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Certain Characterization of m-Polar Fuzzy Graphs by Level Graphs

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Abstract. Zadeh introduced the concept of fuzzy sets as a mathematical tool to deal with uncertainty, imprecision and vagueness. Since then, many higher order fuzzy sets, including intuitionistic fuzzy sets, bipolar fuzzy sets and m-polar fuzzy set, have been reported in literature to solve many real life problems, involving ambiguity and uncertainty. In this paper, we present certain characterization of m-polar fuzzy graphs by level graphs.

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1. Introduction

Graph theory is a enjoyable playground for the research of proof techniques in discrete mathematics. There are many applications of graph theory in different fields. The world of theoretical physics discovered graph theory for its own purposes. In the study of statistical mechanics, the points represent molecules and two adjacent points indicate nearest neighbor interaction of some physical kind, like magnetic interaction or repulsion. The study of Markov chains in probability theory involves directed graphs in the sense that events are given by points and a directed line from one point to another shows a positive probability of direct succession of these two events. Job assignments problem is solved by bipartite graphs.
In 1994, Zhang [19] initiated the idea of bipolar fuzzy sets, which is a generalization of fuzzy set [17]. The membership degree range in a bipolar fuzzy set is $[-1, 1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the

implicit counter-property. The idea of m-polar fuzzy set which is an extension of a bipolar fuzzy set, studied by Chen *et al*. [9] and exposed that $2-$polar and bipolar fuzzy set are cryptomorphic mathematical notions. The background of this concept is that "multipolar information" (not like the bipolar information which give two-valued logic) arise because information for a natural world are frequently from n factors $(n \geq 2)$. The statement 'Pakistan is a good country', consider as an example. The truth value of this statement may not a real number in $[0, 1]$. Being good country may have several components: good in public transport system, good in political awareness, good in medical facilities, etc. The each component may be a real number in $[0, 1]$. If n is the number of such components under consideration, then the truth value of fuzzy statement is a n-tuple of real numbers in $[0, 1]$, that is, an element of $[0, 1]^n$. In 1973, Kauffmann [12] illustrated the notion of fuzzy graphs based on Zadeh's fuzzy relations [18]. The fuzzy graphs structure was described by Rosenfeld [16]. Later, Bhattacharya [8] gave some remarks on fuzzy graphs. 1994, Mordeson and Chang-Shyh [14] defined some operations on fuzzy graphs. In 2011, Akram introduced the notion of bipolar fuzzy graphs in [1]. Dudek and Talebi [10] described operations on level graphs of bipolar fuzzy graphs. Recently, Akram *et al.* $[3-7]$ has discussed several new concepts, including m-polar fuzzy graphs, certain metrics in m-polar fuzzy graphs, certain types of edge m-polar fuzzy graphs and m-polar fuzzy hypergraphs. In this research paper, we present characterization of m-polar fuzzy graphs by level graphs.

2. CHARACTERIZATION OF m-POLAR FUZZY GRAPHS BY LEVEL GRAPHS

Definition 2.1. [9] An m-polar fuzzy set in a universe Y is a function $C : Y \to [0, 1]^m$. The degree of each element $a \in Y$ is written as $C(a) = (P_1oC(a), P_2oC(a), \cdots, P_moC(a))$, where $P_koC : [0, 1]^m \to [0, 1]$ is the kth projection mapping.
Note that $[0, 1]^m$ (m-th power of $[0, 1]$) is considered as a poset with the point-wise order $\leq$, where m is an arbitrary ordinal number (we make an appointment that $m = \{n|n < m\}$ when $m > 0$), $\leq$ is defined by $a \leq b \Leftrightarrow P_k(a) \leq P_k(b)$ for each $k \in m$ ($a, b \in [0, 1]^m$), and $P_k : [0, 1]^m \to [0, 1]$ is the k-th projection mapping ($k \in m$). $\mathbf{1} = (1, 1, \cdots, 1)$ is the greatest value and $\mathbf{0} = (0, 0, \cdots, 0)$ is the smallest value in $[0, 1]^m$.

Definition 2.2. [4] Let C be an m-polar fuzzy subset of a non-empty Y. An m-polar fuzzy relation on C is an m-polar fuzzy subset D of $Y \times Y$ defined by the mapping $D : Y \times Y \to [0, 1]^m$ such that for all $a, b \in Y$

$$P_koD(ab) \leq \inf\{P_koC(a), P_koC(b)\}$$

$1 \leq k \leq m$, where $P_koC(a)$ denotes the k-th degree of membership of a vertex a and $P_koD(ab)$ denotes the k-th degree of membership of the edge ab.

Definition 2.3. [4, 9] An m-polar fuzzy graph is a pair $G = (C, D)$, where $C : Y \to [0, 1]^m$ is an m-polar fuzzy set in Y and $D : Y \times Y \to [0, 1]^m$ is an m-polar fuzzy relation on Y such that

$$P_koD(ab) \leq \inf\{P_koC(a), P_koC(b)\}$$

$1 \leq k \leq m$, for all $a, b \in Y$ and $P_k oD(ab) = 0$ for all $ab \in Y \times Y - F$ for all $k = 1, 2, \cdots, m$. C is called the m-polar fuzzy vertex set of G and D is called the m-polar fuzzy edge set of G, respectively.

We now define t-level set on Y and $F \subseteq Y \times Y$.

Definition 2.4. Let $C : Y \to [0,1]^m$ be an m-polar fuzzy set on Y. The set

$$C_t = \{a \in Y \mid P_k oC(a) \geq \alpha_k, 1 \leq k \leq m\}$$

where $t \in [0,1]^m$ and $t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$, is called the t-level set of C. Let $D : Y \times Y \to [0,1]^m$ be an m-polar fuzzy relation on Y. The set

$$D_t = \{ab \in Y \times Y \mid P_k oD(ab) \geq \alpha_k, 1 \leq k \leq m\}$$

where $t \in [0,1]^m$ and $t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ is called t-level set of D. $G_t = (C_t, D_t)$ is called t-level graph.

Example 2.5. Consider a 3-polar fuzzy graph on $Y = \{s, t, u, v\}$.

