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Approximate Solution for Systems of Fuzzy Differential Equations by Variational Iteration Method

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Abstract. This paper, presents a new perspective on the numerical solution of a fuzzy linear system of differential equations, where initial values and constant coefficients are fuzzy numbers. To do this, the matrix of coefficients was first decomposed into two matrices, then the variational iteration method was presented and applied to them. Afterwards, the convergence of the above mentioned method was proved. Finally, some examples are presented which confirm the applicability, accuracy, and efficiency of the method.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
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1. Introduction

Consider the following linear system of ordinary differential equations (ODEs)

$$\begin{cases} U'(t) = QU(t) + G(t)\; 0 \leq t \leq T, \\ U(0) = U_0. \end{cases} \tag{1. 1}$$

To find the exact solution of ODEs (1. 1) in general, the matrix exponential should be calculated which may be too complex [20, 36]. If the initial values and elements of matrix Q are fuzzy numbers, then calculations will be more complicated. Thus, it seems necessary to employ an approximation method for solving the equation. Several methods like [23, 4, 14, 22, 15, 19, 10, 44] have been proposed to calculate a numerical solution for linear and nonlinear differential equations, among which the variational iteration method (VIM) is an effective and efficient method for finding the solutions of such problems.
It is common to use approximate values for everyday uses. In 1965, Zadeh presented the concept of fuzzy sets for these issues [47]. Nowadays, fuzzy mathematics has been widely adopted in most branches of mathematics [3, 5, 26, 30, 31, 39, 41, 42, 43]. S. L. Chang and L. A. Zadeh have been the first to present the meaning of a fuzzy derivative in [13]. D. Dubois and H. Prade defined and applied the extension principle [16]. Many studies like

[45, 27, 28, 11, 18, 40] have been conducted on fuzzy initial value problems (FIVPs). The fuzzy Cauchy problem was investigated by J.J. Nieto in [38]. L. Stefanini and B. Bede in [46] introduced the generalized fuzzy differentiability definition.
Some useful papers have been presented for numerical solution of different types of fuzzy differential equations (FDEs) using VIM [2, 25, 32, 33, 6, 48, 24]. In [17], VIM was employed to solve a fuzzy system (1. 1) but it has disregarded the sign of $U(t)$ which has been noted in [37]. There are also some special examples in this regard [21]. The convergence of the method is proved in certain states [17]. This paper aims to eliminate the disadvantages stated above and prove the convergence of the VIM in general cases, as well as to apply the VIM to the Bloch equations.
The organization of the paper is as follows: In Section 2, some preliminary considerations for fuzzy numbers and fuzzy derivative are presented. In Section 3, the VIM for a fuzzy system of ordinary differential equations (FSODEs) is discussed. The convergence of the proposed method is given in Section 4 and some problems are solved by utilized approach in Section 5. In Section 6, fuzzy Bloch equations are introduced and their numerical solution is obtained by the VIM. Finally, conclusion is given in Section 7.

2. PRELIMINARIES

This section, reviews required definitions of fuzzy mathematics.
$\mathbb{R}$ and $\mathbb{R}_F$ symbolize the real and fuzzy numbers sets on $\mathbb{R}$, respectively. Properties of a fuzzy number as a mapping $p : \mathbb{R} \longrightarrow [0,1]$ can be introduced as follows:
(a) p is upper semi-continuous on $\mathbb{R}$.
(b) p is fuzzy convex, i.e. for $0 \leq \rightarrow \leq 1$ and any $x, y \in \mathbb{R}$,
$p(\rightarrow x + (1- \rightarrow)y) \geq \min\{p(x), p(y)\}$.
(c) p is normal, i.e. $\exists x_0 \in \mathbb{R}$ for which $p(x_0) = 1$.
(d) $[p]^0 = cl\{x \in \mathbb{R} : p(x) > 0\}$ is the support of p and its closure is compact.
For $r \in (0,1]$, define $p^r = \{x \in \mathbb{R} : p(x) \geq r\} = [\underline{p}^r, \overline{p}^r]$.
Noting (a) to (d), we can see that for all $r \in [0,1]$ the $r-$level set p^r is a closed bounded interval.

Supposing I be a real interval, $r-$level set of $\Psi : I \to \mathbb{R}_F$ is symbolized by

$$\Psi^r(t) = [\underline{\Psi}^r(t), \overline{\Psi}^r(t)],\ t \in I,\ r \in [0,1]\,.$$

A triangular fuzzy number $p = (\rho_1, \rho_2, \rho_3)$ in which $\rho_1 \leq \rho_2 \leq \rho_3$ and $\rho_1, \rho_2, \rho_3 \in \mathbb{R}$ is a triangular with the base on the interval $[\rho_1, \rho_3]$ and vertex at $x = \rho_2$. The $r-$cut of p is $p^r = [\rho_1 + (\rho_2 - \rho_1)r, \rho_3 + (\rho_2 - \rho_3)r]$.
It can be said: (1) $p > 0$ if $\rho_1 > 0$: (2) $p \geq 0$ if $\rho_1 \geq 0$: (3) $p < 0$ if $\rho_3 < 0$: (4) $p \leq 0$ if $\rho_3 \leq 0$ [1].
Let $p, q \in \mathbb{R}_F$ and μ is a positive scalar. Then, for $r \in (0,1]$
$\{p+q\}^r = [\underline{p}^r + \underline{q}^r, \overline{p}^r + \overline{q}^r]$
$\{p-q\}^r = [\underline{p}^r - \overline{q}^r, \overline{p}^r - \underline{q}^r]$
$\{p.q\}^r = [\min\left\{\underline{p}^r.\underline{q}^r, \underline{p}^r.\overline{q}^r, \overline{p}^r.\underline{q}^r, \overline{p}^r.\overline{q}^r\right\},$
$\max\left\{\underline{p}^r.\underline{q}^r, \underline{p}^r.\overline{q}^r, \overline{p}^r.\underline{q}^r, \overline{p}^r.\overline{q}^r\right\}],$

