability and steady state of model (1) is challenging, however we are interested in the models ability to exhibit locally asymptotically stable and steady states.

**Table 1:** Parameters value and description (Marco, etal 2008, Shammakh etal,2006, and Sirikum, etal. 2007)

| Parameter | Value | Description |
|---|---|---|
| $X_{ij}(t)$ | 0.9384m/tons | Concentration of $Co_2$ emission from electric power generating plant. |
| $E_{1j}(t)$ | 800 MW | Power output from fossil fuel energy types |
| $E_{2j}(t)$ | 500 MW | Power output from non-fossil fuel energy |
| $r_j$ | 0.17 | Emission rate of $Co_2$ from power plants |
| $c$ | 0.75 | Carrying capacity of atmosphere in terms of $Co_2$ |
| $\alpha_{ik}$ | 0.46 | Efficiency of fossil fuel plant when technology is applied |
| $\beta$ | 0.05 | Emission rate of $Co_2$ from gross domestic product |
| $u_2$ | 0.005 | Efficiency technology applied to gross domestic product |
| $q_F$ | 0.55 | Generation rate of fossil fuel power plants |
| $q_{NF}$ | 0.60 | Generation rate of non-fossil fuel power plants. |
| $h$ | 0.001 | The rate of switching to another type of fuel |
| $l_1$ | 0.0002 | Loss rate during generation with fossil fuel plant. |
| $l_2$ | 0.0002 | Loss rate during generation with non-fossil fuel plants . |
| $u_1$ | 0.008 | Efficiency technology towards power generation with fossil fuel plant. |

3.1. **The emission free equilibrium and Stability.** The emission free equilibrium (EFE) of model system (1) is denoted by $E(0)$ is expressed as

$$E(0) = (X_{ij}(0), E_{1j}(0), E_{2j}(0)) = (r_j, 0, 0)$$

For all non-negative value of parameter, model system (1) has equilibrium E0 in the boundary of G and the endemic equilibrium point occurs inside G.

**Theorem 3.1:** The emission free equilibrium is locally asymptotically stable when the basic Emission is less than zero and unstable when its greater than zero (that is stable if the Eigen value $\lambda_i < 0$, and unstable for $\lambda_i > 1$.

**Proof**

Let $(X_{ij}(0), E_{1j}(0), E_{2j}(0)) = (r_j, 0, 0) = E(0)$

$$J(X_{ij}, E_{1j}, E_{2j}) = \begin{pmatrix} r_j - \frac{2r_jX_{ij}}{c} + \beta + u_2 & -\alpha_{ik} & 0 \\ u_1 & q_F - l_1 + \alpha_{ik} & 0 \\ 0 & h & q_{NF} - l_2 \end{pmatrix}$$

$$J(E(0)) = \begin{pmatrix} r_j - \frac{2r_j^2}{c} + \beta + u_2 & -\alpha_{ik} & 0 \\ u_1 & q_F - l_1 + \alpha_{ik} & 0 \\ 0 & h & q_{NF} - l_2 \end{pmatrix}$$

where $J$ is the Jacobian matrix of a function. Finding the eigen-value of the Jacobian matrix at equilibrium point $E_(0)$ we have $\det[J(E(0)) - \lambda I]$.

Thus we obtain

$$\left(r_j - \frac{2r_j^2}{c} + \beta + u_2\right) - \lambda_1 \begin{vmatrix} (q_F - l_1 + \alpha_{ik}) - \lambda_2 & 0 \\ h & (q_{NF} - l_2) - \lambda_3 \end{vmatrix}$$

$$\Longrightarrow \left[\left(r_j - \frac{2r_j^2}{c} + \beta + u_2\right) - \lambda_1\right] [(q_F - l_1 + \alpha_{ik}) - \lambda_2] [(q_{NF} - l_2) - \lambda_3] = 0$$

$$\lambda_1 = r_j - \frac{2r_j^2}{c} + \beta + u_2,\ \lambda_2 = (q_F - l_1 + \alpha_{ik}),\ \lambda_3 = (q_{NF} - l_2)$$