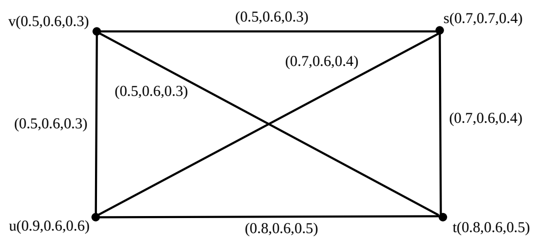

FIGURE 1. 3-polar fuzzy graph $G = (C, D)$

Take $t = (0.6, 0.5, 0.4)$. It is easy to see that $C_{(0.6,0.5,0.4)} = \{s, t, u\}$, $D_{(0.6,0.5,0.4)} = \{st, su, tu\}$. Clearly, the $(0.6, 0.5, 0.4)$-level graph = $G_{(0.6,0.5,0.4)}$ is a subgraph of crisp graph $G^* = (Y, F)$.

We formulate a proposition.

Proposition 2.6. *The level graph $G_t = (C_t, D_t)$ is a crisp graph.*

Theorem 2.7. *G is an m-polar fuzzy graph if and only if $G_t = (C_t, D_t)$ is a crisp graph for each $t \in [0,1]^m$, $t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$.*

Proof. For every $t \in [0,1]^m$, $t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$. Take $ab \in D_t$. Then $P_k oD(ab) \geq \alpha_k$, $1 \leq k \leq m$. Since G is an m-polar fuzzy graph, it follows that

$$\alpha_k \leq P_k oD(ab) \leq \inf\{P_k oC(a), P_k oC(b)\}.$$

This shows that $\alpha_k \leq P_k oC(a)$, $\alpha_k \leq P_k oC(b)$, for $k = 1, 2, \cdots, m$, that is, $a, b \in C_t$. Therefore, $G_t = (C_t, D_t)$ is a graph for each $t \in [0,1]^m$, $t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$. Conversely, let $G_t = (C_t, D_t)$ be a graph for all $t \in [0,1]^m$, $t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$. For every $ab \in Y \times Y$, let $P_k oD(ab) = \alpha_k$, $1 \leq k \leq m$. Then $ab \in D_t$. Since $G_t = (C_t, D_t)$ is a graph, we have $a, b \in C_t$; hence $P_k oC(a) \geq \alpha_k$, $P_k oC(b) \geq \alpha_k$, $1 \leq k \leq m$.

$$P_k oD(ab) = \alpha_k \leq \inf\{P_k oC(a), P_k oC(b)\}.$$

Thus, G is an m-polar fuzzy graph. □

Definition 2.8. Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be m-polar fuzzy graphs of $G_1^* = (Y_1, F_1)$ and $G_2^* = (Y_2, F_2)$, respectively. The Cartesian product $G_1 \times G_2$ is the pair (C, D) of m-polar fuzzy sets defined on the Cartesian product $G_1^* \times G_2^*$ such that

(i) $P_k oC(a_1, a_2) = \inf(P_k oC_1(a_1), P_k oC_2(a_2))$ for all $(a_1, a_2) \in Y_1 \times Y_2$,

(ii) $P_k oD((a, a_2)(a, b_2)) = \inf(P_k oC_1(a), P_k oD_2(a_2 b_2)$ for all $a \in Y_1$ and for all $a_2 b_2 \in F_2$,

(iii) $P_k oD((a_1, c)(b_1, c)) = \inf(P_k oD_1(a_1 b_1), P_k oC_2(c))$ for all $c \in Y_2$ and for all $a_1 b_1 \in F_1$.

Theorem 2.9. *Let $G_1 = (C_1, D_1)$and $G_2 = (C_2, D_2)$ be m-polar fuzzy graphs of $G_1^* = (Y_1, F_1)$ and $G_2^* = (Y_2, F_2)$, respectively. Then $G = (C, D)$ is the Cartesian product of G_1 and G_2 if and only if for each $t \in [0, 1]^m$, $t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ the t-level graph G_t is the Cartesian product of $(G_1)_t$ and $(G_2)_t$.*

Proof. For each $t \in [0, 1]^m$, $t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$, if $(a, b) \in C_t$, then

$$\inf(P_k oC_1(a), P_k oC_2(b)) = P_k oC(a, b) \geq \alpha_k,$$

$1 \leq k \leq m$, so $a \in (C_1)_{(t)}$ and $b \in (C_2)_{(t)}$, that is, $(a, b) \in (C_1)_{(t)} \times (C_2)_{(t)}$. Therefore, $C_t \subseteq (C_1)_t \times (C_2)_t$. Let $(a, b) \in (C_1)_t \times (C_2)_t$, then $a \in (C_1)_t$ and $b \in (C_2)_t$. It follows that $\inf(P_k oC_1(a), P_k oC_2(b)) \geq \alpha_k$, $1 \leq k \leq m$. Since (C, D) is the Cartesian product of G_1 and G_2, $P_k oC(a, b) \geq \alpha_k$, that is, $(a, b) \in C_t$. Therefore, $(C_1)_t \times (C_2)_t \subseteq C_t$ and so $(C_1)_t \times (C_2)_t = C_t$. We now prove $D_t = F$, where F is the edge set of the Cartesian product $(G_1)_t \; and \; (G_2)_t$ for each $t \in [0, 1]^m$, $t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$. Let $(a_1, a_2)(b_1, b_2) \in D_t$. Then, $P_k oD((a_1, a_2)(b_1, b_2)) \geq \alpha_k$, $1 \leq k \leq m$. Since (C, D) is the Cartesian product of G_1 and G_2, one of the following cases hold:

(i) $a_1 = b_1$ and $a_2 b_2 \in F_2$.

(ii) $a_2 = b_2$ and $a_1 b_1 \in F_1$.