$\{\mu p\}^r = \mu p^r$ [18].
The metric space $(\mathbb{R}_F, d_\infty)$ is complete [21], where

$$d_\infty[p_1, p_2] = \sup_{0 \le r \le 1} \max\{|\underline{p_1}^r - \underline{p_2}^r|, |\overline{p_1}^r - \overline{p_2}^r|\}.$$

is called Hausdorff distance.

Definition 2.1. *(see [9]) Assume $p, q \in \mathbb{R}_F$. If there exists $w \in \mathbb{R}_F$ such that $p = q + w$, then w is called the H-difference of p and q and it is displayed by $p \overset{H}{\ominus} q$.*

Definition 2.2. *(see [9]) Suppose $g : (a, b) \to \mathbb{R}_F$ and $\xi_0 \in (a, b)$. f is strongly generalized differentiable at ξ_0, if there exists an element $g'(\xi_0) \in \mathbb{R}_F$ such that*
(i) for all $h > 0$ small enough, $\exists g(\xi_0 + h) \overset{H}{\ominus} g(\xi_0), \exists g(\xi_0) \overset{H}{\ominus} g(\xi_0 - h)$ and limits:

$$\lim_{h \to 0} \frac{g(\xi_0 + h) \overset{H}{\ominus} g(\xi_0)}{h} = \lim_{h \to 0} \frac{g(\xi_0) \overset{H}{\ominus} g(\xi_0 - h)}{h} = g'(\xi_0)$$

or
(ii) for all $h > 0$ small enough, $\exists g(\xi_0) \overset{H}{\ominus} g(\xi_0 + h), \exists g(\xi_0 - h) \overset{H}{\ominus} g(\xi_0)$ and limits:

$$\lim_{h \to 0} \frac{g(\xi_0) \overset{H}{\ominus} g(\xi_0 + h)}{-h} = \lim_{h \to 0} \frac{g(\xi_0 - h) \overset{H}{\ominus} g(\xi_0)}{-h} = g'(\xi_0)$$

or
(iii) for all $h > 0$ small enough, $\exists g(\xi_0 + h) \overset{H}{\ominus} g(\xi_0), \exists g(\xi_0 - h) \overset{H}{\ominus} g(\xi_0)$ and limits:

$$\lim_{h \to 0} \frac{g(\xi_0 + h) \overset{H}{\ominus} g(\xi_0)}{h} = \lim_{h \to 0} \frac{g(\xi_0 - h) \overset{H}{\ominus} g(\xi_0)}{-h} = g'(\xi_0)$$

or
(iiii) for all $h > 0$ small enough, $\exists g(\xi_0) \overset{H}{\ominus} g(\xi_0 + h), \exists g(\xi_0) \overset{H}{\ominus} g(\xi_0 - h)$ and limits:

$$\lim_{h \to 0} \frac{g(\xi_0) \overset{H}{\ominus} g(\xi_0 + h)}{-h} = \lim_{h \to 0} \frac{g(\xi_0) \overset{H}{\ominus} g(\xi_0 - h)}{h} = g'(\xi_0).$$

If g be differentiable with (i) of Definition 2.2 then g is (I)-differentiable and differentiable and it is (II)-differentiable if it is differentiable with (ii) of Definition 2.2.

Theorem 2.3. *(see [12]) Suppose $g : (a, b) \to \mathbb{R}_F$ and $g^r(t) = [\underline{g}^r(t), \overline{g}^r(t)]$ for any $r \in [0, 1]$. Then*
if g be (I)-differentiable, then $\underline{g}^r(t)$ and $\overline{g}^r(t)$ are differentiable functions and

$$[g'^r(t)] = [\underline{g'}^r(t), \overline{g'}^r(t)].$$

if g be (II)-differentiable, then $\underline{g}^r(t)$ and $\overline{g}^r(t)$ are differentiable functions and

$$[g'^r(t)] = [\overline{g'}^r(t), \underline{g'}^r(t)].$$

3. VIM TO SOLVE FSODEs

Recently, a method has been presented to transform a FDE in to a system of ODEs [8, 29].

Let in Eq. (1. 1) the initial values and elements of Q matrix (q_{ij}) be fuzzy numbers. Assume q_{ij} as the following cases: $q_{ij} > 0$ *or* $q_{ij} < 0$ *or* $q_{ij} = \widetilde{0}$. If $U(t)$ is (I)-differentiable; then,

$$\begin{cases} \underline{U'}^r(t) = [\underline{QU}]^r(t) + \underline{G}^r(t) \\ \overline{U'}^r(t) = [\overline{QU}]^r(t) + \overline{G}^r(t),\ 0 \le t \le T, \\ \underline{U}^r(0) = \underline{U_0}^r,\ \overline{U}^r(0) = \overline{U_0}^r \end{cases} \tag{3. 2}$$

in which

$$\begin{aligned} [\underline{QU}]^r(t) &= \min\{ML | M \in [\underline{Q}, \overline{Q}]^r, L \in [\underline{U}(t), \overline{U}(t)]^r\} \\ [\overline{QU}]^r(t) &= \max\{ML | M \in [\underline{Q}, \overline{Q}]^r, L \in [\underline{U}(t), \overline{U}(t)]^r\} \end{aligned} \tag{3. 3}$$

and in a case that $U(t)$ is (II)-differentiable, we have

$$\begin{cases} \underline{U'}^r(t) = [\overline{QU}]^r(t) + \overline{G}^r(t),\ 0 \le t \le T, \\ \overline{U'}^r(t) = [\underline{QU}]^r(t) + \underline{G}^r(t) \\ \underline{U}^r(0) = \underline{U_0}^r,\ \overline{U}(0)^r = \overline{U_0}^r. \end{cases} \tag{3. 4}$$

