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SEMT Labelings and Deficiencies of Forests with Two Components (II)

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Abstract. Given a simple graph $G(V, E)$, consider a bijective function Γ from $V(G) \cup E(G)$ to $[\nu + \varepsilon]$, where $\nu = |V(G)| = order\ of\ G$, $\varepsilon = |E(G)| = size\ of\ G$. If for all $e = xy \in E(G), \Gamma(x) + \Gamma(e) + \Gamma(y)$ is a constant, then Γ is called an *edge-magic total (EMT) labeling*. Moreover, if $\Gamma(V(G)) = [\nu]$, then Γ is a *super edge-magic total (SEMT) labeling* of G and G is a SEMT graph. If a graph G has at least one SEMT labeling then the smallest of the magic constants for all possible distinct SEMT labelings of G describes *super edge-magic total (SEMT) strength*, $sm(G)$, of G. For any graph G, *SEMT deficiency* is the least number of isolated vertices which when uniting with G yields a SEMT graph. This paper focuses on finding SEMT strength of generalized comb $Cb_\tau(2, 3, \ldots, \tau+1)$ and evaluating SEMT labeling and deficiency of forests be composed of two components, where one of the components for each forest is aforesaid generalized comb and other component is star, bistar, comb, path respectively.

AMS (MOS) Subject Classification Codes: 05C78
Key Words: SEMT labeling, SEMT strength, SEMT Deficiency, generalized comb, star, bistar, comb, path.

1. Basic Terminologies

The graphs throughout our discussion will be planar, finite, without having; directions, multiple edges, loops. A (ν, ε)-graph G determines an *edge-magic total (EMT) labeling* when $\Gamma : V(G) \cup E(G) \to \{1, \nu + \varepsilon\}$ is bijective so as the weights at every edge are same constant (say) c i.e., for $x, y \in V(G)$; $\Gamma(x) + \Gamma(xy) + \Gamma(y) = c$, independent of the choice of any $xy \in E(G)$, such a number is interpreted as a *magic constant*. If all vertices gain smallest of the labels then an EMT labeling is called a *super edge-magic total (SEMT)* *labeling*. Kotzig and Rosa [17] and Enomoto *et al.* [7] first introduced the notions of EMT and SEMT graphs respectively and presented the conjectures: *every tree is EMT* [17], and *every tree is SEMT* [7].

If a graph G has at least one SEMT labeling then smallest of the magic constants for all possible distinct SEMT labelings of G is called a *super edge-magic total (SEMT) strength*, $sm(G)$, of G. Avadayappan *et al.* first introduced the notion of SEMT strength [4] and found exact values of SEMT strength for some graphs.

In [17], the notion of *EMT deficiency* was proposed by authors and Figueroa-Centeno *et al.* [8] continued it to the SEMT graphs. For any graph G, the *SEMT deficiency*, denoted $\mu_s(G)$, is the least number n of isolated vertices that we have to take in union with G s.t. the resulting graph $G \cup nK_1$ is SEMT, the case $+\infty$ will arise if no number of isolated vertices meet this criterion.

Exact values for SEMT deficiencies of several classes of graphs are provided in [9, 8], authors also proposed a conjecture which tells us about the confined deficiencies of the forests. In [10], an assumption was made as a special case of previous one that says, the deficiency of each two-tree forest is not more than 1. Baig *et al.* [6] determined SEMT deficiencies of various forests made up of banana trees, stars etc. In [13, 20], S. Javed *et al.* and Ngurah *et al.* gave some upper bounds for SEMT deficiency of forests be composed of stars, fans, comb, double fans, wheels and generalized comb. The results in [1, 2, 3, 5, 15, 16, 18, 19] migth found useful in the aspect of examined labeling here. A general reference to graph theoretic terminologies can be found in [21]. For more review, see the recent survey of graph labelings by Gallian [12].

SEMT strength of generalized comb $Cb_\tau(2, 3, \dots, \tau+1)$ and SEMT labeling, SEMT deficiency of forests formed by aforesaid generalized comb with star, bistar, comb and path respectively, are enveloped in our present work. The values of parameters of star, bistar, comb and path are totally dependant on the parameters involved in generalized comb.

2. The Results

Star on n vertices is isomorphic to $K_{1,n-1}$. When we join two stars $K_{1,g}$, $K_{1,h}$ through a bridge, where $g, h \geq 1$ and $g + h = n - 2$, the resulting tree is termed as a *bistar* $BS(g, h)$. P_n denotes the *path* of order n and size $n-1$, with vertices labelled from x_1 to x_n along P_n. The *comb* Cb_n [6] is an acyclic graph consists of P_n together with $n-1$ new pendant vertices $y_1, y_2, ..., y_{n-1}$ adjacent to $x_2, x_3, ..., x_n$ respectively, thus the new edges obtained are $\{x_{\imath+1}y_\imath : \imath \in \{\overline{1, n-1}\}\}$. A *generalized comb* [13] is basically a detailing (or subdivision) of a comb's pendant vertices hanging from main horizontal path to form τ hanging paths of order $\ell_\imath$, this is denoted by $Cb_\tau(\ell_1, \ell_2, \dots, \ell_\tau)$. This work deals with the generalized comb $Cb_\tau(2, 3, \dots, \tau+1)$, in which each next hanging path has one more vertex than the previous one, as shown in fig. 1.

