Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 50(2)(2018) pp. 1-10

Improved Complexity of a Homotopy Method for Locating an Approximate Zero

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Received: 25 August, 2017 / Accepted: 21 November, 2017 / Published online: 18 January, 2018

Abstract. The goal of this study is to extend the applicability of a homotopy method for locating an approximate zero using Newton's method. The improvements are obtained using more precise Lipschitz-type functions than in earlier works and our new idea of restricted convergence regions. Moreover, these improvements are found under the same computational effort.

AMS Subject Classifications: 65G99, 65H10, 47H17, 49M15,
Key words: Approximate zero, Banach space, Homotopy method, Newton's method, local, semi-local convergence, Lipschitz-type conditions.

1 Introduction

The convergence region and error analysis of iterative methods are very pessimistic in general for both the semi-local and local case [1–5, 11–16]. The aim of the paper is to extend the convergence region using the homotopy method. This goal is achieved using the same Lipschitz-type functions as before [4, 6–10, 13]. We achieve this goal, since we find a more precise location for the Newton iterates leading to at least as tight Lipschitz-type functions [4, 6, 7]. Let $F : D \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be differentiable in the sense of Fréchet, D be a convex and open subset of $\mathcal{B}_1$ and $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces.

Let F' is one-to-one and onto, we introduce the Newton operator

$$N_F(x) := x - F'(x)^{-1}F(x) \tag{1.1}$$

and the corresponding Newton iteration

$$x_{n+1} = N_F(x_n) \text{ for all } n = 0, 1, 2, \ldots \tag{1.2}$$

where $x_0 \in D$ is an initial point. We are concerned with the problem of approximating a regular (to be precised in Section 2) solution w of

$$F(x) = 0 \tag{1.3}$$

utilizing a homotopy method of the form

$$\mathcal{H}(x, t) := F(x) - tF(x_0) \tag{1.4}$$

where $x_0 \in D$ is a given initial point and $t \in [0, 1]$. Clearly this is a geometrical way of solving equation (1.3). Consider the line segment $M = \{tF(x_0) : t \in [0, 1]\}$ and the set $F^{-1}(M)$. Suppose that $F'(x_0)$ is one-to-one and onto. Then, it follows by the implicit function theorem applied in a neighbourhood of x_0 that there exists a curve $x(t)$ solving the equation $F(x(t)) = tF(x_0)$ for $t \in [1 - \varepsilon, 1]$ and $\varepsilon > 0$. This curve solves the initial value problem (IVP)

$$\dot{x}(t) = -DF(x(t))^{-1}F(x_0), \; x(1) = x_0. \tag{1.5}$$

It is well known that (1.5) has no solution on $[0, 1]$, in general. But if it has a solution, one must follow $x(t)$ (numerically), which is given by $\mathcal{H}(x(t), t) = 0$ using the operator related to $\mathcal{H}(., t)$. That is consider the sequence $\{s_n\}$ given by $s_0 = 1 > s_1 > \ldots > s_n > \ldots > 0$ such that

$$x_{n+1} = N_{\mathcal{H}(., s_{n+1})}(x_k)$$

is an approximate zero of $x(s_{n+1})$, with

$$\mathcal{H}(x(s_{n+1}), s_{n+1}) = 0.$$

A convergence analysis of Newton sequence $\{x_n\}$ was given in the elegant work by Guttierrez et al. [10]. Here, we improve their results as already mentioned previously.

The study is structured as: The convergence of Newton's method is presented in Section 2 whereas Section 3 contains the special cases. Finally, in Section 4, we present the numerical examples.

1 Convergence Analysis

We need the Definition of an approximate zero.

Definition 1.1 *[14] A $G-$regular ball is open so that $G'(x)$ is one-to-one and onto. A point x_0 is a regular approximate zero of G, provided there exists a ball $G-$regular containing a zero w of G and a sequence $\{x_n\}$ converging to w.*

Let $L_0, \bar{L}, L : [0, +\infty) \longrightarrow [0, +\infty)$ be continuous and non-decreasing functions. These functions are needed for the introduction of the Lipschitz conditions that follow (see (1.7), (1.9) and (1.11)). We shall also suppose that there exists $z \in D$ so $G'(z)$ is continuous, one-to-one, onto and $G'(z)^{-1}$ exists. We need to introduce the following two Lipschitz conditions that follow.