We assumed that $U(t)$ is (I)-differentiable. Now, suppose $Q = H + K$ where H, K are matrices as follows:

$$h_{ij} = \begin{cases} q_{ij} \text{ if } q_{ij} > 0, \\ 0 \text{ if } otherwise, \end{cases} \tag{3. 5}$$

and

$$k_{ij} = \begin{cases} q_{ij} \text{ if } q_{ij} < 0, \\ 0 \text{ if } otherwise, \end{cases} \tag{3. 6}$$

then

$$\begin{cases} U'(t) = HU(t) + KU(t) + G(t)\ 0 \le t \le T, \\ U(0) = U_0. \end{cases} \tag{3. 7}$$

By employing relation (3. 2) on (3. 7) we have

$$\begin{aligned} \underline{U}'(t) &= \underline{H}\ \underline{U}(t) + \underline{K}\overline{U}(t),\ \underline{U}(0) = \underline{U_0} \\ \overline{U}'(t) &= \overline{H}\ \overline{U}(t) + \overline{K}\underline{U}(t),\ \overline{U}(0) = \overline{U_0}. \end{aligned} \tag{3. 8}$$

The relations (3. 8) are valid without any restrictions on the matrix Q when $U^r(t)$ is nonnegative for all $0 \le t \le T$ and $0 \le r \le 1$.

To use the VIM, consider the following differential equation in the formal form

$$L[w(t)] + N[w(t)] = g(t)$$

where L, N and $g(t)$ are the linear operator, the nonlinear operator and the inhomogeneous term respectively. We construct correctional functional as follows:

$$w_{m+1}(t) = w_m(t) + \int_0^t \lambda[Lw_m(s) + N\,\tilde{w}_m(s) - g(s)]\,ds, \qquad (m = 0, 1, 2, ...)$$

where λ is the general Lagrange multiplier [22, 15, 44] which can be determined via the variational theory. Specifying λ can pave the way to compute the successive approximations, $m = 0, 1, \ldots$. Now, consider the FSODEs (3. 8). To solve this system by the VIM, assume that both matrices $\underline{H} = (\underline{h_{ij}})$ and $\overline{H} = (\overline{h_{ij}})$ are decomposed into two matrices $\underline{D}$, $\underline{B}$ and $\overline{D}$, $\overline{B}$ respectively, such that $\underline{H} = \underline{D} + \underline{B}$, $\underline{B} = \underline{H} - \underline{D}$ and $\overline{H} = \overline{D} + \overline{B}$, $\overline{B} = \overline{H} - \overline{D}$ where $\underline{D} = diag(\underline{h_{11}}, \underline{h_{22}}, \ldots, \underline{h_{nn}})$ and $\overline{D} = diag(\overline{h_{11}}, \overline{h_{22}}, \ldots, \overline{h_{nn}})$. Correction functional for $\underline{U}, \overline{U}$ can be presented as:

$$\underline{U}_{m+1}(t) = \underline{U}_m(t) + \int_0^t \Lambda[\underline{U'}_m(s) - \underline{D}\,\underline{U}_m(s) - \underline{B}\,\widetilde{\underline{U}}_m(s) - \underline{K}\,\widetilde{\overline{U}}_m(s) - \underline{G}(s)]\,ds, \quad (3.\ 9)$$

$$\overline{U}_{m+1}(t) = \overline{U}_m(t) + \int_0^t \Upsilon[\overline{U'}_m(s) - \overline{D}\,\overline{U}_m(s) - \overline{B}\,\widetilde{\overline{U}}_m(s) - \overline{K}\,\widetilde{\underline{U}}_m(s) - \overline{G}(s)]\,ds \quad (3.\ 10)$$

where $m = 0, 1, 2, \ldots$, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots\ldots, \lambda_n)$, and $\Upsilon = \text{diag}(\gamma_1, \gamma_2, \ldots\ldots, \gamma_n)$ in which λ_i, γ_i for $i = 1, 2, \ldots, n$ are the Lagrange multipliers and $\widetilde{\underline{U}}_m, \widetilde{\overline{U}}_m$ denotes the restrictive variation, i.e. $\delta\,\widetilde{\underline{U}}_m = \delta\,\widetilde{\overline{U}}_m = 0$. Note that although $\underline{B}\ \underline{U}$, $\underline{K}\ \overline{U}$, $\overline{B}\ \overline{U}$, and $\overline{K}\ \underline{U}$ are not nonlinear terms, we consider them as such.