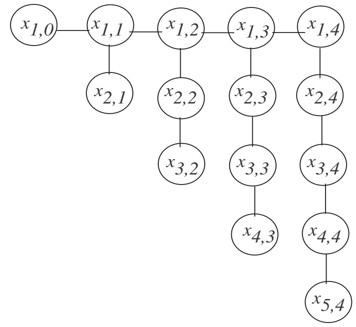

FIGURE 1. Generalized comb $Cb_4(2,3,4,5)$

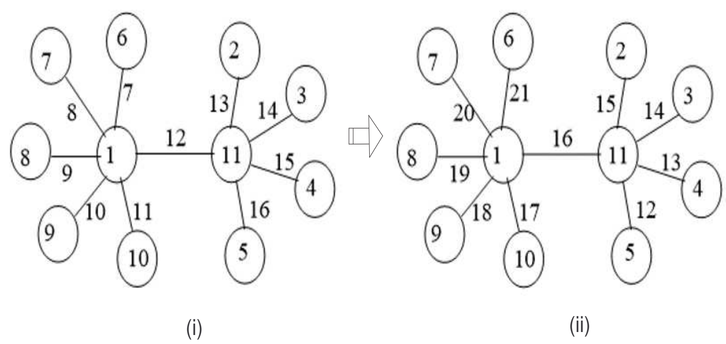

FIGURE 2. (i) A Bistar $BS(5,4)$ with consecutive edge-sums, (ii) A SEMT Bistar $BS(5,4)$ with magic constant $c = 28$

The following Lemma is an elementary tool for proving graphs to be SEMT. It will be used as a base in each result presented in this work.

Lemma 2.1. [11] *A (ν, ε)-graph G is SEMT if and only if $\exists$ a bijective map $\Gamma : V(G) \to \{\overline{1,\nu}\}$ s.t. the set of edge-sums*

$$S = \{\Gamma(l) + \Gamma(m) : lm \in E(G)\}$$

constructs ε consecutive Z^+. In that case, G can elongate to a SEMT labeling of G with magic constant $c = \nu + \varepsilon + min(S)$ and

$$S = \{c - (\nu + \varepsilon), c - (\nu + \varepsilon) + 1, \dots, c - (\nu + 1)\}.$$

To understand the lemma 2.1, we consider an example, see fig. 2, where it is shown that if a graph constitutes consecutive edge-sums then its super edge-magicness is assured.

It can be seen easily that following result about SEMT graphs also holds i.e.,

Note. [4] Let $c(\Gamma)$ be a magic constant of a SEMT labeling of Γ of $G(V, E)$, then we end up on this statement:

$$\varepsilon\ c(\Gamma) = \sum_{v \in V} deg_G(v)\Gamma(v) + \sum_{p \in E} \Gamma(p), \qquad \varepsilon = |E(G)| \tag{2. 1}$$

For a single graph, many SEMT labelings might exist and of-course for a different labeling, there will be a different magic constant. In our another paper [14], we already have established the bounds for magic constants of generalized comb $Cb_\tau(\ell_1, \ell_2, \ldots, \ell_\tau)$; $\tau \geq 2$ with $\varepsilon = \tau\ell$ and SEMT strength for balanced generalized comb $Cb_\tau(\ell, \ell, \ldots, \ell)$. Hence, for generalized comb $Cb_\tau(2, 3, \ldots, \tau + 1)$, the bounds for SEMT magic constants will be same with $\varepsilon = \frac{\tau(\tau+3)}{2}$ but SEMT strength will vary with some differences as described under.

3. SEMT Strength of Generalized Comb

From SEMT labeling for generalized comb $Cb_\tau(2, 3, \ldots, \tau + 1)$; $\tau \geq 2$, [13], we have magic constant $c = \frac{2\tau^2+6\tau+2}{2} + \lceil \frac{\tau^2+4\tau+11}{4} \rceil$ and by given lower bound of magic constants for SEMT labelings of generalized comb in Lemma 2.3 of [14], we have:

Theorem 3.1. *The SEMT strength for generalized comb $G \cong Cb_\tau(2, 3, \ldots, \tau+1)$; $\tau \geq 2$ is, for $\varepsilon = \frac{\tau(\tau+3)}{2}$:*

$$\frac{5\varepsilon^2 + 2\tau^2 - 2\varepsilon\tau + 7\varepsilon - 2\tau + 2}{2\varepsilon} \leq sm(G) \leq \frac{2\tau^2 + 6\tau + 2}{2} + \lceil \frac{\tau^2 + 4\tau + 11}{4} \rceil$$

4. SEMT Labeling and Deficiency of Forests formed by Generalized Comb and Star, Generalized Comb and Bistar

Theorem 4.1. *For $\tau \geq 2$*
(a): $Cb_\tau(2, 3, \ldots, \tau + 1) \cup K_{1,\varpi}$ is SEMT.
(b): $\mu_s(Cb_\tau(2, 3, \ldots, \tau + 1) \cup K_{1,\varpi-1}) \leq 1$; $\tau \neq 2$, where $\varpi \geq 2$ and is given by

$$\varpi = \begin{cases} \frac{(\tau-1)(\tau+3)}{4} & ;\tau \ odd \\ \frac{\tau(\tau+2)-4}{4} & ;\tau \ even \end{cases}$$