Definition 1.2 *The function $G'(z)^{-1}G'$ is L_0- center Lipschitz at z if there exist positive quantities v_0 and*

$$\gamma_0 := \gamma_0(G, z) \tag{1.6}$$

satisfying for $a \in D, \gamma_0(\|a - z\|) \le v_0$

$$\|G'(z)^{-1}(G'(a) - G'(z))\| \le \int_0^{\gamma_0 \|a-z\|} L_0(\tau)d\tau. \tag{1.7}$$

Definition 1.3 *The function $G'(z)^{-1}G'$ is $\bar{L}-$center Lipschitz restricted at z, if there exist positive quantities $\bar{v}$ and*

$$\bar{\gamma} := \bar{\gamma}(G, z) \tag{1.8}$$

satisfying for $a, b \in D_0 := D \cap \bar{U}(z, \frac{\bar{v}}{\bar{\gamma}})$

$$\bar{\gamma}(\|a - z\| + \tau\|a - b\|) \le \bar{v}$$

and

$$\|G'(z)^{-1}(G'((1-\tau)a + \tau b) - G'(a))\| \le \int_{\bar{\gamma}\|a-z\|}^{\bar{\gamma}(\|a-z\|+\tau\|b-a\|)} \bar{L}(\tau)d\tau \tag{1.9}$$

for all $\tau \in [0, 1]$.

Definition 1.4 *[10] The function $G'(y_0)^{-1}G'$ is $L-$center Lipschitz at z if there exist positive quantities v and*

$$\gamma := \gamma(G, z) \tag{1.10}$$

satisfying for $a, b \in D$

$$\gamma(\|a - z\| + \tau\|a - b\|) \le v$$

and

$$\|G'(z)^{-1}(G'((1-\tau)a + \tau b) - G'(a))\| \le \int_{\gamma\|a-z\|}^{\gamma(\|a-z\|+\tau\|b-a\|)} L(\tau)d\tau \tag{1.11}$$

for each $\tau \in [0, 1]$.

REMARK 1.5 *Notice that (1.11) implies (1.7) and (1.9). We can certainly take $v_0 = v = \bar{v}$, $L_0(\tau) = L(\tau) = \bar{L}(\tau)$ for each $\tau \ge 0$, so for all $\tau \in [0, v]$*

$$\gamma_0(\tau) \le \gamma(\tau) \tag{1.12}$$

and

$$\bar{\gamma}(\tau) \le \gamma(\tau), \tag{1.13}$$

since $D_0 \subset D$.

In what follows we shall assume that

$$\gamma_0(\tau) \leq \bar{\gamma}(\tau). \tag{1.14}$$

If instead of (1.14)

$$\bar{\gamma}(\tau) \leq \gamma_0(\tau), \tag{1.15}$$

holds then the following results are true with $\bar{L}$ replacing L_0 in all of them.

LEMMA 1.6 *Suppose that v_0 is the least positive number such that*

$$\int_0^{v_0} L_0(\tau)d\tau = 1. \tag{1.16}$$

Then $F'(x)$ is one-to-one, onto and

$$\|F'(x)^{-1}F'(z)\| \leq \left(1 - \int_0^{\gamma_0\|x-z\|} L_0(\tau)d\tau\right)^{-1} \text{ for all } x \in U(z, \frac{v_0}{\gamma_0}). \tag{1.17}$$

The set $U(z, \frac{v_0}{\gamma_0})$ is called the γ_0-ball of z. We define similarly, the $\bar{\gamma}$ and $\gamma-$ balls. As in [10], we assume the existence of $\bar{\varphi} : [0, \bar{v}) \longrightarrow [0, +\infty)$ satisfying $\bar{\varphi}(0) = 1$, where

$$\bar{\gamma}(F, x) = \bar{\varphi}(\bar{\gamma}(F, z)\|x - z\|)\bar{\gamma} \text{ for each } x \text{ in } U(z, \frac{\bar{v}}{\bar{\gamma}}). \tag{1.18}$$