For the case (i), we have

$$P_k oD((a_1, a_2)(b_1, b_2)) = \inf(P_k oC_1(a_1), P_k oD_2(a_2 b_2)) \geq \alpha_k,$$

so $P_k oC_1(a_1) \geq \alpha_k$, $P_k oD_2(a_2 b_2) \geq \alpha_k$. It follows that $a_1 = b_1 \in (C_1)_t, \; a_2 b_2 \in (D_2)_t$, that is, $(a_1, a_2)(b_1, b_2) \in F$. Similarly, for the case (ii), we conclude that $(a_1, a_2)(b_1, b_2) \in F$. Therefore, $D_t \subseteq F$. For every $(a, a_2)(a, b_2) \in F$, $P_k oC_1(a) \geq \alpha_k$, $P_k oD_2(a_2 b_2) \geq \alpha_k$, $1 \leq k \leq m$. Since (C, D) is the Cartesian product of G_1 and G_2, we have

$$P_k oD((a, a_2)(a, b_2)) = \inf(P_k oC_1(a), P_k oD_2(a_2 b_2)) \geq \alpha_k,$$

$1 \leq k \leq m$. Therefore $(a, a_2)(a, b_2) \in D_t$. Similarly, for every $(a_1, c)(b_1, c) \in F$, we have $(a_1, c)(b_1, c) \in D_t$. Therefore, $F \subseteq D_t$, and so $D_t = F$.

Conversely, suppose that $G_t = (C_t, D_t)$ is the Cartesian product of $(G_1)_t = ((C_1)_t, (D_1)_t)$ and $(G_2)_t = ((C_2)_t, (D_2)_t)$ for all $t \in [0, 1]^m$, $t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$. Let $\inf(P_k oC_1(a_1), P_k oC_2(a_2)) = \alpha_k$, $1 \leq k \leq m$ for some $(a_1, a_2) \in Y_1 \times Y_2$. Then $a_1 \in (C_1)_t$ and $a_2 \in (C_2)_t$. By hypothesis, $(a_1, a_2) \in C_t$, hence

$$P_k oC(a_1, a_2) \geq \alpha_k = \inf(P_k oC_1(a_1), P_k oC_2(a_2))$$

Take $P_koC(a_1, a_2) = \beta_k, 1 \leq k \leq m$, then $(a_1, a_2) \in C_{t'}$ where $t' \in [0, 1]^m$, $t' = (\beta_1, \beta_2, \cdots, \beta_m)$. Since $(C_{t'}, D_{t'})$ is the Cartesian product of $((C_1)_{t'}, (D_1)_{t'})$ and $((C_2)_{t'}, (D_2)_{t'})$, then $a_1 \in (C_1)_{t'}$ and $a_2 \in (C_2)_{t'}$. Hence,

$$P_koC_1(a_1) \geq \beta_k, P_koC_2(a_2) \geq \beta_k$$

It follows that

$$\inf(P_koC_1(a_1), P_koC_2(a_2)) \geq P_koC(a_1, a_2)$$

Therefore,

$$P_koC(a_1, a_2) = \inf(P_koC_1(a_1), P_koC_2(a_2)) \text{ for all } (\mathrm{a}_1, \mathrm{a}_2) \in \mathrm{Y}_1 \times \mathrm{Y}_2.$$

Similarly, for every $a \in Y_1$ and every $a_2b_2 \in F_2$, let

$$\inf(P_koC_1(a), P_koD_2(a_2b_2)) = \alpha_k,$$

$$P_koD((a, a_1)(a, b_2)) = \beta_k, 1 \leq k \leq m.$$

Then we have $P_koC_1(a) \geq \alpha_k$, $P_koD_2(a_2b_2) \geq \alpha_k$, that is, $a \in (C_1)_t$, $a_2b_2 \in (D_2)_t$, $t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ and $(a, a_2)(a, b_2) \in D_{t'}$, $t' = (\beta_1, \beta_2, \cdots, \beta_m)$. Since $(C_t, D_t)(\mathrm{resp.}(\mathrm{C}_{t'}, \mathrm{D}_{t'}))$ is the Cartesian product of $((C_1)_t, (D_1)_t)$ and $((C_2)_t, (D_2)_t)$ $(\mathrm{resp.}(\mathrm{C}_1)_{t'}, (\mathrm{D}_1)_{t'})$ and $((C_2)_{t'}, (D_2)_{t'})$ we have $(a, a_2)(a, b_2) \in D_t$, $a \in (C_1)_{t'}$ and $a_2b_2 \in (D_2)_{t'}$, which implies $P_koC_1(a) \geq \beta_k$, $P_koD_2(a_2b_2) \geq \beta_k$. It follows that

$$P_koD((a, a_2)(a, b_2)) \geq \alpha_k = \inf(P_koC_1(a), P_koD_2(a_2b_2)),$$

$$\inf(P_koC_1(a), P_koD_2(a_2b_2)) \geq \beta_k = P_koD((a, a_2)(a, b_2)).$$

Therefore,

$$P_koD((a, a_2)(a, b_2)) = \inf(P_koC_1(a), P_koD_2(a_2b_2))$$

for all $a \in Y_1$ and $a_2b_2 \in F_2$. Similarly, we can show that

$$P_koD((a_1, c)(b_1, c)) = \inf(P_koD_1(a_1b_1), P_koC_2(c))$$

for all $c \in Y_2$ and $a_1b_1 \in F_1$. This completes the proof. □

Definition 2.10. Let G_1 and G_2 be m-polar fuzzy graphs of G_1^* and G_2^*, respectively. The composition $G_1[G_2]$ is the pair (C, D) of m-polar fuzzy sets defined on the composition $G_1^*[G_2^*]$ such that

(i) $P_koC(a_1, a_2) = \inf(P_koC_1(a_1), P_koC_2(a_2))$ for all $(a_1, a_2) \in Y_1 \times Y_2$,

(ii) $P_koD((a, a_2)(a, b_2)) = \inf(P_koC_1(a), P_koD_2(a_2b_2))$ for all $a \in Y_1$ and for all $a_2b_2 \in F_2$,

(iii) $P_koD((a_1, c)(b_1, c)) = \inf(P_koD_1(a_1b_1), P_koC_2(c))$ for all $c \in Y_2$ and for all $a_1b_1 \in F_1$,

(iv) $P_koD((a_1, a_2)(b_1, b_2)) = \inf(P_koD_1(a_1b_1), P_koC_2(a_2), P_koC_2(b_2))$ for all $a_2, b_2 \in Y_2$, where $a_2 \neq b_2$ and for all $a_1b_1 \in F_1$.