Hence, stability at this point depends critically on the following conditions:

$$u_2 < \frac{2r_j^2}{c} - (r_j + \beta),\ or\ cu_2 < 2r_j^2 - c(r_j + \beta),\ or\ c < \frac{1}{u_2}[2r_j^2 - c(r_j + \beta)]$$

$l_1 < q_F + \alpha_{ik}$ and $l_2 < q_{NF}$

The eigen - values of $\det[J(E_0) - \lambda I]$, have negative real part whenever the above three conditions hold. This implies that $E_0$ is locally asymptotically stable. After substituting the value of the parameters in Table 1, we obtained 3 eigen-values of $J$ in which none of them is negative; $\lambda_1 = 0.14793$, $\lambda_2 = 1.0098$, $\lambda_3 = 0.5998$.

Hence, $E_0$ is not asymptotically stable. For the prove of similar result see Bhunu, et al (2008).

**Lemma 3.1**

Let $f : [0, \infty] \rightarrow \mathbb{R}$ be a bounded function and twice differentiable with bounded second derivative. Let $t_n \rightarrow \infty$ and $f(t_n)$ converge to $f^\infty$ or $f_\infty$ for $n \rightarrow \infty$, then $\lim_{f(t_n)} = 0$ where $f_\infty = \liminf_{t\rightarrow\infty} f(t)$, $f^\infty = \limsup_{t\rightarrow\infty} f(t)$ and $f$ is a real valued function, Burghes and Graham (2002).

**Theorem 3.2:**

The emission free equilibrium is globally asymptotically stable for $E_0 < 0$.

**Proof:** Let $\lambda_i$ choose a sequence $t_n \rightarrow \infty$ such that

$$X_{ij}(t_n) \rightarrow X_{ij}^\infty,\ \frac{d}{dt}X_{ij}(t_n) \rightarrow 0$$

Using $\frac{d}{dt}X_{ij}$ in (1) and Lemma 3.1, we have

$$0 \leq (\beta + u_2)X_{ij}^\infty - \alpha_{ik}E_{1j}^\infty + r_jX_{ij}^\infty\left(1 - \frac{X_{ij}^\infty}{c}\right)$$

$$\Longrightarrow X_{ij}^{\infty} \leq (\beta + u_2 + r_j) \pm \frac{\sqrt{(\beta + u_2 + r_j)^2 - 4(r_j c^{-1}\alpha_{ik} E_{1j}^{\infty})}}{2r_j c^{-1}} \tag{2}$$

Similarly, choose a sequence $r_n \to \infty$ such that $E_{1j}(r_n) \to E_{1j}^{\infty}$, $\frac{d}{dt}E_{1j}(r_n) \to 0$ and using $\frac{d}{dt}E_{1j}$ and Lemma 3.1

$$0 \leq (q_F - l_1 + \alpha_{ik})E_{1j}^{\infty} + u_1 X_{ij}^{\infty}$$

$$\Longrightarrow E_{1j}^{\infty} \leq \frac{-u_1 X_{ij}^{\infty}}{(q_F - l_1 + \alpha_{ik})} \tag{3}$$

Also, choosing sequence $s_n \to \infty$ such that $E_{2j}(s_n) \to E_{1j}^{\infty}$, $\frac{d}{dt}E_{1j}(s_n) \to 0$ and using equation (1) and Lemma 3.1 we have,

$$0 \leq (q_{NF} - l_2)E_{2j}^{\infty} + hE_{1j}^{\infty}$$

$$\Longrightarrow E_{2j}^{\infty} \leq \frac{-hE_{1j}^{\infty}}{(q_{NF} - l_2)} \tag{4}$$