Proof. **(a):** Consider the graph $G \cong Cb_\tau(2, 3, \ldots, \tau + 1) \cup K_{1,\varpi}$
Here $V(K_{1,\varpi}) = \{y_t; 1 \leq t \leq \varpi + 1\}$ and $E(K_{1,\varpi}) = \{y_1 y_t; 2 \leq t \leq \varpi + 1\}$
Let $\nu = |V(G)|$ and $\varepsilon = |E(G)|$, so we get $\nu = \frac{(\tau+1)(\tau+2)}{2} + \varpi + 1$ and $\varepsilon = \frac{\tau(\tau+3)}{2} + \varpi$.
Valuation $\Gamma : V(Cb_\tau(2, 3, \ldots, \tau + 1)) \to \{\overline{1, \frac{(\tau+1)(\tau+2)}{2}}\}$ is described as follows:

$$\Gamma(x_{\imath,\jmath}) = \begin{cases} \frac{(\jmath+1)^2}{4} - \frac{\imath-1}{2} & ;\imath, \jmath \ odd \\ \frac{\jmath^2}{4} + \frac{\imath}{2} & ;\imath, \jmath \ even \end{cases}$$

Now consider the labeling $\Omega : V(G) \to \{\overline{1, \nu}\}$
For τ even:

$$\Omega(y_t) = \begin{cases} \frac{\tau(\tau+2)}{4} + 1 & ; t = 11 \\ \frac{(\tau+1)(\tau+2)}{2} + t - 1 & ; t \neq 1 \end{cases}$$

For τ odd:

$$\Omega(y_t) = \begin{cases} \frac{(\tau+1)^2}{4} + 1 & ; t = 11 \\ \frac{(\tau+1)(\tau+2)}{2} + t - 1 & ; t \neq 1 \end{cases}$$

Let

$$A = \begin{cases} \frac{\tau(\tau+2)}{4} + 1 & ; \tau \ even \\ \frac{(\tau+1)^2}{4} + 1 & ; \tau \ odd \end{cases}$$

and $B = \frac{(\tau+1)(\tau+2)}{2} + \varpi$.
For τ even:

$$\Gamma(x_{\imath,\jmath}) = \begin{cases} A + \frac{(\jmath+1)(\jmath+3)}{4} - \frac{\imath}{2} & ; \imath \ even, \ \jmath \ odd \\ A + \frac{\jmath(\jmath+2)}{4} + \frac{\imath-1}{2} & ; \imath \ odd, \ \jmath \ even \end{cases}$$

For τ odd:

$$\Gamma(x_{\imath,\jmath}) = \begin{cases} A + \frac{(\jmath+1)(\jmath+3)}{4} - \frac{\imath-2}{2} - 1 & ; \imath \ even, \ \jmath \ odd \\ A + \frac{(\jmath+1)^2+2\imath+1}{4} - 1 & ; \imath \ odd, \ \jmath \ even \end{cases}$$

$\Gamma(x_{1,0}) = B + 1 = \frac{(\tau+1)(\tau+2)}{2} + \varpi + 1$
$\Omega(x_{\imath,\jmath}) = \Gamma(x_{\imath,\jmath}); 1 \leq \imath \leq \jmath + 1, 0 \leq \jmath \leq \tau.$

The edge-sums of G induced by above labeling Ω form consecutive integers starting from $\hbar + 1$ and ending on $\hbar + \varepsilon$, where

$$\hbar = \begin{cases} \frac{\tau(\tau+2)+8}{4} & ; \tau \ even \\ \frac{(\tau+1)^2+8}{4} & ; \tau \ odd \end{cases}$$

Hence from Lemma 2.1, we end up on a SEMT graph with $c = \frac{(\tau+1)(\tau+2)+4\varpi+4+2\hbar+\tau(\tau+3)}{2}$.

(b): Let $\acute{G} \cong Cb_\tau(2, 3, \ldots, \tau + 1) \cup K_{1,\varpi-1} \cup K_1; \tau \neq 2$
so, $V(\acute{G}) = V(Cb_\tau(2, 3, \ldots, \tau + 1)) \cup V(K_{1,\varpi-1}) \cup \{z\}$
and $V(K_{1,\varpi-1}) = \{y_t; 1 \leq t \leq \varpi\}$ and $E(K_{1,\varpi-1}) = \{y_1 y_t; 2 \leq t \leq \varpi\}$
Let $\acute{\nu} = |V(\acute{G})| = \frac{(\tau+1)(\tau+2)}{2} + \varpi + 1$ and $\acute{\varepsilon} = |E(\acute{G})| = |E(G)| = \frac{\tau(\tau+3)}{2} + \varpi - 1$

Keeping in mind the valuation Γ defined in (a), we describe the labeling $\acute{\Omega} : V(\acute{G}) \to \{\overline{1, \acute{\nu}}\}$ as