Moreover, for $b = b(F, z) := \|F'(z)^{-1}F(z)\|$ we set

$$\bar{\alpha} := \bar{\alpha}(F, z) := \bar{\gamma}\bar{b}. \tag{1.19}$$

By simply using (1.17) instead of the less precise estimate (since $\gamma_0(\tau) \leq \gamma(\tau)$)

$$\|F'(x)^{-1}F'(x_0)\| \leq \left(1 - \int_0^{\gamma_0\|x-x_0\|} L(\tau)d\tau\right)^{-1} \text{ for all } x \in U(x_0, \frac{v}{\gamma}). \tag{1.20}$$

as well as $\bar{\gamma}, \bar{v}$ instead of γ, v, respectively, we can reproduce the proofs of the results of [10] in this setting.

The following result improves Theorem 1 in [10] which in turn generalizes the corresponding result by Meyer [13].

THEOREM 1.7 *Suppose: $F'(x_0)^{-1}F$ is $\bar{L}-$ and L_0- Lipschitz restricted at $x_0 \in D$;*

$$\bar{\alpha}(F, x_0) \leq \int_0^{\bar{v}} \bar{L}(\tau)\tau d\tau \tag{1.21}$$

and

$$\bar{U}(x_0, \bar{v}) \subseteq D, \tag{1.22}$$

where $\bar{\alpha}$ is given by (1.19) and $\bar{v}$ is the smallest positive number such that

$$\int_0^{\bar{v}} \bar{L}(\tau)d\tau = 1. \tag{1.23}$$

Then, the solution of the IVP (1.5) exists in $U(x_0, \frac{\bar{v}_1}{\bar{\gamma}})$ *for each* $t \in [0,1]$*, where* $\bar{v}_1$ *is the first positive root of* $g_{\bar{a}}(t)$ *less than or equal to* $u_{\bar{L}/\bar{c}}$ *where* $g_{\bar{a}}(t) = \bar{a} - t + \int_0^t \bar{L}(\tau)(t-\tau)d\tau$*. Therefore,* $x(0)$ *is a solution of equation (1.3).*

Condition (1.21) is the usual Newton-Kantorovich type criterion [2, 3, 15].

REMARK 1.8 *If* $L_0(s) \geq \bar{L}(s)$ *for all* $s \in [0, \bar{v}]$*, then the results of Theorem 1.7 hold with* $\bar{L}$ *replacing* L*.*

The Theorem 1.7 does not apply, if $\bar{\alpha} > \int_0^{\bar{v}} \bar{L}(s)ds$. That is why as in [10], we suppose that the solution of the IVP (1.5) is inside the $\bar{\gamma}-$ball of x_0. Then, we ask: How many $k-$steps are needed to approximate the zero x_k of $F = h(.,0)$?

THEOREM 1.9 *Let* x_0 *be an element of the* $\bar{\gamma}-$*ball of* z*. Set* $v^* = \bar{\gamma}\|x_0 - z\|$ *for* $0 \leq u < \bar{v}$*, where* $\bar{v}$ *satisfies (1.23). Define function* $\bar{q}$ *on* $[0, \bar{v}]$ *by*

$$\bar{q}(t) = \frac{\int_0^t \bar{L}(\tau)d\tau}{t(1 - \int_0^t L_0(\tau)d\tau)}. \tag{1.24}$$

Let $u_{\bar{L}}$ *be such that*

$$\bar{q}(u_{\bar{L}}) = 1. \tag{1.25}$$

Let $\bar{c} \geq 1$ *and define function* $g_{\bar{a}}$ *on* $[0, \bar{v}]$ *by*

$$g_{\bar{a}}(t) = \bar{a} - t + \int_0^t \bar{L}(\tau)(t-\tau)d\tau, \tag{1.26}$$

so that

$$\min\{u_{\bar{L}/\bar{c}} - \int_0^{u_{\bar{L}/\bar{c}}} \bar{L}(\tau)(u_{\bar{L}/\bar{c}} - \tau)d\tau, \int_0^{\bar{v}} \bar{L}(\tau)\tau d\tau\} \geq \bar{a} \tag{1.27}$$

with the smallest positive solution of equation $g_{\bar{a}}(t) = 0$ *is not exceeding* $u_{\bar{L}/\bar{c}}$*. Set*