substituting (3) and (4) into (2), we have

$$\begin{aligned}
X_{ij}^{\infty} &\leq (\beta + u_2 + r_j) \pm \frac{\sqrt{(\beta + u_2 + r_j)^2 - 4(r_j c^{-1}\alpha_{ik})\left(\frac{-u_1 X_{ij}^{\infty}}{(q_F - l_1 + \alpha_{ik})}\right)}}{2r_j c^{-1}} \\
&= (\beta + u_2 + r_j) \pm \frac{\sqrt{\frac{c(\beta + u_2 + r_j)^2 (q_F - l_1 + \alpha_{ik}) + 4r_j \alpha_{ik} u_1 X_{ij}^{\infty}}{c(q_F - l_1 + \alpha_{ik})}}}{2r_j c^{-1}} \\
&= (\beta + u_2 + r_j)^2 \pm \frac{\frac{c(\beta + u_2 + r_j)^2 (q_F - l_1 + \alpha_{ik}) + 4r_j \alpha_{ik} u_1 X_{ij}^{\infty}}{c(q_F - l_1 + \alpha_{ik})}}{4r_j^2 c^{-2}} \\
&= \frac{2c^3(\beta + u_2 + r_j)^2 (q_F - l_1 + \alpha_{ik}) + 4r_j \alpha_{ik} u_1 X_{ij}^{\infty}}{4r_j^2 c(q_F - l_1 + \alpha_{ik})}
\end{aligned}$$

$$E_0 = \frac{E + PX_{ij}^{\infty}}{Q} \tag{5}$$

This implies $X_{ij}^{\infty} \leq 0$ whenever $E_0 < 0$. Substituting the data in table 1, we have the following results. $E_0 = 0.519496 > 0$ meaning that $X_{ij}^{\infty}$ does not less or equal to zero. Hence, without loss of generality, $E_0$ is not globally asymptotically stable.

## 4. Control Formulation

We consider a control problem given by

$$J(u_1, u_2) = \int_{t_0}^{t_f} (AX_{ij} + Bu_1^2 + Cu_2^2)dt \tag{6}$$

Subject to the state equation (1), where A, B and C are adjusted weights to reflect the relative importance of the variables $X_{ij}$, $u_1$, and $u_2$. Variables $u_1$ and $u_2$, are the control variables such as (i) introduction of power plant efficiency through a variety of adjustment in the plants, (ii) Restore Plants to design condition (iii) retrofit improvement, e.t.c.

The objective function (6) expresses our goal to minimize both the cost of retrofit for switching from one type of fuel to another and cost of applying technology for efficiency improvement, as well the atmospheric Co2 emissions while maximizing electric power output.
Therefore, we seek an optimal control pair $(u_1^*, u_2^*)$ such that
$J(u_1^*, u_2^*) = \min\{J(u_1, u_2)|(u_1, u_2) \in U\}$ subject to the system of ordinary differential equation (1), where $u = (u, u)|u_i$ is measurable, $a_i \leq u_i \leq b_i$, $t \in [t_0, t_f], \forall i = 1, 2$ is the control set.
Many researchers have used a control theoretical approach to formulate and study problems. Adams et al (2004) used optimal control approaches on dynamic multidrug therapies for HIV. Sirikum (2007) worked on power generation expansion planning with emission controls. Fister et al (1998) use control strategies to optimize chemotherapy in an HIV model. However, the present work is based on the use of optimal control approach applied to electric power generation with $Co_2$ emission.

## 5. The Optimality System and Applications

We established the existence of an optimal control pair for the optimality system using results from literatures. That is by showing that the right hand sides of equation (1) are bounded by the state and control variables and that the integrand of the objective function (6) is concave in u and is bounded. These bounds give the compactness to establish the existence of the optimal controls.
**Theorem 5.1:** Consider the optimal controls and the state system (1), establish that there exists an optimality system for the optimal control pair.
**Proof:**
By the application of Pontryagin Maximum / Minimum Principle, we define the Lagrangian function to be:

$$\left.\begin{array}{c} L(X_{ij}, E_{1j}, E_{2j}, u_1, u_2, \lambda_1, \lambda_2, \lambda_3) = AX_{ij} + Bu_1^2 + Cu_2^2 \\ +\lambda_1\left(r_j + r_j X_{ij}\left(1 - \frac{X_{ij}}{c}\right) - \alpha_{ik}E_{1j} + (\beta + u_2)X_{ij}\right) \\ +\lambda_2(q_F E_{1j} - l_1 E_{1j} + u_1 X_{ij} + \alpha_{ik}E_{1j}) + \lambda_3(q_{NF}E_{2j} - l_2 E_{2j} + hE_{2j}) \\ -\omega_{11}(u_1 - a_1) - \omega_{12}(b_1 - u_2) - \omega_{21}(u_2 - a_2) - \omega_{22}(b_2 - u_2) \end{array}\right\} \tag{7}$$

where $\lambda_i$ are Langrange Multipliers and $W_{ij} \geq 0$ are penalty Multipliers satisfying $\omega_{11}(t)(u_1(t) - a_1) = \omega_{12}(t)(b_1 - u_2(t)) = 0$ at $u_1 = u_1^*$
and $\omega_{21}(t)(u_2(t) - a_2) = \omega_{22}(t)(b_2 - u_2(t)) = 0.$
By Differentiating the Lagrangian with respect to state variables $X_{ij}$, $E_{1j}$ and $E_{2j}$ respectively, the following equations were obtained for the adjoint variables $\lambda_i$.

$$\lambda_1 = \frac{\partial L}{\partial X_{ij}},\ \lambda_2 = \frac{\partial L}{\partial E_{1j}},\ \lambda_3 = \frac{\partial L}{\partial E_{2j}}$$

The adjoint variables $\lambda_1, \lambda_2, \lambda_3$ satisfy the following ordinary differential equations from (7).

$$\lambda_1' = -\left\{A + \lambda_1\left(r_j(1 - \frac{2X_{ij}}{c}) + \beta + u_2\right) + \lambda_2 u_1\right\}$$
$$\lambda_2' = -\{\lambda_1(-\alpha_{ik}) + \lambda_2(q_F - l_1 + \alpha_{ik}) + \lambda_3 l_2\}$$

$$\lambda_3' = -\left\{\lambda_3(q_{NF} - l_2)\right\}$$
$$\lambda_1(t_1) = \lambda_2(t_1) = \lambda_3(t_1) = 0 \tag{8}$$

The optimality of the control variables $u_1$ and $u_2$ requires that we find the derivatives of the Langrangian with respect to the controls $u_1$ and $u_2$ respectively.
Thus, we have,

$$\frac{\partial L}{\partial u_1} = 2Bu_1 + \lambda_2 X_{ij} - \omega_{11} + \omega_{22} = 0,\ at\ u_1^*$$
$$\frac{\partial L}{\partial u_2} = 2Cu_2 + \lambda_1 X_{ij} - \omega_{21} + \omega_{12} = 0,\ at\ u_2^*$$

Solving for the controls, we obtain

$$u_1^* = \frac{1}{2B}\left(\lambda_2 X_{ij} - \omega_{11} + \omega_{12}\right) \tag{9}$$
$$u_2^* = \frac{1}{2C}\left(\lambda_1 X_{ij} - \omega_{21} + \omega_{22}\right) \tag{10}$$

To determine an explicit expansion for the optimal control without $\omega_{11}, \omega_{12}, \omega_{21}$ and $\omega_{22}$ we consider the following cases.
(i) On $\{t|a_1 = u_1^*(t)\}$, since $u_1^*(t) \neq b_1, \omega_{11} = 0$, then $a_1 = u_1^* = \frac{1}{2B}\left(\lambda_2 X_{ij} + \omega_{12}\right)$.
Solving for $\omega_{12}$ gives
$2Ba_1 - \lambda_2 X_{ij} = \omega_{12} \geq 0 \Rightarrow 2Ba_1 \geq \lambda_2 X_{ij}$ and $a_1 \geq \frac{1}{2B}\lambda_2 X_{ij}$
(ii) On $\{t|a_1 < u_1^*(t) < b_1\}$. By definition of penalty multipliers $\omega_{12} = \omega_{11} = 0$, we have
$u_1^*(t) = \frac{1}{2B}\left(\lambda_2 X_{ij}\right)$.
(iii) On $\{t|b_1 = u_1^*(t)\}$, since $u_1^*(t) \neq a_1, \omega_{12} = 0$, we have