For $1 \leq t \leq \varpi$, $\acute{\Omega}(y_t) = \Omega(y_t)$ with A as in (a) and $B = \frac{(\tau+1)(\tau+2)}{2} + \varpi - 1$.
$\acute{\Omega}(x_{1,0}) = B + 2 = \frac{(\tau+1)(\tau+2)}{2} + \varpi + 1$, $\acute{\Omega}(z) = B + 1 = \frac{(\tau+1)(\tau+2)}{2} + \varpi$
$\acute{\Omega}(x_{\imath,\jmath}) = \Gamma(x_{\imath,\jmath}); 1 \leq \imath \leq \jmath + 1, 0 \leq \jmath \leq \tau.$

The edge-sums of $\acute{G}$ induced by above labeling $\acute{\Omega}$ form consecutive integers starting from $\acute{\hbar} + 1$ and ending on $\acute{\hbar} + \acute{\varepsilon}$, where $\acute{\hbar} = \hbar$. Hence from Lemma 2.1, we end up on a

SEMT graph.

□

In formulation of all next results in this work, we will use the labeling Γ provided in previous theorem 4.1.

Theorem 4.2. *For* $\tau \geq 3;\ \omega, \varpi \geq 1$
(a)*:* $Cb_\tau(2, 3, \ldots, \tau+1) \cup BS(\omega, \varpi)$ *is SEMT.*
(b)*:* $\mu_s(Cb_\tau(2, 3, \ldots, \tau+1) \cup BS(\omega, \varpi - 1)) \leq 1; \tau \neq 3, \varpi \geq 2$
where

$$\varpi = \begin{cases} \frac{(\tau-1)(\tau+3)-4}{4} & ;\tau\ odd \\ \frac{\tau(\tau+2)-8}{4} & ;\tau\ even \end{cases}$$

Proof. **(a):** Consider the graph $G \cong Cb_\tau(2, 3, \ldots, \tau+1) \cup BS(\omega, \varpi); \tau \geq 3, \omega, \varpi \geq 1$ $V(BS(\omega, \varpi)) = \{z_{ut} : u = 1, 2; 0 \leq t \leq \rho\}$, where

$$\rho = \begin{cases} \omega & ;u = 1 \\ \varpi & ;u = 2 \end{cases}$$

and $E(BS(\omega, \varpi)) = \{z_{10}z_{1t}; 1 \leq t \leq \omega\} \cup \{z_{10}z_{20}\} \cup \{z_{20}z_{2t}; 1 \leq t \leq \varpi\}$. Let $\nu = |V(G)|$ and $\varepsilon = |E(G)|$, so we get $\nu = \frac{(\tau+1)(\tau+2)}{2} + \omega + \varpi + 2$ and $\varepsilon = \frac{\tau(\tau+3)}{2} + \omega + \varpi + 1$. Keeping in mind the valuation Γ defined in Theorem 4.1 with

$$A = \begin{cases} \frac{\tau(\tau+2)}{4} + 1 + \omega & ;\tau\ even \\ \frac{(\tau+1)^2}{4} + 1 + \omega & ;\tau\ odd \end{cases}$$

and $B = \frac{(\tau+1)(\tau+2)}{2} + \omega + \varpi + 1$. We describe the labeling $\Omega : V(G) \to \{\overline{1, \nu}\}$ as

$$\Omega(z_{ut}) = \begin{cases} \frac{\tau(\tau+2)}{4} + t & ;u = 1; t = r, 1 \leq r \leq \omega, \tau\ even \\ \frac{(\tau+1)^2}{4} + t & ;u = 1; t = r, 1 \leq r \leq \omega, \tau\ odd \\ \frac{\tau(\tau+2)}{4} + \omega + 1 & ;u = 2, t = 0, \tau\ even \\ \frac{(\tau+1)^2}{4} + \omega + 1 & ;u = 2, t = 0, \tau\ odd \\ \frac{(\tau+1)(\tau+2)}{2} + \omega + 1 & ;u = 1, t = 0 \\ \frac{(\tau+1)(\tau+2)}{2} + \omega + 1 + t & ;u = 2, t = r, 1 \leq r \leq \varpi \end{cases}$$

$\Omega(x_{\imath,\jmath}) = \Gamma(x_{\imath,\jmath}); 1 \leq \imath \leq \jmath + 1, 0 \leq \jmath \leq \tau$
$\Omega(x_{1,0}) = \frac{(\tau+1)(\tau+2)}{2} + \omega + \varpi + 2 = B + 1.$

The edge-sums of G induced by above labeling Ω form consecutive integers starting from $\hbar + 1$ and ending on $\hbar + \varepsilon$, where

$$\hbar = \begin{cases} \frac{\tau(\tau+2)+8}{4} + \omega & ;\tau\ even \\ \frac{(\tau+1)^2+8}{4} + \omega & ;\tau\ odd \end{cases}$$

Hence from Lemma 2.1, we end up on a SEMT graph.