$$p = \frac{\bar{\varphi}(u)(\bar{\alpha} + \int_0^{v^*} \bar{L}(\tau)(v^* - \tau)d\tau + v^*)}{(1 - \int_0^{u_{\bar{L}/\bar{c}}} L_0(\tau)d\tau)(1 - \int_0^u L_0(\tau)d\tau)}$$

$$q = \frac{\int_0^{u_{\bar{L}/\bar{c}}} \bar{L}(\tau)(u_{\bar{L}/\bar{c}} - \tau)d\tau + u_{\bar{L}/\bar{c}}}{1 - \int_0^{u_{\bar{L}/\bar{c}}} L_0(\tau)d\tau},$$

where $\bar{\varphi}$ *is given in (1.18). Moreover, suppose* $x(t)$ *is the solution of the IVP is inside the* $\bar{\gamma}-$*ball of* z*. Let us also define sequence* $\{s_n\}$ *by*

$$s_0 = 1, s_n > 0, s_{n-1} - s_n > s_n - s_{n+1} > 0,\ n \geq 0, \lim_{n \longrightarrow +\infty} s_n = 0, \tag{1.28}$$

where

$$s_1 = 1 - \frac{1 - \int_0^u L_0(\tau)d\tau}{\bar{\varphi}(u)(\bar{\alpha} + \int_0^{v^*} \bar{L}(\tau)(v^* - \tau)d\tau + v^*)}.$$

Set w_n *such that* $F(w_n) = s_n F(x_0)$*. Then, the following assertions hold:*

(i) Points w_n and w_{n+1}, are such that

$$\bar{\gamma}\varphi(u)\|w_{n+1} - w_n\| \leq \bar{a}.$$

(ii) Newton sequence $\{x_n\}$ generated by (1.28) and w_n are such that

$$\bar{\gamma}\varphi(v^*)\|x_n - w_n\| \leq u_{\bar{L}/\bar{c}}.$$

(iii) Set $\bar{N} = \frac{\int_0^{\bar{v}} \bar{L}(\tau)\tau d\tau - q}{p}$. The steps n required for x_n to be an approximate zero of w_n exceeds or is equal to

$$\left[\frac{1-\bar{N}}{1-s_1}\right], \text{ if } s_n := \max\{0, 1 - n(1-s_1)\},$$

$$\left[\frac{log\bar{N}}{logs_1}\right], \text{ if } s_n := s_1^n$$

$$\left[log_2\left(\frac{log\bar{N}}{logs_1}+1\right)\right], \text{ if } s_n := s_1^{2^k-1}.$$

$$\|x_n - \bar{w}\| \leq \bar{q}(\bar{u})^{2^n-1}\|x_0 - \bar{w}\|,$$

where $\bar{u} = \gamma(F, \bar{w})\|x_0 - \bar{w}\| < u_{\bar{L}}$ and $\bar{q}$ is given in (1.24).

REMARK 1.10 *If $L = L_0 = \bar{L}, \gamma_0 = \gamma = \bar{\gamma}$, then the preceding items coincide with the ones in [10]. But , if (1.12) or (1.8) hold as strict inequalities, then the new results constitute an improvement over the ones in [10]. These improvements are deduced using the same effort as in [10], because finding function L requires finding functions L_0 and $\bar{L}$. If $L_0 > \bar{L}$, then, the preceding results hold with $\bar{L}$ replacing L_0.*

2 Special Cases

We consider specializations of the preceding results in the general (Kantorovich) case $\bar{L}(s) = 1$ and the analytic case $\bar{L}(s) = \frac{2}{(1-s)^3}$, respectively. Examples, where (1.14) and (1.15) hold as strict inequalities in the Kantorovich case can be found in [2,3] whereas the examples in the analytic case can be found in [4]. To avoid repetitions, we refer the reader to [10], where $\alpha(F, x_0), \varphi, v, N, L$ are replaced by $\bar{\alpha}(F, x_0), \bar{\varphi}, u, \bar{N}\ \bar{L}$, respectively.

Next, we present the α and γ Theorems improving the works in [10] which in turn improved the works by X. Wang [16] and Traub and Wozniakowski [15], respectively.