Theorem 2.11. *Let $G_1$and G_2 be m-polar fuzzy graphs of G_1^* and G_2^*, respectively. Then G is the composition of G_1 and G_2 if and only if for each $t \in [0, 1]^m$, $t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ the t-level graph G_t is the composition of $(G_1)_t$ and $(G_2)_t$.*

Proof. By the definition of $G_1[G_2]$ and in the same way as in the proof of Theorem 2.9, we have $C_t = (C_1)_t \times (C_2)_t$. We prove $D_t = F$, where F is the edge set of the composition $(G_1)_t[(G_2)_t]$ for all $t \in [0,1]^m$, $t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$. Let $(a_1, a_2)(b_1, b_2) \in D_t$. Then $P_k oD((a_1, a_2)(b_1, b_2)) \geq \alpha_k$, $1 \leq k \leq m$. Since G is the composition $G_1[G_2]$, one of the following cases hold:

(i) $a_1 = b_1$ and $a_2 b_2 \in F_2$.
(ii) $a_2 = b_2$ and $a_1 b_1 \in F_1$.
(iii) $a_2 \neq b_2$ and $a_1 b_1 \in F_1$.

For the cases (i) and (ii), similarly as in the cases (i) and (ii) in the proof of Theorem 2.9, we obtain $(a_1, a_2)(b_1, b_2) \in F$. For the case (iii), we have

$$P_k oD((a_1, a_2)(b_1, b_2)) = \inf(P_k oD_1(a_1 b_1), P_k oC_2(a_2), P_k oC_2(b_2)) \geq \alpha_k$$

Thus, $P_k oC_2(a_2) \geq \alpha_k$, $P_k oC_2(b_2) \geq \alpha_k$, $P_k oD_1(a_1 b_1) \geq \alpha_k$, $1 \leq k \leq m$. It follows that $a_2, b_2 \in (C_2)_t$ and $a_1 b_1 \in (D_1)_t$, that is, $(a_1, a_2)(b_1, b_2) \in F$. Therefore, $D_t \subseteq F$. For every $(a, a_2)(a, b_2) \in F, P_k oC_1(a) \geq \alpha_k, P_k oD_2(a_2 b_2) \geq \alpha_k$, $1 \leq k \leq m$. Since $G = (C, D)$ is the composition $G_1[G_2]$, we have

$$P_k oD((a, a_2)(a, b_2)) = \inf(P_k oC_1(a), P_k oD_2(a_2 b_2)) \geq \alpha_k,$$

$1 \leq k \leq m$. Therefore, $(a, a_2)(a, b_2) \in D_t$. Similarly, for every $(a_1, c)(b_1, c) \in F$, we have $(a_1, c)(b_1, c) \in D_t$. For every $(a_1, a_2)(b_1, b_2) \in F$ where $a_2 \neq b_2$, $a_1 \neq b_1$, $P_k oD_1(a_1 b_1) \geq \alpha_k$, $P_k oC_2(a_2) \geq \alpha_k$, $P_k oC_2(b_2) \geq \alpha_k$, $1 \leq k \leq m$. Since G is the composition $G_1[G_2]$, we have

$$P_k oD((a_1, a_2)(b_1, b_2)) = \inf(P_k oD_1(a_1 b_1), P_k oC_2(a_2), P_k oC_2(b_2)) \geq \alpha_k$$

$1 \leq k \leq m$. Thus, $(a_1, a_2)(b_1, b_2) \in D_t$. Therefore, $F \subseteq D_t$, and so $F = D_t$.
Conversely, suppose that $G_t = (C_t, D_t)$, where $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ is the composition of $(G_1)_t = ((C_1)_t, (D_1)_t)$ and $(G_2)_t = ((C_2)_t, (D_2)_t)$. By the definition of the composition and the proof of Theorem 2.9, we have

(i) $P_k oC(a_1, a_2) = \inf(P_k oC_1(a_1), P_k oC_2(a_2))$ for all $(a_1, a_2) \in Y_1 \times Y_2$,
(ii) $P_k oD((a, a_2)(a, b_2)) = \inf(P_k oC_1(a), P_k oD_2(a_2 b_2))$ for all $a \in Y_1$ and for all $a_2 b_2 \in F_2$,
(iii) $P_k oD((a_1, c)(b_1, c)) = \inf(P_k oD_1(a_1 b_1), P_k oC_2(c))$ for all $c \in Y_2$ and for all $a_1 b_1 \in F_1$.

Similarly, by using same arguments as in the proof of Theorem 2.9, we obtain

$$P_k oD((a_1, a_2)(b_1, b_2)) = \inf(P_k oD_1(a_1 b_1), P_k oC_2(a_2), P_k oC_2(b_2))$$

for all $a_2, b_2 \in Y_2$ $(a_2 \neq b_2)$ and for all $a_1 b_1 \in F_1$. This completes the proof. □

Definition 2.12. Let G_1 and G_2 be m-polar fuzzy graphs of G_1^* and G_2^*, respectively. The union $G_1 \cup G_2$ is defined as the pair (C, D) of m-polar fuzzy sets determined on the union of graphs G_1^* and G_2^* such that

(i) $P_k oC(a) = \begin{cases} P_k oC_1(a) & \text{if } a \in Y_1 \text{ and } a \notin Y_2, \\ P_k oC_2(a) & \text{if } a \in Y_2 \text{ and } a \notin Y_1, \\ \sup(P_k oC_1(a), P_k oC_2(a)) & \text{if } a \in Y_1 \cap Y_2. \end{cases}$

(ii) $P_k oD(ab) = \begin{cases} P_k oD_1(ab) & \text{if } ab \in F_1 \text{ and } ab \notin F_2, \\ P_k oD_2(ab) & \text{if } ab \in F_2 \text{ and } ab \notin F_1, \\ \sup(P_k oD_1(ab), P_k oD_2(ab)) & \text{if } ab \in F_1 \cap F_2. \end{cases}$

Theorem 2.13. *Let G_1 and G_2 be m-polar fuzzy graphs of G_1^* and G_2^*, respectively, and $Y_1 \cap Y_2 = \emptyset$. Then G is the union of G_1 and G_2 if and only if each t-level graph G_t is the union of $(G_1)_t$ and $(G_2)_t$.*