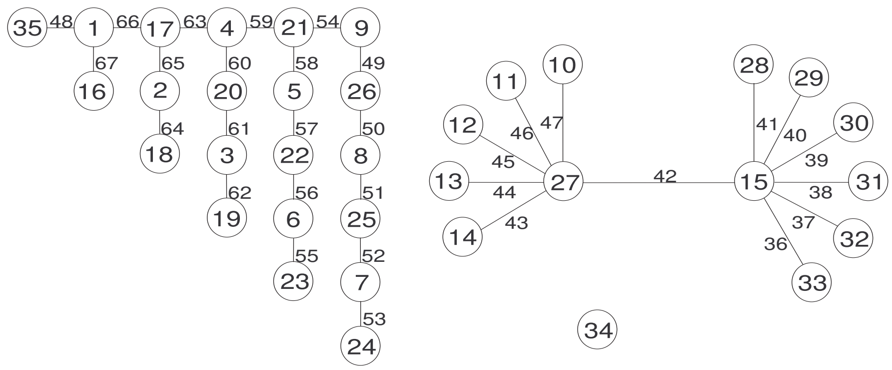

FIGURE 3. SEMT deficiency; $Cb_5(2,3,4,5,6) \cup BS(5,6) \cup K_1$

(b): Let $\acute{G} \cong Cb_\tau(2,3,\ldots,\tau+1) \cup BS(\omega,\varpi-1) \cup K_1$; $\tau \geq 3$, $omega \geq 1, \varpi \geq 2$ Here $V(\acute{G}) = V(Cb_\tau(2,3,\ldots,\tau+1)) \cup V(BS(\omega,\varpi-1)) \cup \{z\}$ and where $V(BS(\omega,\varpi-1)) = \{z_{ut} : u = 1,2; 0 \leq t \leq \rho\}$, where

$$\rho = \begin{cases} \omega & ; u = 1 \\ \varpi - 1 & ; u = 2 \end{cases}$$

and $E(BS(\omega,\varpi-1)) = \{z_{10}z_{1t}; 1 \leq t \leq \omega\} \cup \{z_{10}z_{20}\} \cup \{z_{20}z_{2t}; 1 \leq t \leq \varpi-1\}$. Let $\acute{\nu} = |V(\acute{G})|$ and $\acute{\varepsilon} = |E(\acute{G})|$, so we get $\acute{\nu} = \nu$ and $\acute{\varepsilon} = \varepsilon - 1$. Keeping in mind the valuation Γ defined in Theorem 4.1 with A as in (a) and $B = \frac{(\tau+1)(\tau+2)}{2} + \omega + \varpi$, we describe the labeling $\acute{\Omega} : V(\acute{G}) \to \{\overline{1,\acute{\nu}}\}$ as

$$\acute{\Omega}(x_{1,0}) = B + 2, \acute{\Omega}(z) = B + 1$$
$$\acute{\Omega}(x_{\imath,\jmath}) = \Gamma(x_{\imath,\jmath}); 1 \leq \imath \leq \jmath + 1, 0 \leq \jmath \leq \tau.$$

The edge-sums of $\acute{G}$ induced by above labeling $\acute{\Omega}$ form consecutive integers starting from $\acute{\hbar} + 1$ and ending on $\acute{\hbar} + \acute{\varepsilon}$, where $\acute{\hbar} = \hbar$. Hence from Lemma 2.1, we end up on a SEMT graph. □

5. SEMT Labeling and Deficiency of Forests formed by Generalized Comb and Comb, Generalized Comb and Path

Theorem 5.1. *For* $\tau \geq 2, \omega \geq 1$
(a)*:* $Cb_\tau(2,3,\ldots,\tau+1) \cup Cb_\omega$ *is SEMT.*
(b)*:* $\mu_s(Cb_\tau(2,3,\ldots,\tau+1) \cup Cb_{\omega-1}) \leq 1$*;* $\omega \geq 2$*,* $\tau \neq 2$*, where*

$$\omega = \begin{cases} \frac{(\tau+1)^2}{4} - 1 & ; \tau \ odd \\ \frac{\tau(\tau+2)}{4} - 1 & ; \tau \ even \end{cases}.$$

Proof. **(a):** Consider the graph $G \cong Cb_\tau(2,3,\ldots,\tau+1) \cup Cb_\omega$, where

$$V(Cb_\omega) = \{x_p; 0 \leq p \leq \omega\} \cup \{y_q; 1 \leq q \leq \omega\}$$

and

$$E(Cb_\omega) = \{x_p x_{p+1}; 0 \le p \le \omega - 1\} \cup \{x_p y_p; 1 \le p \le \omega\}.$$

Let $\nu = |V(G)|$ and $\varepsilon = |E(G)|$, so we get $\nu = \frac{(\tau+1)(\tau+2)}{2} + 2\omega + 1$ and $\varepsilon = \frac{\tau(\tau+3)}{2} + 2\omega$. Keeping in mind the valuation Γ defined in Theorem 4.1 with

$$A = \begin{cases} \frac{\tau(\tau+2)}{4} + \omega + 1 & ; \tau \ even \\ \frac{(\tau+1)^2}{4} + \omega + 1 & ; \tau \ odd \end{cases}$$

and $B = \frac{(\tau+1)(\tau+2)}{2} + 2\omega$. We describe the labeling $\Omega : V(G) \to \{\overline{1, \nu}\}$ as