THEOREM 2.1 *Suppose: $F'(x_0)^{-1}F$ is $\bar{L}$ and L_0- center-Lipschitz restricted at x_0;*

$$\bar{\alpha}(F, x_0) \leq \int_0^{\bar{v}} \bar{L}(\tau)\tau d\tau,$$

where $\bar{v}$ is given in (1.23). Specialize function $\bar{g}_{\alpha(F,x_0)}$ by

$$\bar{g}_{\bar{\alpha}(F,x_0)}(t) := \bar{g}(t) = \bar{\alpha}(F,x_0) - t + \int_0^t \bar{L}(\tau)(t-\tau)d\tau. \qquad (2.1)$$

Then, the following items hold

(i) There exist $\rho_1, \rho_2 \in \mathbb{R}$ with $\rho_1 \neq \rho_2$ such that $\bar{g}(\rho_1) = \bar{g}(\rho_2) = 0$ with $\bar{g}$ strictly convex and

$$\bar{g}(t) = (t-\rho_1)(t-\rho_2)\psi(t),$$

where

$$\psi(t) = \int_0^1 \int_0^1 \theta(\bar{L}(1-\theta) + \theta s\rho_2 + \theta\tau t)d\tau d\theta.$$

and for $r_0 = 0$, $\lim_{n \longrightarrow +\infty} r_n = \lim_{n \longrightarrow +\infty} N_{\bar{g}}(r_{n-1}) = \bar{v}_1$.

(ii) Equation $F(x) = 0$ has a solution $\bar{w}$ which is unique in $U(x_0, \frac{\bar{v}}{\bar{\gamma}(F,x_0)})$.

(iii) Newton sequence $\{x_n\}$ defined by $x_{n+1} = N_F(x_n)$ exists, stays in $\bar{U}(x_0, \frac{\rho_1}{\bar{\gamma}(F,x_0)})$ and converges to $\bar{w}$, so that

$$\|x_n - \bar{w}\| \leq \|r_n - \rho_1\|$$

(iv) If $\bar{g}(t) \geq \frac{\bar{\alpha}(F,x_0)}{\rho_1\rho_2}$, then

$$\|x_n - \bar{w}\| \leq \frac{1}{\bar{\gamma}(F,x_0)z^n} \left(\frac{\rho_1}{\rho_2}\right)^{2^n-1} \rho_1.$$

THEOREM 2.2 *Suppose:*

(i) $\bar{w}$ solves $F(x) = 0$ and is a regular solution:

(ii) $F'(\bar{w})^{-1}F'(\bar{w})$ is $\bar{L}$ and L_0 center Lipschitz restricted for all $x \in U(\bar{w}, \frac{\bar{v}}{\bar{\gamma}(F,\bar{w})})$. Then, Newton sequence $\{x_n\}$ generated by $x_0 = x, x_{n+1} = N_F(x_n)$ converges to $\bar{w}$ for all $x \in U(\bar{w}, \frac{u_{\bar{L}}}{\bar{\gamma}(F,\bar{w})})$, where $u_{\bar{L}}$ is given in (1.25). Moreover, we have the following:

$$\|x_n - \bar{w}\| \leq \bar{q}(\bar{u})^{2^n-1}\|x_0 - \bar{w}\|.$$

REMARK 2.3 *If $L_0 = L = \bar{L}, \gamma_0 = \gamma = \bar{\gamma}$, the two preceding results reduce to Theorem 3 and Theorem 4 in [10], respectively, i.e., if (1.14) or (1.15) hold as strict inequalities, then the earlier results are improved (see also the numerical examples).*

3 Numerical examples

We provide two examples for the Kantorovich case, where function has no positive roots. Hence the older results can not apply, but function $\bar{g}$ has roots, so the new results apply to solve equations.