Proof. We show that $C_t = (C_1)_t \cup (C_2)_t$ for each $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$. Let $a \in C_t$, then $a \in Y_1 \setminus Y_2$ or $a \in Y_2 \setminus Y_1$. If $a \in Y_1 \setminus Y_2$, then $P_k oC_1(a) = P_k oC(a) \geq \alpha_k, 1 \leq k \leq m$ which implies $a \in (C_1)_t$. Analogously $a \in Y_2 \setminus Y_1$ implies $a \in (C_2)_t$. Therefore, $a \in (C_1)_t \cup (C_2)_t$, and so $C_t \subseteq (C_1)_t \cup (C_2)_t$. Now let $a \in (C_1)_t \cup (C_2)_t$. Then $a \in (C_1)_t,\ a \notin (C_2)_t$ or $a \in (C_2)_t,\ a \notin (C_1)_t$. For the first case, we have $P_k oC_1(a) = P_k oC(a) \geq \alpha_k, 1 \leq k \leq m$ which implies $a \in C_t$. For the second case, we have $P_k oC_2(a) = P_k oC(a) \geq \alpha_k, 1 \leq k \leq m$. Hence $a \in C_t$. Consequently, $(C_1)_t \cup (C_2)_t \subseteq C_t$.
To prove that $D_t = (D_1)_t \cup (D_2)_t$, for all $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$, consider $ab \in D_t$. Then $ab \in F_1 \setminus F_2$ or $ab \in F_2 \setminus F_1$. For $ab \in F_1 \setminus F_2$ we have $P_k oD_1(ab) = P_k oD(ab) \geq \alpha_k, 1 \leq k \leq m$. Thus $ab \in (D_1)_t$. Similarly $ab \in F_2 \setminus F_1$ gives $ab \in (D_2)_t$. Therefore $D_t \subseteq (D_1)_t \cup (D_2)_t$. If $ab \in (D_1)_t \cup (D_2)_t$, then $ab \in (D_1)_t \setminus (D_2)_t$ or $ab \in (D_2)_t \setminus (D_1)_t$. For the first case $P_k oD(ab) = P_k oD_1(ab) \geq \alpha_k, 1 \leq k \leq m$, hence $ab \in D_t$. In the second case we obtain $ab \in D_t$. Therefore, $(D_1)_t \cup (D_2)_t \subseteq D_t$.
Conversely, let for all $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ the level graph $G_t = (C_t, D_t)$ be the union of $(G_1)_t = ((C_1)_t, (D_1)_t)$ and $(G_2)_t = ((C_2)_t, (D_2)_t)$. Let $a \in Y_1, P_k oC_1(a) = \alpha_k, P_k oC(a) = \beta_k, 1 \leq k \leq m$, Then $a \in (C_1)_t$ where $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ and $a \in C_{t'}$ where $t' \in [0,1]^m, t' = (\beta_1, \beta_2, \cdots, \beta_m)$. But by the hypothesis $a \in (C_1)_{t'}$ and $a \in C_t$. Thus, $P_k oC_1(a) \geq \beta_k, P_k oC(a) \geq \alpha_k, 1 \leq k \leq m$. Therefore, $P_k oC_1(a) \leq P_k oC(a)$ and $P_k oC_1(a) \geq P_k oC(a)$. Hence $P_k oC_1(a) = P_k oC(a)$. Similarly, for every $a \in Y_2$, we get $P_k oC_2(a) = P_k oC(a)$. Thus we conclude that

$$\text{(i)} \begin{cases} P_k oC(a) = P_k oC_1(a) & \text{if } a \in Y_1, \\ P_k oC(a) = P_k oC_2(a) & \text{if } a \in Y_2. \end{cases}$$

By a similar method as above, we obtain

$$\text{(ii)} \begin{cases} P_k oD(ab) = P_k oD_1(ab) & \text{if } ab \in F_1, \\ P_k oD(ab) = P_k oD_2(ab) & \text{if } ab \in F_2. \end{cases}$$

This completes the proof. □

Definition 2.14. Let G_1 and G_2 be m-polar fuzzy graphs of G_1^* and G_2^*, respectively. The join $G_1 + G_2$ is the pair (C, D) of m-polar fuzzy sets defined on the join $G_1^* + G_2^*$ such that

$$\text{(i)}\ P_k oC(a) = \begin{cases} P_k oC_1(a) & \text{if } a \in Y_1 \text{ and } a \notin Y_2, \\ P_k oC_2(a) & \text{if } a \in Y_2 \text{ and } a \notin Y_1, \\ \sup(P_k oC_1(a), P_k oC_2(a)) & \text{if } a \in Y_1 \cap Y_2. \end{cases}$$

$$\text{(ii)}\ P_k oD(ab) = \begin{cases} P_k oD_1(ab) & \text{if } ab \in F_1 \text{ and } ab \notin F_2, \\ P_k oD_2(ab) & \text{if } ab \in F_2 \text{ and } ab \notin F_1, \\ \sup(P_k oD_1(ab), P_k oD_2(ab)) & \text{if } ab \in F_1 \cap F_2, \\ \inf(P_k oC_1(a), P_k oC_2(b)) & \text{if } ab \in F'. \end{cases}$$

Theorem 2.15. *Let G_1 and G_2 be m-polar fuzzy graphs of G_1^* and G_2^*, respectively, and $Y_1 \cap Y_2 = \emptyset$. Then G is the join of G_1 and G_2 if and only if each t-level graph G_t is the join of $(G_1)_t$ and $(G_2)_t$.*