For $0 \le p \le \omega$, $1 \le q \le \omega$

$$\Omega(x_p) = \begin{cases} \frac{\tau(\tau+2)}{4} + p + 1 & ; p, \tau \ are \ even \\ \frac{(\tau+1)^2}{4} + p + 1 & ; p \ even, \ \tau \ odd \\ \frac{(\tau+1)(\tau+2)}{2} + \omega + p & ; p \ odd \end{cases}$$

and

$$\Omega(y_q) = \begin{cases} \frac{(\tau+1)(\tau+2)}{2} + \omega + q & ; q \ even \\ \frac{(\tau+1)^2}{4} + q + 1 & ; q, \tau \ are \ odd \\ \frac{\tau(\tau+2)}{4} + q + 1 & ; q \ odd, \ \tau \ even \end{cases}$$

$\Omega(x_{\imath,\jmath}) = \Gamma(x_{\imath,\jmath}); 1 \le \imath \le \jmath + 1, 0 \le \jmath \le \tau$ and $\Omega(x_{1,0}) = B + 1$.

The edge-sums of G induced by above labeling Ω form consecutive integers starting from $\hbar + 1$ and ending on $\hbar + \varepsilon$, where

$$\hbar = \begin{cases} \frac{\tau(\tau+2)+8}{4} + \omega & ; \tau \ even \\ \frac{(\tau+1)^2+8}{4} + \omega & ; \tau \ odd \end{cases}$$

Hence from Lemma 2.1, we end up on a SEMT graph.

(b): Let $\acute{G} \cong Cb_\tau(2, 3, \ldots, \tau+1) \cup Cb_{\omega-1} \cup K_1; \tau \ne 2$, where $V(Cb_{\omega-1}) = \{x_p; 0 \le p \le \omega - 1\} \cup \{y_q; 1 \le q \le \omega - 1\}$, $V(K_1) = \{z\}$ and $E(Cb_{\omega-1}) = \{x_p x_{p+1}; 0 \le p \le \omega - 2\} \cup \{x_p y_p; 1 \le p \le \omega - 1\}$. Let $\acute{\nu} = |V(\acute{G})|$ and $\acute{\varepsilon} = |E(\acute{G})|$, so we get $\acute{\nu} = \frac{(\tau+1)(\tau+2)}{2} + 2\omega$ and $\acute{\varepsilon} = \frac{\tau(\tau+3)}{2} + 2\omega - 2$. Keeping in mind the valuation Γ defined in Theorem 4.1 with

$$A = \begin{cases} \frac{\tau(\tau+2)}{4} + \omega & ; \tau \ even \\ \frac{(\tau+1)^2}{4} + \omega & ; \tau \ odd \end{cases}$$

and $B = \frac{(\tau+1)(\tau+2)}{2} + 2\omega - 2$. We describe the labeling $\acute{\Omega} : V(\acute{G}) \to \{\overline{1, \acute{\nu}}\}$ as

For $0 \le p \le \omega - 1$, $1 \le q \le \omega - 1$

$$\acute{\Omega}(x_p) = \begin{cases} \frac{\tau(\tau+2)}{4} + p + 1 & ; p, \tau \ are \ even \\ \frac{(\tau+1)^2}{4} + p + 1 & ; p \ even, \ \tau \ odd \\ \frac{(\tau+1)(\tau+2)}{2} + \omega + p - 1 & ; p \ odd \end{cases}$$

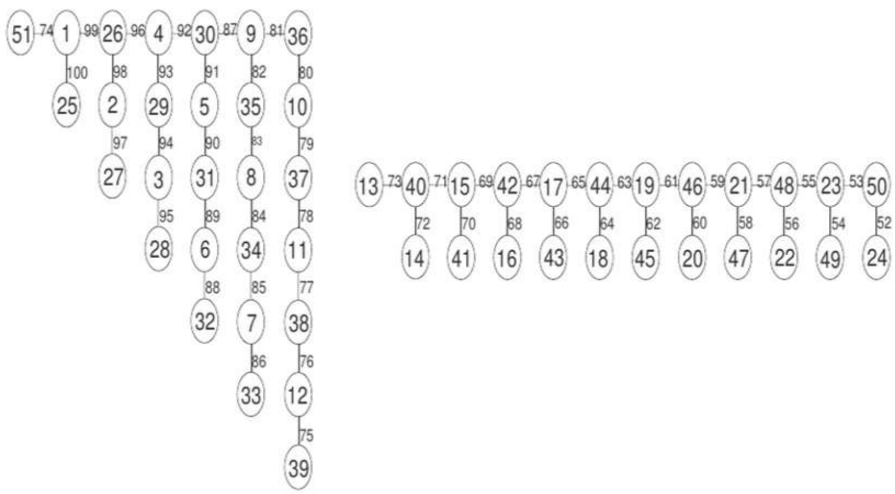

FIGURE 4. SEMT labeling of $Cb_6(2,3,4,5,6,7) \cup Cb_{11}$

and

$$\acute{\Omega}(y_q) = \begin{cases} \frac{(\tau+1)(\tau+2)}{2} + \omega + q - 1 & ; q \ even \\ \frac{(\tau+1)^2}{4} + q + 1 & ; q, \tau \ are \ odd \\ \frac{\tau(\tau+2)}{4} + q + 1 & ; q \ odd, \ \tau \ even \end{cases}$$

$\acute{\Omega}(x_{\imath,\jmath}) = \Gamma(x_{\imath,\jmath}); 1 \leq \imath \leq \jmath + 1, 0 \leq \jmath \leq \tau$
$\acute{\Omega}(z) = B + 1$ and $\acute{\Omega}(x_{1,0}) = B + 2$.