$$b_1 = u_1^*(t) = \frac{1}{2B}\left(\lambda_2 X_{ij} - \omega_{11}\right)$$

$0 \leq \omega_{11} = 2Bb_1 - \lambda_2 X_{ij}$ and $b_1 \leq \frac{1}{2B}\lambda_2 X_{ij}$
Hence, we have

$$u_1^* = \begin{cases} \frac{1}{2B}\left(\lambda_2 X_{ij}\right)\ if\ a_1 < \frac{1}{2B}\left(\lambda_2 X_{ij}\right) < b_1 \\ a_1\ if\ \frac{1}{2B}\left(\lambda_2 X_{ij}\right) \leq a_1 \\ b_1\ if\ \frac{1}{2B}\left(\lambda_2 X_{ij}\right) \geq b_1 \end{cases}$$

In compact form, we have

$$u_1^* = \min\left\{\max\left\{a_1, \frac{1}{2B}\left(\lambda_2 X_{ij}\right)\right\}, b_1\right\} \tag{11}$$

Using similar arguments, we obtain the following expression for the second optimal control function. (i) On $\{t|a_2 = u_2^*(t)\}, \omega_{21} = 0$, then $a_2 = u_2^*(t) = \frac{1}{2C}\left(\lambda_1 X_{ij}^* + \omega_{22}\right)$.
$\Rightarrow 2Ca_2 - \lambda_1 X_{ij^*} = \omega_{22} \geq 0$
$\Rightarrow 2Ca_2 \geq \lambda_1 X_{ij}^*$ and $a_2 \geq \frac{1}{2C}\lambda_1 X_{ij}^*$
(ii) On $\{t|a_2 < u_2^*(t) < b_2\}, \omega_{21} = \omega_{22} = 0$ and $u_2^*(t) = \frac{1}{2C}\left(\lambda_1 X_{ij}^*\right)$.
(iii) On $\{t|b_2 = u_2^*(t)\}$, since $u_2^*(t) \neq a_2, \omega_{22} = 0$, we have

$$b_2 = u_2^*(t) = \frac{1}{2C}\left(\lambda_1 X_{ij}^* - \omega_{11}\right)$$

$\Rightarrow 0 \leq \omega_{21} = \lambda_1 X_{ij}^* - 2Cb_2$ since $\omega \geq 0, 1 = 1, 2$ and $b_2 \leq \frac{1}{2C}\lambda_1 X_{ij}^*$

Hence, $u_2^* = \begin{cases} \frac{1}{2C}\left(\lambda_1 X_{ij}^*\right) \; if \; a_2 < \frac{1}{2C}\left(\lambda_1 X_{ij}^*\right) < b_2 \\ a_2 \; if \; \frac{1}{2C}\left(\lambda_1 X_{ij}^*\right) \leq a_2 \\ b_2 \; if \; \frac{1}{2C}\left(\lambda_1 X_{ij}^*\right) \geq b_2 \end{cases}$

Combining these three cases, we have

$$u_2^* = \min\left\{\max\left\{a_2, \frac{1}{2C}\left(\lambda_1 X_{ij}^*\right)\right\}, b_2\right\} \tag{12}$$

The optimality system comprises of the state system with the adjoint system (8) with initial and transversality conditions together with the characterization of the optimal control pair (11) and (12).

The optimality system can then be solved via an iterative numerical method. The state equation is solved using Runge Kutta forward order scheme and the adjoint system is solved backward in time. The controls are updated at the end of each iteration using (11) and (12).

## 6. CONCLUSION

The mathematical model for better understanding of Co2 emission with application to electricity power generating system model was formulated. We further characterize the model using the existing results. Finally, the optimality system for the model was established.

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**Author's Contributions:** The first author gave the idea of the main results. Both authors contributed to the writing of this paper. Both authors read and approved the final manuscript.

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