By Using the method presented in [7] we obtain Lagrange multipliers Λ, Υ.
Lagrange multipliers can be written as: $\Lambda = \Lambda(t-s), \Upsilon = \Upsilon(t-s)$.
We apply the Laplace transform on the both side of equations (3. 9) and (3. 10)

$$\begin{aligned} L[\underline{U}_{m+1}(t)] &= L[\underline{U}_m(t)] + L\Big[\int_0^t \Lambda(t-s)\Big\{\underline{U'}_m(s) - \underline{D}\,\underline{U}_m(s) - \underline{B}\,\widetilde{\underline{U}}_m(s) - \underline{K}\,\widetilde{\overline{U}}_m(s) - \widetilde{\underline{G}}(s)\Big\}\,ds\Big] \\ &= L[\underline{U}_m(t)] + L[\Lambda(t)]L\Big[\underline{U'}_m(t) - \underline{D}\,\underline{U}_m(t) - \underline{B}\,\widetilde{\underline{U}}_m(t) - \underline{K}\,\widetilde{\overline{U}}_m(t) - \widetilde{\underline{G}}(t)\Big] \\ &= L[\underline{U}_m(t)] + L[\Lambda(t)]\Big\{\tau L[\underline{U}_m(t)] - \underline{U}_m(0) - \underline{D}\,L[\underline{U}_m(t)] - \underline{B}\,L[\widetilde{\underline{U}}_m(t)] - \underline{K}\,L[\widetilde{\overline{U}}_m(t)] - L[\widetilde{\underline{G}}(t)]\Big\}, \end{aligned} \quad (3.\ 11)$$

$$\begin{aligned} L[\overline{U}_{m+1}(t)] &= L[\overline{U}_m(t)] + L\Big[\int_0^t \Upsilon(t-s)\Big\{\overline{U'}_m(s) - \overline{D}\,\overline{U}_m(s) - \overline{B}\,\widetilde{\overline{U}}_m(s) - \overline{K}\,\widetilde{\underline{U}}_m(s) - \widetilde{\overline{G}}(s)\Big\}\,ds\Big] \\ &= L[\overline{U}_m(t)] + L[\Upsilon(t)]L\Big[\overline{U'}_m(t) - \overline{D}\,\overline{U}_m(t) - \overline{B}\,\widetilde{\overline{U}}_m(t) - \overline{K}\,\widetilde{\underline{U}}_m(t) - \widetilde{\overline{G}}(t)\Big] \\ &= L[\overline{U}_m(t)] + L[\Upsilon(t)]\Big\{\tau L[\overline{U}_m(t)] - \overline{U}_m(0) - \overline{D}\,L[\overline{U}_m(t)] - \overline{B}\,L[\widetilde{\overline{U}}_m(t)] - \overline{K}\,L[\widetilde{\underline{U}}_m(t)] - L[\widetilde{\overline{G}}(t)]\Big\}. \end{aligned} \quad (3.\ 12)$$

The optimal value of Λ, Υ can be obtained by making equations (3. 11) and (3. 12) stationary with respect to $\overline{U}_m(t), \underline{U}_m(t)$ respectively and so

$$\begin{aligned} &\frac{\delta}{\underline{U}_m} L[\underline{U}_{m+1}(t)] = \frac{\delta}{\underline{U}_m} L[\underline{U}_m(t)] + \frac{\delta}{\underline{U}_m}\Big[L[\Lambda(t)]\Big\{\tau L[\underline{U}_m(t)] - \underline{U}_m(0) - \underline{D}\,L[\underline{U}_m(t)] - \underline{B}\,L[\widetilde{\underline{U}}_m(t)] - \\ &\underline{K}\,L[\widetilde{\overline{U}}_m(t)] - L[\widetilde{\underline{G}}(t)]\Big\}\Big] = \Big\{I + L[\Lambda(t)](\tau I - \underline{D})\Big\}\frac{\delta}{\underline{U}_m} L[\underline{U}_m(t)] = 0, \end{aligned} \quad (3.\ 13)$$

$$\begin{aligned} &\frac{\delta}{\overline{U}_m} L[\overline{U}_{m+1}(t)] = \frac{\delta}{\overline{U}_m} L[\overline{U}_m(t)] + \frac{\delta}{\overline{U}_m}\Big[L[\Upsilon(t)]\Big\{\tau L[\overline{U}_m(t)] - \overline{U}_m(0) - \overline{D}\,L[\overline{U}_m(t)] - \overline{B}\,L[\widetilde{\overline{U}}_m(t)] - \\ &\overline{K}\,L[\widetilde{\underline{U}}_m(t)] - L[\widetilde{\overline{G}}(t)]\Big]\Big\} = \Big\{I + L[\Upsilon(t)](\tau I - \overline{D})\Big\}\frac{\delta}{\overline{U}_m} L[\overline{U}_m(t)] = 0. \end{aligned} \quad (3.\ 14)$$

Given the relations (3. 13) and (3. 14), we can write

$$\mathcal{L}[\lambda_i] = \frac{-1}{\tau - \underline{h_{ii}}} \Longrightarrow \lambda_i(t) = -e^{\underline{h_{ii}}t},$$

$$\mathcal{L}[\gamma_i] = \frac{-1}{\tau - \overline{h_{ii}}} \Longrightarrow \gamma_i(t) = -e^{\overline{h_{ii}}t}.$$

Therefore

$$\Lambda = -e^{\underline{D}(t-s)}, \Upsilon = -e^{\overline{D}(t-s)}.$$

Therefore, from (3. 9) and (3. 10), the following iteration formula for computing $\underline{Y}_m(t), \overline{Y}_m(t)$ may be obtained

$$\underline{U}^r_{m+1}(t) = \underline{U}^r_m(t) - \int_0^t e^{-\underline{D}^r(s-t)}[\underline{U'}^r_m(s) - \underline{H}^r\ \underline{U}^r_m(s) - \underline{K}^r\overline{U}^r_m(s) - \underline{G}^r(s)]\, ds, \quad (3.\ 15)$$

$$\overline{U}^r_{m+1}(t) = \overline{U}^r_m(t) - \int_0^t e^{-\overline{D}^r(s-t)}[\overline{U'}^r_m(s) - \overline{H}^r\ \overline{U}^r_m(s) - \overline{K}^r\underline{U}^r_m(s) - \overline{G}^r(s)]\, ds \quad (3.\ 16)$$