Proof. By the definition of union and the proof of Theorem 2.13, $C_t = (C_1)_t \cup (C_2)_t$, for all $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$. We show that $D_t = (D_1)_t \cup (D_2)_t \cup F'_t$ for all $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$, where F'_t is the set of all edges joining the vertices of $(C_1)_t$ and $(C_2)_t$.
From the proof of Theorem 2.13, it follows that $(D_1)_t \cup (D_2)_t \subseteq D_t$. If $ab \in F'_t$, then $P_k o C_1(a) \geq \alpha_k, P_k o C_2(b) \geq \alpha_k, 1 \leq k \leq m$. Hence

$$P_k o D(ab) = \inf(P_k o C_1(a), P_k o C_2(b)) \geq \alpha_k$$

It follows that $ab \in D_t$. Therefore, $(D_1)_t \cup (D_2)_t \cup F'_t \subseteq D_t$. For every $ab \in D_t$, if $ab \in F_1 \cup F_2$, then $ab \in (D_1)_t \cup (D_2)_t$, by the proof of Theorem 2.13. If $a \in Y_1$ and $b \in Y_2$, then

$$\inf(P_k o C_1(a), P_k o C_2(b)) = P_k o D(ab) \geq \alpha_k,$$

so $a \in (C_1)_t$ and $b \in (C_2)_t$. Thus $ab \in F'_t$. Therefore, $D_t \subseteq (D_1)_t \cup (D_2)_t \cup F'_t$.
Conversely, let each level graph $G_t = (C_t, D_t)$ be the join of $(G_1)_t = ((C_1)_t, (D_1)_t)$ and $(G_2)_t = ((C_2)_t, (D_2)_t)$. From the proof of the Theorem 2.13, we have

(i) $\begin{cases} P_k o C(a) = P_k o C_1(a) & \text{if } a \in Y_1, \\ P_k o C(a) = P_k o C_2(a) & \text{if } a \in Y_2. \end{cases}$

(ii) $\begin{cases} P_k o D(ab) = P_k o D_1(ab) & \text{if } ab \in F_1, \\ P_k o D(ab) = P_k o D_2(ab) & \text{if } ab \in F_2. \end{cases}$

let $a \in Y_1, b \in Y_2, \inf(P_k o C_1(a), P_k o C_2(b)) = \alpha_k, P_k o D(ab) = \beta_k$. Then $a \in (C_1)_t, b \in (C_2)_t$ where $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ and $ab \in D_{t'}$ where $t' \in [0,1]^m, t' = (\beta_1, \beta_2, \cdots, \beta_m)$. It follows that $ab \in D_t, a \in (C_1)_{t'}$ and $b \in (C_2)_{t'}$. So, $P_k o D(ab) \geq \alpha_k, P_k o C_1(a) \geq \beta_k$ and $P_k o C_2(b) \geq \beta_k$. Therefore,

$$P_k o D(ab) \geq \alpha_k = \inf(P_k o C_1(a), P_k o C_2(b)) \geq \beta_k = P_k o D(ab).$$

Thus,

$$P_k o D(ab) = \inf(P_k o C_1(a), P_k o C_2(b)).$$

□

Definition 2.16. Let G_1 and G_2 be m-polar fuzzy graphs of G_1^* and G_2^*, respectively. The cross product $G_1 * G_2$ is the pair (C, D) of m-polar fuzzy sets defined on the cross product $G_1^* * G_2^*$ such that

(i) $P_k o C(a_1, a_2) = \inf(P_k o C_1(a_1), P_k o C_2(a_2))$ for all $(a_1, a_2) \in Y_1 \times Y_2$,
(ii) $P_k o D((a_1, a_2)(b_1, b_2)) = \inf(P_k o D_1(a_1 b_1), P_k o D_2(a_2 b_2))$ for all $a_1 b_1 \in F_1$ and for all $a_2 b_2 \in F_2$.

Theorem 2.17. *Let G_1 and G_2 be m-polar fuzzy graphs of G_1^* and G_2^*, respectively. Then $G = (C, D)$ is the cross product of G_1 and G_2 if and only if each level graph G_t is the cross product of $(G_1)_t$ and $(G_2)_t$.*

Proof. By the definition of the Cartesian product and the proof of Theorem 2.9, we have $C_t = (C_1)_t \times (C_2)_t$, for all $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$. We show that

$$D_t = \{(a_1, a_2)(b_1, b_2) \mid a_1 b_1 \in (D_1)_t,\ a_2 b_2 \in (D_2)_t\}$$

for all $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$. Infact, if $(a_1, a_2)(b_1, b_2) \in D_t$, then

$$P_k o D((a_1, a_2)(b_1, b_2)) = \inf(P_k o D_1(a_1 b_1), P_k o D_2(a_2 b_2)) \geq \alpha_k$$

so $P_k o D_1(a_1 b_1) \geq \alpha_k$ and $P_k o D_2(a_2 b_2) \geq \alpha_k, 1 \leq k \leq m$. So, $a_1 b_1 \in (D_1)_t$ and $a_2 b_2 \in (D_2)_t$. Now if $a_1 b_1 \in (D_1)_t$ and $a_2 b_2 \in (D_2)_t$, then $P_k o D_1(a_1 b_1) \geq \alpha_k$ and $P_k o D_2(a_2 b_2) \geq \alpha_k, 1 \leq k \leq m$. It follows that

$$P_k o D((a_1, a_2)(b_1, b_2)) = \inf(P_k o D_1(a_1 b_1), P_k o D_2(a_2 b_2)) \geq \alpha_k$$

Since $G = (C, D)$ is the cross product of $G_1 * G_2$. Therefore, $(a_1, a_2)(b_1, b_2) \in D_t$. Conversely, let each t-level graph $G_t = (C_t, D_t)$ be the cross product of $(G_1)_t = ((C_1)_t, (D_1)_t)$ and $(G_2)_t = ((C_2)_t, (D_2)_t)$. In view of the fact that the cross product (C_t, D_t) has the same vertex set as the Cartesian product of $((C_1)_t, (D_1)_t)$ and $((C_2)_t, (D_2)_t)$, and by the proof of Theorem 2.9, we have

$$P_k o C((a_1, a_2)) = \inf(P_k o C_1(a_1), P_k o C_2(a_2)) \text{ for all } (a_1, a_2) \in Y_1 \times Y_2.$$