The edge-sums of $\acute{G}$ induced by above labeling $\acute{\Omega}$ form consecutive integers starting from $\acute{\hbar} + 1$ and ending on $\acute{\hbar} + \acute{\varepsilon}$, where $\acute{\hbar} = \hbar - 1$. Hence from Lemma 2.1, we end up on a SEMT graph. □

Theorem 5.2. *For* $\tau \geq 2$,
$(a)(i)$*:* $Cb_\tau(2,3,\ldots,\tau+1) \cup P_\varpi$ *is SEMT,*
$(a)(ii)$*:* $Cb_\tau(2,3,\ldots,\tau+1) \cup P_{\varpi-1}$ *is SEMT,*
$(b)(i)$*:* $\mu_s(Cb_\tau(2,3,\ldots,\tau+1) \cup P_{\varpi-2}) \leq 1$,
$(b)(ii)$*:* $\mu_s(Cb_\tau(2,3,\ldots,\tau+1) \cup P_{\varpi-3}) \leq 1$*;* $\tau \neq 2$*, where*

$$\varpi = \begin{cases} \frac{\tau^2+2\tau-2}{2} & ; \tau \ even \\ \frac{\tau^2+2\tau-1}{2} & ; \tau \ odd \end{cases}$$

Proof. **(a):** Consider the graph $G \cong Cb_\tau(2,3,\ldots,\tau+1) \cup P_t$, where $V(P_t) = \{x_p; 1 \leq p \leq t\}$ and $E(P_t) = \{x_p x_{p+1}; 1 \leq p \leq t-1\}$. Let $\nu = |V(G)|$ and $\varepsilon = |E(G)|$, so we get $\nu = \frac{(\tau+1)(\tau+2)}{2} + t$ and $\varepsilon = \frac{\tau(\tau+3)}{2} + t - 1$, where

$$t = \begin{cases} \varpi & ; for(a(i)) \\ \varpi - 1 & ; for(a(ii)) \end{cases}$$

Keeping in mind the valuation Γ defined in Theorem 4.1 with, For τ even

$$A = \begin{cases} \frac{\tau(\tau+2)}{4} + \frac{t+1}{2} & ; for(a(i)) \\ \frac{\tau(\tau+2)}{4} + \frac{t}{2} & ; for(a(ii)) \end{cases}$$

and for τ odd

$$A = \begin{cases} \frac{(\tau+1)^2}{4} + \frac{t+1}{2} & ; for(a(i)) \\ \frac{(\tau+1)^2}{4} + \frac{t}{2} & ; for(a(ii)) \end{cases}$$

and $B = \frac{(\tau+1)(\tau+2)}{2} + t - 1$. We describe the labeling $\Omega : V(G) \to \{\overline{1, \nu}\}$ as

$$\Omega(x_p) = \begin{cases} \frac{\tau(\tau+2)}{4} + r & ; p = 2r-1, 1 \le r \le \lfloor \frac{t+1}{2} \rfloor, \tau \ even \\ \frac{(\tau+1)^2}{4} + r & ; p = 2r-1, 1 \le r \le \lfloor \frac{t+1}{2} \rfloor, \tau \ odd \\ \frac{(\tau+1)(\tau+2)}{2} + \frac{t-1}{2} + \acute{t} & ; p = 2\acute{t}, 1 \le \acute{t} \le \lfloor \frac{t}{2} \rfloor, \tau \ even, for(a(i)) \\ \frac{(\tau+1)(\tau+2)}{2} + \frac{t-1}{2} + \acute{t} & ; p = 2\acute{t}, 1 \le \acute{t} \le \lfloor \frac{t}{2} \rfloor, \tau \ odd, for(a(i)) \\ \frac{(\tau+1)(\tau+2)}{2} + \frac{t-2}{2} + \acute{t} & ; p = 2\acute{t}, 1 \le \acute{t} \le \lfloor \frac{t}{2} \rfloor, \tau \ even, for(a(ii)) \\ \frac{(\tau+1)(\tau+2)}{2} + \frac{t-2}{2} + \acute{t} & ; p = 2\acute{t}, 1 \le \acute{t} \le \lfloor \frac{t}{2} \rfloor, \tau \ odd, for(a(ii)) \end{cases}$$

Furthermore, $\Omega(x_{\imath,\jmath}) = \Gamma(x_{\imath,\jmath})$; $1 \le \imath \le \jmath + 1, 0 \le \jmath \le \tau$ and $\Omega(x_{1,0}) = B + 1$.

The edge-sums of G induced by above labeling Ω form consecutive integers starting from $\hbar + 1$ and ending on $\hbar + \varepsilon$, where for $a(i)$

$$\hbar = \begin{cases} \frac{\tau(\tau+2)}{4} + \frac{t+3}{2} & ; \tau \ even \\ \frac{(\tau+1)^2}{4} + \frac{t+3}{2} & ; \tau \ odd \end{cases}$$

and for $a(ii)$

$$\delta = \begin{cases} \frac{\tau(\tau+2)}{4} + \frac{t+2}{2} & ; \tau \ even \\ \frac{(\tau+1)^2}{4} + \frac{t+2}{2} & ; \tau \ odd \end{cases}$$

Hence from Lemma 2.1, we end up on a SEMT graph.