4. Convergence of VIM for fuzzy linear systems

Theorem 4.1. *If $\underline{U}^r(t), \overline{U}^r(t), \underline{U}^r_m(t)$, and $\overline{U}^r_m(t) \in (C^1[0,T])^n, m = 0, 1, ...$ then the sequences defined by (3. 15) and (3. 16) are convergent to the exact solution of (3. 8).*

Proof. Based on the equations in (3. 8) we can write

$$\underline{U}^r(t) = \underline{U}^r(t) - \int_0^t e^{-\underline{D}^r(s-t)}[\underline{U'}^r(s) - \underline{H}^r\ \underline{U}^r(s) - \underline{K}^r\overline{U}^r(s) - \underline{G}^r(s)]\, ds, \quad (4.\ 17)$$

$$\overline{U}^r(t) = \overline{U}^r(t) - \int_0^t e^{-\overline{D}^r(s-t)}[\overline{U'}^r(s) - \overline{H}^r\ \overline{U}^r(s) - \overline{K}^r\underline{U}^r(s) - \overline{G}^r(s)]\, ds. \quad (4.\ 18)$$

Now, for $j = 1, 2, ...$ we denote $E^r_j(t) = \underline{U}^r_j(t) - \underline{U}^r(t)$ and $\Delta^r_j(t) = \overline{U}^r_j(t) - \overline{U}^r(t)$ and by subtracting (4. 17) and (4. 18) from (3. 15) and (3. 16), respectively, we can write

$$E^r_{m+1}(t) = E^r_m(t) - \int_0^t e^{-\underline{D}^r(s-t)}[(E^r_m(s))' - \underline{H}^r\ E^r_m(s) - \underline{K}^r\Delta^r_m(s)]\, ds, \quad (4.\ 19)$$

$$\Delta^r_{m+1}(t) = \Delta^r_m(t) - \int_0^t e^{-\overline{D}^r(s-t)}[(\Delta^r_m(s))' - \overline{H}^r\ \Delta^r_m(s) - \overline{K}^r E^r_m(s)]\, ds. \quad (4.\ 20)$$

By integrating (4. 19) and (4. 20) and considering $E^r_m(0) = \Delta^r_m(0) = 0$ for $m = 0, 1, ...$, we have

$$E^r_{m+1}(t) = \underline{B}^r \int_0^t e^{-\underline{D}^r(s-t)} E^r_m(s) ds + \int_0^t e^{-\underline{D}^r(s-t)}\underline{K}^r\Delta^r_m(s)\, ds, \quad (4.\ 21)$$

$$\Delta^r_{m+1}(t) = \overline{B}^r \int_0^t e^{-\underline{D}^r(s-t)} \Delta^r_m(s) ds + \int_0^t e^{-\underline{D}^r(s-t)}\overline{K}^r E^r_m(s)\, ds. \quad (4.\ 22)$$

Therefore

$$\begin{aligned} ||E^r_{m+1}(t)|| \leq ||\underline{B}^r|| \int_0^t ||e^{-\underline{D}^r(s-t)}||\ ||E^r_m(s)||ds + \\ \int_0^t ||e^{-\underline{D}^r(s-t)}||\ ||\underline{K}^r||\ ||\Delta^r_m(s)||\, ds, \end{aligned} \quad (4.\ 23)$$

$$||\Delta^r_{m+1}(t)|| \leq ||\overline{B}^r|| \int_0^t ||e^{-\underline{D}^r(s-t)}|| \, ||\Delta^r_m(s)||ds + \int_0^t ||e^{-\underline{D}^r(s-t)}|| \, ||\overline{K}^r|| \, ||E^r_m(s)|| \, ds. \tag{4. 24}$$

Since $s \leq t \leq T$, we can conclude that $||e^{-\underline{D}^r(s-t)}|| \leq M^r_1$ and $||e^{-\overline{D}^r(s-t)}|| \leq M^r_2$. If we set, $\max\{||\underline{B}^r||, ||\overline{B}^r||\} = W^r_1$, $\max\{||M^r_1||, ||M^r_2||\} = W^r_2$ and $\max\{||\underline{K}^r||, ||\overline{K}^r||\} = W^r_3$, then

$$||E^r_{m+1}(t)|| \leq W^r_1 \int_0^t W^r_2 \, ||E^r_m(s)||ds + \int_0^t W^r_2 \, W^r_3 \, ||\Delta^r_m(s)|| \, ds, \tag{4. 25}$$

$$||\Delta^r_{m+1}(t)|| \leq W^r_1 \int_0^t W^r_2 \, ||\Delta^r_m(s)||ds + \int_0^t W^r_2 \, W^r_3 \, ||E^r_m(s)|| \, ds. \tag{4. 26}$$

If we set, $W^r = \max\{W^r_1 W^r_2, W^r_2 \; W^r_3\}$, then Equations (4. 25) and (4. 26) can be written as

$$||E^r_{m+1}(t)|| \leq W^r \int_0^t (||E^r_m(s)|| + ||\Delta^r_m(s)||) \, ds, \tag{4. 27}$$

$$||\Delta^r_{m+1}(t)|| \leq W^r \int_0^t (||E^r_m(s)|| + ||\Delta^r_m(s)||) \, ds. \tag{4. 28}$$