Let $\inf(P_k o D_1(a_1 b_1), P_k o D_2(a_2 b_2)) = \alpha_k$ and $P_k o D((a_1, a_2)(b_1, b_2)) = \beta_k, 1 \leq k \leq m$ for $a_1 b_1 \in F_1,\ a_2 b_2 \in F_2$. Then $P_k o D_1(a_1 b_1) \geq \alpha_k, P_k o D_2(a_2 b_2) \geq \alpha_k$ and $(a_1, a_2)(b_1, b_2) \in D_{t'}$ where $t' \in [o,1]^m, t' = (\beta_1, \beta_2, \cdots, \beta_m)$, hence $a_1 b_1 \in (D_1)_t,\ a_2 b_2 \in (D_2)_t$, where $t \in [o,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ and consequently $a_1 b_1 \in (D_1)_{t'},\ a_2 b_2 \in (D_2)_{t'}$, since $D_{t'} = \{(a_1, a_2)(b_1, b_2) \mid a_1 b_1 \in (D_1)_{t'},\ a_2 b_2 \in (D_2)_{t'}\}$. It follows that $(a_1, a_2)(b_1, b_2) \in D_t,\ P_k o D_1(a_1 b_1) \geq \beta_k, P_k o D_2(a_2 b_2) \geq \beta_k, 1 \leq k \leq m$. Therefore, $P_k o D((a_1, a_2)(b_1, b_2)) = \beta_k \leq \inf(P_k o D_1(a_1 b_1), P_k o D_2(a_2 b_2)) = \alpha_k \leq P_k o D((a_1, a_2)(b_1, b_2))$. Hence

$$P_k o D((a_1, a_2)(b_1, b_2)) = \inf(P_k o D_1(a_1 b_1), P_k o D_2(a_2 b_2)).$$

This completes the proof. □

Definition 2.18. Let G_1 and G_2 be m-polar fuzzy graphs. The lexicographic product $G_1 \bullet G_2$ is the pair (C, D) of m-polar fuzzy sets defined on the lexicographic product $G_1^* \bullet G_2^*$ such that

(i) $P_k o C(a_1, a_2) = \inf(P_k o C_1(a_1), P_k o C_2(a_2))$ for all $(a_1, a_2) \in Y_1 \times Y_2$,

(ii) $P_k o D((a, a_2)(a, b_2)) = \inf(P_k o C_1(a), P_k o D_2(a_2 b_2))$ for all $a \in Y_1$ and for all $a_2 b_2 \in F_2$,

(iii) $P_k o D((a_1, a_2)(b_1, b_2)) = \inf(P_k o D_1(a_1 b_1), P_k o D_2(a_2 b_2))$ for all $a_1 b_1 \in F_1$ and for all $a_2 b_2 \in F_2$.

Theorem 2.19. *Let G_1 and G_2 be m-polar fuzzy graphs. Then G is the lexicographic product of G_1 and G_2 if and only if $G_t = (G_1)_t \bullet (G_2)_t$ for all $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$.*

Proof. By the definition of Cartesian product $G_1 \times G_2$ and the proof of Theorem 2.9, we have $C_t = (C_1)_t \times (C_2)_t$ for all $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$. We show that $D_t = F_t \cup F'_t$ for all $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$, where $F_t = \{(a, a_2)(a, b_2) \mid a \in Y_1,\ a_2 b_2 \in (D_2)_t\}$ is the subset of the edge set of the cross product $(G_1)_t \times (G_2)_t$, and $F'_t = \{(a_1, a_2)(b_1, b_2) \mid a_1 b_1 \in (D_1)_t,\ a_2 b_2 \in (D_2)_t\}$ is the edge set of the cross product $(G_1)_t * (G_2)_t$. For every $(a_1, a_2)(b_1, b_2) \in D_t,\ a_1 = b_1,\ a_2 b_2 \in F_2$ or $a_1 b_1 \in F_1,\ a_2 b_2 \in F_2$. If $a_1 = b_1,\ a_2 b_2 \in F_2$, then $(a_1, a_2)(b_1, b_2) \in F_t$, by the definition of the Cartesian product and the proof of Theorem 2.9. If $a_1 b_1 \in F_1,\ a_2 b_2 \in F_2$, then $(a_1, a_2)(b_1, b_2) \in F'_t$, by the definition

of cross product and the proof of Theorem 2.17. Therefore, $D_t \subseteq F_t \cup F'_t$. From the definition of the Cartesian product and the proof of Theorem 2.9, we conclude that $F_t \subseteq D_t$, and also from the definition of cross product and the proof of Theorem 2.17, we obtain $F'_t \subseteq D_t$. Therefore, $F_t \cup F'_t \subseteq D_t$.
Conversely, let $G_t = (C_t, D_t) = (G_1)_t \bullet (G_2)_t$ for all $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$. We know that $(G_1)_t \bullet (G_2)_t$ has the same vertex set as the Cartesian product $(G_1)_t \times (G_2)_t$. Now by the proof of Theorem 2.9, we have

$$P_k oC((a_1, a_2)) = \inf(P_k oC_1(a_1), P_k oC_2(a_2)) \text{ for all } (a_1, a_2) \in Y_1 \times Y_2.$$

Let for $a \in Y_1$ and $a_2b_2 \in F_2$ will be $\inf(P_k oC_1(a), P_k oD_2(a_2b_2)) = \alpha_k$ and $P_k oD((a, a_2)(a, b_2)) = \beta_k, 1 \leq k \leq m$. Then, in view of the definitions of the Cartesian product and lexicographic product, we have

$$(a, a_2)(a, b_2) \in (D_1)_t \bullet (D_2)_t \Longleftrightarrow (a, a_2)(a, b_2) \in (D_1)_t \times (D_2)_t,$$

$$(a, a_2)(a, b_2) \in (D_1)_{t'} \bullet (D_2)_{t'} \Longleftrightarrow (a, a_2)(a, b_2) \in (D_1)_{t'} \times (D_2)_{t'}.$$

From this, by the same way as in the proof of Theorem 2.9, we conclude

$$P_k oD((a, a_2)(a, b_2)) = \inf(P_k oC_1(a), P_k oD_2(a_2b_2)).$$

Now let $P_k oD((a_1, a_2)(b_1, b_2)) = \alpha_k$, $\inf(P_k oD_1(a_1b_1), P_k oD_2(a_2b_2)) = \beta_k, 1 \leq k \leq m$ for $a_1b_1 \in F_1$ and $a_2b_2 \in F_2$. Then in view of the definitions of cross product and the lexicographic product, we have

$$(a_1, a_2)(b_1, b_2) \in (D_1)_t \bullet (D_2)_t \Longleftrightarrow (a_1, a_2)(b_1, b_2) \in (D_1)_t * (D_2)_t,$$

$$(a_1, a_2)(b_1, b_2) \in (D_1)_{t'} \bullet (D_2)_{t'} \Longleftrightarrow (a_1, a_2)(b_1, b_2) \in (D_1)_{t'} * (D_2)_{t'}.$$