(b): Let $\acute{G} \cong Cb_\tau(2, 3, \ldots, \tau+1) \cup P_t \cup K_1$, where $V(P_t) = \{x_p; 1 \le p \le t\}$, $V(K_1) = \{z\}$ and $E(P_t) = \{x_p x_{p+1}; 1 \le p \le t-1\}$. Let $\acute{\nu} = |V(\acute{G})|$ and $\acute{\varepsilon} = |E(\acute{G})|$, so we get $\acute{\nu} = \frac{(\tau+1)(\tau+2)}{2} + t + 1$ and $\acute{\varepsilon} = \frac{\tau(\tau+3)}{2} + t - 1$, where

$$t = \begin{cases} \varpi - 2 & ; for(b(i)) \\ \varpi - 3 & ; for(b(ii)) \end{cases}$$

Keeping in mind the valuation Γ defined in Theorem 4.1 with, For τ even

$$A = \begin{cases} \frac{\tau(\tau+2)}{4} + \frac{t+1}{2} & ; for(b(i)) \\ \frac{\tau(\tau+2)}{4} + \frac{t}{2} & ; for(b(ii)) \end{cases}$$

For τ odd

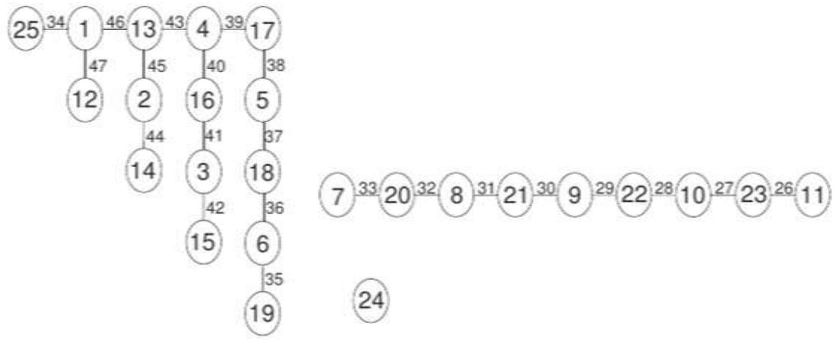

FIGURE 5. SEMT deficiency; $Cb_4(2,3,4,5) \cup P_9 \cup K_1$

$$A = \begin{cases} \frac{(\tau+1)^2}{4} + \frac{t+1}{2} & ; for(b(i)) \\ \frac{(\tau+1)^2}{4} + \frac{t}{2} & ; for(b(ii)) \end{cases}$$

and $B = \frac{(\tau+1)(\tau+2)}{2} + t - 1$. We describe the labeling $\acute{\Omega} : V(\acute{G}) \to \{\overline{1, \acute{\nu}}\}$ as

$$\acute{\Omega}(x_p) = \begin{cases} \frac{\tau(\tau+2)}{4} + r & ; p = 2r-1, 1 \le r \le \lfloor \frac{t+1}{2} \rfloor, \tau \ even \\ \frac{(\tau+1)^2}{4} + r & ; p = 2r-1, 1 \le r \le \lfloor \frac{t+1}{2} \rfloor, \tau \ odd \\ \frac{(\tau+1)(\tau+2)}{2} + \frac{t-1}{2} + \acute{t} & ; p = 2\acute{t}, 1 \le \acute{t} \le \lfloor \frac{t}{2} \rfloor, \tau \ even, for(b(i)) \\ \frac{(\tau+1)(\tau+2)}{2} + \frac{t-1}{2} + \acute{t} & ; p = 2\acute{t}, 1 \le \acute{t} \le \lfloor \frac{t}{2} \rfloor, \tau \ odd, for(b(i)) \\ \frac{(\tau+1)(\tau+2)}{2} + \frac{t-2}{2} + \acute{t} & ; p = 2\acute{t}, 1 \le \acute{t} \le \lfloor \frac{t}{2} \rfloor, \tau \ even, for(b(ii)) \\ \frac{(\tau+1)(\tau+2)}{2} + \frac{t-2}{2} + \acute{t} & ; p = 2\acute{t}, 1 \le \acute{t} \le \lfloor \frac{t}{2} \rfloor, \tau \ odd, for(b(ii)) \end{cases}$$

Furthermore,
$\acute{\Omega}(x_{\imath,\jmath}) = \Gamma(x_{\imath,\jmath}); 1 \le \imath \le \jmath + 1, 0 \le \jmath \le \tau$
$\acute{\Omega}(z) = B + 1$, $\acute{\Omega}(x_{1,0}) = B + 2$.

The edge-sums of $\acute{G}$ induced by above labeling $\acute{\Omega}$ form consecutive integers starting from $\acute{(\hbar)} + 1$ and ending on $\acute{(\hbar)} + \acute{(\varepsilon)}$, where for both $b(i)$ and $b(ii)$, $\acute{(\hbar)} = \hbar$. Hence from Lemma 2.1, we end up on a SEMT graph. □

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