EXAMPLE 3.1 *Let* $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$, $x_0 = 1$, $D = \{x : |x - x_0| \leq \lambda\}, \lambda \in [0, 1/2)$ *and* F *defined by*

$$F(x) = x^3 - \lambda. \tag{3.2}$$

Then, for $L_0(\tau) = L(\tau) = \bar{L}(\tau) = 1, v_0 = \bar{v} = v = 1$, *we have*

$$\bar{b} = b = \frac{1-\lambda}{3}, \gamma_0(\tau) = 3 - \lambda, \gamma(\tau) = 2(2-\lambda) \text{ and } \bar{\gamma}(\tau) = 2(1 + \frac{1}{3-\lambda}).$$

Notice that

$$\gamma_0 < \gamma < \bar{\gamma}.$$

The functions g *and* $\bar{g}$ *are then given, respectively by*

$$g(t) = \frac{t^2}{2} - t + \frac{2}{3}(1-\lambda)(2-\lambda)$$

and

$$\bar{g}(t) = \frac{t^2}{2} - t + \frac{2}{3}(1-\lambda)(1 + \frac{1}{3-\lambda}).$$

The Newton-Kantorovich condition (i.e., the discriminant d_g *of* g*) is given by*

$$d_g = 1 - \frac{4}{3}(1-\lambda)(2-\lambda) < 0 \quad \text{for each } \lambda \in [0, 1/2) \tag{3.3}$$

so function g *has not positive roots. However, function* $\bar{g}$ *has positive roots, since the discriminant*

$$d_{\bar{g}} = 1 - \frac{4}{3}(1-\lambda)(1 + \frac{1}{3-\lambda}) > 0 \quad \text{for each } \lambda \in I = [0.4619832, 1/2). \tag{3.4}$$

Therefore, our Theorem 2.1 can be used to solve equation $F(x) = 0$ *for all* $\lambda \in I$.

EXAMPLE 3.2 *Let* $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{C}[0, 1]$. *Let* $D = \{x \in \mathcal{B}_1 : \|x\| \leq R\}$ *for* $R > 0$. *Define* F *on* D *by*

$$F(x)(s) = x(s) - f(s) - \delta \int_0^1 K(s,t) x(t)^3 dt, x \in \mathcal{B}_1, s \in [0, 1], \tag{3.5}$$

where $f \in \mathcal{B}_1$ *is a fixed function and* λ *is given by*

$$K(s,t) = \begin{cases} (1-s)t, & \text{if } t \leq s, \\ s(1-t), & \text{if } s \leq t. \end{cases}$$

Then, for each $x \in D$, $F'(x)$ *is given by*

$$[F'(x)(v)](s) = v(s) - 3\delta \int_0^1 K(s,t)x(t)^2 v(t)dt, v \in X, s \in [0,1].$$

Set $x_0(s) = f(s) = 1$. *Then, we have* $\|I - F'(x_0)\| \leq 3|\delta|/8$ *if* $|\delta| < 8/3$, *then* $F'(x_0)^{-1}$ *exists and*

$$\|F'(x_0)^{-1}\| \leq \frac{8}{8 - 3|\delta|}.$$

Moreover,

$$\|F(x_0)\| \leq \frac{|\delta|}{8},$$

so

$$b = \|F'(x_0)^{-1}F(x_0)\| \leq \frac{|\delta|}{8 - 3|\delta|}.$$

Furthermore, for $x, y \in D$, *we have*

$$\|F'(x) - F'(y)\| \leq \frac{1 + 3|\delta|\|x + y\|}{8}\|x - y\| \leq \frac{1 + 6R|\delta|}{8}\|x - y\|$$

and

$$\|F'(x) - F'(1)\| \leq \frac{1 + 3|\delta|(\|x\| + 1)}{8}\|x - 1\| \leq \frac{1 + 3|\delta|(1 + R)}{8}\|x - 1\|.$$

Let $\delta = 1.175$ *and* $R = 2$, *we have* $b = 0.26257..., \bar{\gamma}(\tau) = 2.76875..., \gamma_0(\tau) = 1.8875...$ *and* $\gamma(\tau) = 1.47314..., v_0 = \bar{v} = v = 1$. *Using these values, we get that the discriminant* d_g *of* g *is*

$$d_g = 1 - 1.02688 < 0,$$

but the discriminant $d_{\bar{g}}$ *of* $\bar{g}$ *is*

$$d_{\bar{g}} = 1 - 0.986217 > 0.$$

Hence, $\lim_{n \longrightarrow \infty} x_n = x_*$ *by Theorem 2.1, where* x_* *is a solution of equation* $F(x)(s) = 0$, *where* F *is given by (3.5).*

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