Now we proceed as follows:

$$\begin{aligned}
&||E^r_1(t)|| \leq tW^r(||E^r_0(s)|| + ||\Delta^r_0(s)||)\\
&||\Delta^r_1(t)|| \leq tW^r(||E^r_0(s)|| + ||\Delta^r_0(s)||)\\
&||E^r_2(t)|| \leq 2^1 \tfrac{t^2}{2!}(W^r)^2(||E^r_0(s)|| + ||\Delta^r_0(s)||)\\
&||\Delta^r_2(t)|| \leq 2^1 \tfrac{t^2}{2!}(W^r)^2(||E^r_0(s)|| + ||\Delta^r_0(s)||)\\
&||E^r_3(t)|| \leq 2^2 \tfrac{t^3}{3!}(W^r)^3(||E^r_0(s)|| + ||\Delta^r_0(s)||)\\
&||\Delta^r_3(t)|| \leq 2^2 \tfrac{t^3}{3!}(W^r)^3(||E^r_0(s)|| + ||\Delta^r_0(s)||)\\
&||E^r_4(t)|| \leq 2^3 \tfrac{t^4}{4!}(W^r)^4(||E^r_0(s)|| + ||\Delta^r_0(s)||)\\
&||\Delta^r_4(t)|| \leq 2^3 \tfrac{t^4}{4!}(W^r)^4(||E^r_0(s)|| + ||\Delta^r_0(s)||)\\
&\vdots\\
&||E^r_m(t)|| \leq 2^{m-1} \tfrac{t^m}{m!}(W^r)^m(||E^r_0(s)|| + ||\Delta^r_0(s)||)\\
&||\Delta^r_m(t)|| \leq 2^{m-1} \tfrac{t^m}{m!}(W^r)^m(||E^r_0(s)|| + ||\Delta^r_0(s)||).
\end{aligned} \tag{4. 29}$$

Now, if $m \to \infty$, then $||E^r_m(t)|| \to 0, ||\Delta^r_m(t)|| \to 0$. Thus, VIM is convergent to the exact solution. □

5. EXAMPLES

Example 5.1. *Assume the fuzzy decay model* [21]*:*

$$X'(t) = -\eta X(t) \; 0 \leq t \leq 25,$$
$$X^r(0) = [0.35 + 0.1r, \; 0.55 - 0.1r], \; \eta^r = [0.16 + 0.04r, \; 0.24 - 0.04r].$$

Exact solutions for $r = 0, 0.5, 1$ are as follows:

$$X^1(t) = \frac{9}{20}e^{-\frac{t}{50}}$$
$$\underline{X}^{0.5}(t) = \frac{\sqrt{11}}{3}e^{-\frac{3\sqrt{11}t}{500}}(\frac{3\sqrt{11}}{55} + \frac{1}{4}) - \frac{e^{\frac{3\sqrt{11}t}{500}}}{60}(5\sqrt{11} - 12)$$
$$\overline{X}^{0.5}(t) = e^{-\frac{3\sqrt{11}t}{500}}(\frac{3\sqrt{11}}{55} + \frac{1}{4}) + \frac{\sqrt{11}}{220}e^{\frac{3\sqrt{11}t}{500}}(5\sqrt{11} - 12)$$
$$\underline{X}^{0}(t) = \frac{\sqrt{6}}{2}e^{-\frac{\sqrt{6}t}{125}}(\frac{7\sqrt{6}}{120} + \frac{11}{40}) - \frac{e^{\frac{\sqrt{6}t}{125}}}{80}(11\sqrt{6} - 14)$$
$$\overline{X}^{0}(t) = e^{-\frac{\sqrt{6}t}{125}}(\frac{7\sqrt{6}}{120} + \frac{11}{40}) + \frac{\sqrt{6}}{240}e^{\frac{\sqrt{6}t}{125}}(11\sqrt{6} - 14).$$

Exact and approximate solutions for $r = 0, 0.5, 1$ are given in Figure 1 and also numerical results are given in Tables 1, 2 and 3.

Example 5.2. *Suppose that, in the system of differential equation which is given below* [17]*, we have*

$$\begin{cases} X'(t) = AX(t) + F(t)\ 0 \le t \le 1, \\ X(0) = X_0. \end{cases}$$

$$a_{11}^r = [1 + 0.5r,\ 2 - 0.5r],\ a_{12}^r = [0.5 + 0.5r,\ 1.5 - 0.5r],$$
$$a_{21}^r = [0.5 + 0.5r,\ 1.5 - 0.5r],\ a_{22}^r = [2 + 0.5r,\ 3 - 0.5r],$$
$$f_1^r(t) = [1 + t + (0.5 + t)r,\ 2 + 3t - (0.5 + t)r]$$
$$f_2^r(t) = [e^{-t} + (\frac{e^t - e^{-t}}{2})r,\ e^t - (\frac{e^t - e^{-t}}{2})r]$$
$$X_1^r(0) = [1 + 0.5r,\ 2 - 0.5r],\ X_2^r(0) = [0.5r,\ 1 - 0.5r].$$

Exact and approximate solutions for $r = 0, 0.5, 1$ are given in Figures 2 and 3 and also numerical results are given in Tables 4, 5, 6, 7, 8 and 9.