By the same way as in the proof of Theorem 2.17, we can conclude

$$P_k oD((a_1, a_2)(b_1, b_2)) = \inf(P_k oD_1(a_1b_1), P_k oD_2(a_2b_2)),$$

which completes the proof. □

Proposition 2.20. *Let G_1 and G_2 be m-polar fuzzy graphs of $G_1^* = (Y_1, F_1)$ and $G_2^* = (Y_2, F_2)$, respectively, such that $Y_1 = Y_2,\ C_1 = C_2$ and $F_1 \cap F_2 = \emptyset$. Then $G = (C, D)$ is the union of G_1 and G_2 if and only if G_t is the union of $(G_1)_t$ and $(G_2)_t$ for all $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$.*

Proof. Let $G = (C, D)$ be the union of m-polar fuzzy graphs G_1 and G_2. Then by the definition of the union and the fact that $Y_1 = Y_2,\ C_1 = C_2$, we have $C = C_1 = C_2$, hence $C_t = (C_1)_t \cup (C_2)_t$. We now show that $D_t = (D_1)_t \cup (D_2)_t$ for all $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$. For every $ab \in (D_1)_t$ we have $P_k oD(ab) = P_k oD_1(ab) \geq \alpha_k, 1 \leq k \leq m$, hence $ab \in D_t$. Therefore, $(D_1)_t \subseteq D_t$. Similarly we obtain $(D_2)_t \subseteq D_t$. Thus, $(D_1)_t \cup (D_2)_t \subseteq D_t$. For every $ab \in D_t,\ ab \in F_1$ or $ab \in F_2$. If $ab \in F_1$, $P_k oD_1(ab) = P_k oD(ab) \geq \alpha_k, 1 \leq k \leq m$ and hence $ab \in (D_1)_t$. If $ab \in F_2$, we have $ab \in (D_2)_t$. Therefore, $D_t \subseteq (D_1)_t \cup (D_2)_t$.
Conversely, suppose that the t-level graph $G_t = (C_t, D_t)$ be the union of $(G_1)_t = ((C_1)_t, (D_1)_t)$ and $(G_2)_t = ((C_2)_t, (D_2)_t)$. Let $P_k oC(a) = \alpha_k, P_k oC_1(a) = \beta_k, 1 \leq k \leq m$ for some $a \in Y_1 = Y_2$. Then $a \in C_t$ where $t \in [o,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ and $a \in (C_1)_{t'}$ where $t' \in [0,1]^m, t' = (\beta_1, \beta_2, \cdots, \beta_m)$, so $a \in (C_1)_t$ and $a \in C_{t'}$, because $C_t = (C_1)_t$ and $C_{t'} = (C_1)_{t'}$. It follows that $P_k oC_1(a) \geq \alpha_k$, and $P_k oC(a) \geq \beta_k, 1 \leq k \leq m$. Therefore, $P_k oC_1(a) \geq P_k oC(a)$and $P_k oC(a) \geq$

$P_k o C_1(a)$. So, $P_k o C(a) = P_k o C_1(a)$. Since $C_1 = C_2,\ Y_1 = Y_2$, then $C = C_1 = C_1 \cup C_2$.
By a similar method, we conclude that

$$(1) \begin{cases} P_k o D(ab) = P_k o D_1(ab) & \text{if } ab \in F_1, \\ P_k o D(ab) = P_k o D_2(ab) & \text{if } ab \in F_2. \end{cases}$$

This completes the proof. □

Definition 2.21. Let G_1 and G_2 be m-polar fuzzy graphs of G_1^* and G_2^*, respectively. The strong product $G_1 \boxtimes G_2$ is the pair (C, D) of m-polar fuzzy sets defined on the strong product $G_1^* \boxtimes G_2^*$ such that

(i) $P_k o C(a_1, a_2) = \inf(P_k o C_1(a_1), P_k o C_2(a_2))$ for all $(a_1, a_2) \in Y_1 \times Y_2$,
(ii) $P_k o D((a, a_2)(a, b_2)) = \inf(P_k o C_1(a), P_k o D_2(a_2 b_2))$ for all $a \in Y_1$ and for all $a_2 b_2 \in F_2$,
(iii) $P_k o D((a_1, c)(b_1, c)) = \inf(P_k o D_1(a_1 b_1), P_k o C_2(c))$ for all $c \in Y_2$ and for all $a_1 b_1 \in F_1$,
(iv) $P_k o D((a_1, a_2)(b_1, b_2)) = \inf(P_k o D_1(a_1 b_1), P_k o D_2(a_2 b_2))$ for all $a_1 b_1 \in F_1$ and for all $a_2 b_2 \in F_2$.

We state the following Theorem without its proof.

Theorem 2.22. *Let G_1 and G_2 be m-polar fuzzy graphs of G_1^* and G_2^*, respectively. Then G is the strong product of G_1 and G_2 if and only if G_t, where $t \in [0,1]^m, t = (\alpha_1, \alpha_2, \cdots, \alpha_m)$, is the strong product of $(G_1)_t$ and $(G_2)_t$.*

3. Conclusion

An m-polar fuzzy set is an extension of a bipolar fuzzy set. An m-polar fuzzy model is useful for multi-polar information, multi-agent, multi-attribute and multi-object network models which gives more precision, flexibility, and comparability to the system as compared to the classical, fuzzy and bipolar fuzzy models. In this research article, we have presented certain characterization of m-polar fuzzy graphs by level graphs. We have aim to extend our work to (1) single-valued neutrosophic soft graph structures, (2) single-valued neutrosophic rough fuzzy graph structures, (3) single-valued neutrosophic rough fuzzy soft graph structures, and (4) single-valued neutrosophic fuzzy soft graph structures.

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