Example 5.3. *(Irregular Heartbeats and Lidocaine) For a specific body weight, this problem can be written as*

$$Y_1'(t) = aY_1(t) + bY_2(t)$$
$$Y_2'(t) = cY_1(t) + dY_2(t)$$
$$Y_1(0) = e,\ Y_2(0) = f$$

where $Y_1(t)$, $Y_2(t)$ are respectively amount of lidocaine in the bloodstream and amount of lidocaine in body tissue. Assume that in this problem we have

$$a^r = [-0.1 + 0.01r,\ -0.08 - 0.01r],\ b^r = [0.028 + 0.01r,\ 0.048 - 0.01r],$$
$$c^r = [0.056 + 0.01r,\ 0.076 - 0.01r],\ d^r = [-0.048 + 0.01r,\ -0.028 - 0.01r],$$
$$e^r = \widetilde{0},\ f^r = [0.99 + 0.01r,\ 1.01 - 0.01r].$$

Exact and approximate solutions for $r = 0, 0.5, 1$ are given in Figures 4 and 5 and also numerical results are given in Tables 10, 11, 12, 13, 14 and 15.

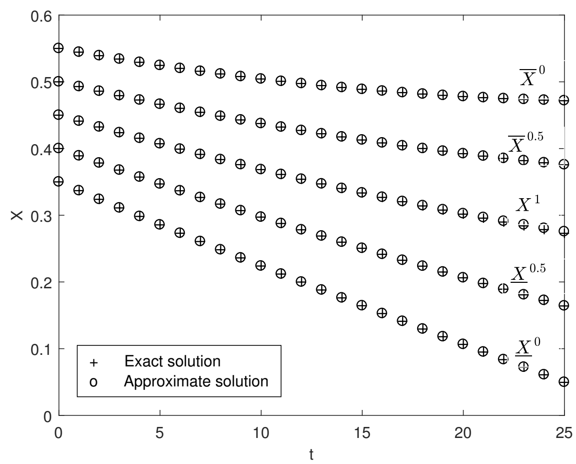

FIGURE 1. The analytical and approximate solutions of X by VIM (m=4).

TABLE 1. Exact values for X in example 5.1

t	$\underline{X}^0$	$\underline{X}^{0.5}$	X^1	$\overline{X}^0$	$\overline{X}^{0.5}$
0	3.5000e-01	4.0000e-01	4.5000e-01	5.5000e-01	5.0000e-01
5	2.8558e-01	3.4689e-01	4.0718e-01	5.2460e-01	4.6642e-01
10	2.2390e-01	2.9722e-01	3.6843e-01	5.0423e-01	4.3746e-01
15	1.6437e-01	2.5049e-01	3.3337e-01	4.8872e-01	4.1283e-01
20	1.0642e-01	2.0625e-01	3.0164e-01	4.7789e-01	3.9229e-01
25	4.9487e-02	1.6404e-01	2.7294e-01	4.7166e-01	3.7564e-01

TABLE 2. Approximate values by VIM (m=4) for X in example 5.1

t	$\underline{X}^0$	$\underline{X}^{0.5}$	X^1	$\overline{X}^0$	$\overline{X}^{0.5}$
0	3.5000e-01	4.0000e-01	4.5000e-01	5.5000e-01	5.0000e-01
5	2.8558e-01	3.4689e-01	4.0718e-01	5.2460e-01	4.6642e-01
10	2.2390e-01	2.9722e-01	3.6845e-01	5.0424e-01	4.3746e-01
15	1.6438e-01	2.5050e-01	3.3354e-01	4.8872e-01	4.1284e-01
20	1.0647e-01	2.0629e-01	3.0240e-01	4.7791e-01	3.9232e-01
25	4.9640e-02	1.6418e-01	2.7537e-01	4.7172e-01	3.7573e-01

TABLE 3. Relative error by VIM (m=4) for X in example 5.1

t	$\underline{X}^0$	$\underline{X}^{0.5}$	X^1	$\overline{X}^0$	$\overline{X}^{0.5}$
0	0	0	0	0	0
5	1.7603e-07	1.2797e-07	1.5498e-06	3.9712e-08	6.1610e-08
10	7.1281e-06	4.7248e-06	5.7689e-05	1.2801e-06	2.0542e-06
15	7.3182e-05	4.2099e-05	5.1000e-04	9.7046e-06	1.6152e-05
20	4.7294e-04	2.1314e-04	2.5040e-03	4.0438e-05	6.9985e-05
25	3.0828e-03	8.0923e-04	8.9099e-03	1.2081e-04	2.1790e-04

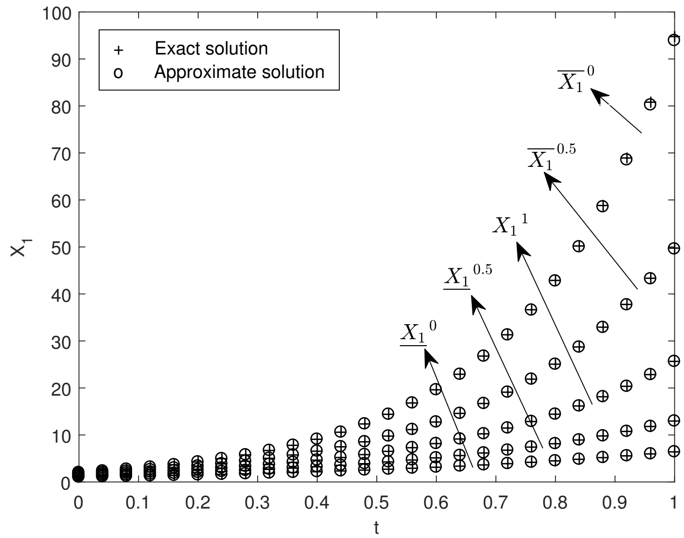

FIGURE 2. The analytical and approximate solutions of X_1 by VIM (m=5).

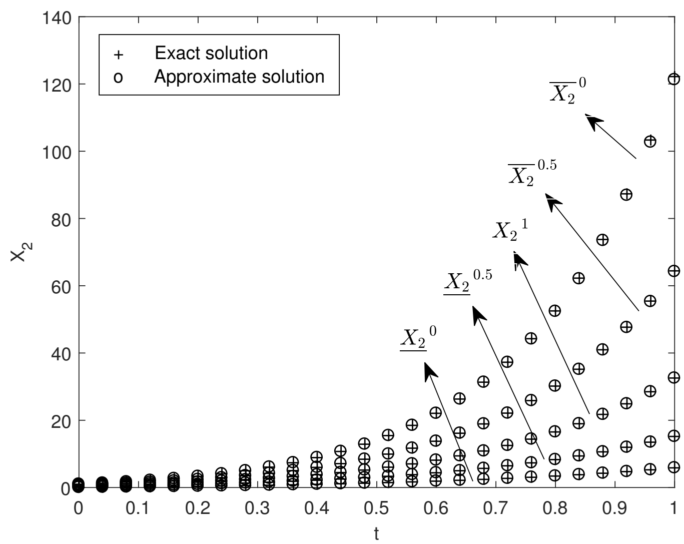

FIGURE 3. The analytical and approximate solutions of X_2 by VIM (m=5).

TABLE 4. Exact values for X_1 in example 5.2

t	$\underline{X_1}^0$	$\underline{X_1}^{0.5}$	${X_1}^1$	$\overline{X_1}^0$	$\overline{X_1}^{0.5}$
0	1.0000e+00	1.2500e+00	1.5000e+00	2.0000e+00	1.7500e+00
0.1	1.2197e+00	1.5882e+00	1.9913e+00	2.9123e+00	2.4318e+00
0.2	1.4827e+00	2.0144e+00	2.6421e+00	4.2465e+00	3.3806e+00
0.3	1.7961e+00	2.5496e+00	3.5028e+00	6.2023e+00	4.7018e+00
0.4	2.1683e+00	3.2209e+00	4.6418e+00	9.0789e+00	6.5453e+00
0.5	2.6095e+00	4.0631e+00	6.1518e+00	1.3325e+01	9.1255e+00
0.6	3.1322e+00	5.1210e+00	8.1588e+00	1.9615e+01	1.2748e+01
0.7	3.7516e+00	6.4526e+00	1.0834e+01	2.8961e+01	1.7851e+01
0.8	4.4862e+00	8.1327e+00	1.4411e+01	4.2885e+01	2.5058e+01
0.9	5.3589e+00	1.0258e+01	1.9207e+01	6.3673e+01	3.5267e+01
1	6.3974e+00	1.2955e+01	2.5654e+01	9.4763e+01	4.9757e+01

TABLE 5. Approximate values by VIM (m=5) for X_1 in example 5.2

t	$\underline{X_1}^0$	$\underline{X_1}^{0.5}$	${X_1}^1$	$\overline{X_1}^0$	$\overline{X_1}^{0.5}$
0	1.0000e+00	1.2500e+00	1.5000e+00	2.0000e+00	1.7500e+00
0.1	1.2197e+00	1.5882e+00	1.9913e+00	2.9123e+00	2.4318e+00
0.2	1.4827e+00	2.0144e+00	2.6421e+00	4.2465e+00	3.3806e+00
0.3	1.7961e+00	2.5496e+00	3.5028e+00	6.2022e+00	4.7017e+00
0.4	2.1683e+00	3.2209e+00	4.6417e+00	9.0781e+00	6.5451e+00
0.5	2.6095e+00	4.0631e+00	6.1515e+00	1.3321e+01	9.1244e+00
0.6	3.1322e+00	5.1209e+00	8.1580e+00	1.9600e+01	1.2744e+01
0.7	3.7515e+00	6.4523e+00	1.0832e+01	2.8913e+01	1.7839e+01
0.8	4.4862e+00	8.1319e+00	1.4405e+01	4.2748e+01	2.5025e+01
0.9	5.3587e+00	1.0256e+01	1.9191e+01	6.3315e+01	3.5182e+01
1	6.3972e+00	1.2951e+01	2.5619e+01	9.3896e+01	4.9556e+01

TABLE 6. Relative error by VIM (m=5) for X_1 in example 5.2

t	$\underline{X_1}^0$	$\underline{X_1}^{0.5}$	${X_1}^1$	$\overline{X_1}^0$	$\overline{X_1}^{0.5}$
0	0	0	0	0	0
0.1	6.0740e-11	6.0919e-10	3.1571e-09	3.2228e-08	1.1339e-08
0.2	3.6434e-09	3.6134e-08	1.8403e-07	1.8009e-06	6.4766e-07
0.3	3.9081e-08	3.8267e-07	1.9128e-06	1.7907e-05	6.5890e-06
0.4	2.0773e-07	2.0049e-06	9.8219e-06	8.7754e-05	3.3074e-05
0.5	7.5279e-07	7.1488e-06	3.4272e-05	2.9160e-04	1.1268e-04
0.6	2.1434e-06	1.9990e-05	9.3636e-05	7.5726e-04	3.0022e-04
0.7	5.1700e-06	4.7261e-05	2.1598e-04	1.6580e-03	6.7470e-04
0.8	1.1048e-05	9.8794e-05	4.3988e-04	3.2025e-03	1.3379e-03
0.9	2.1525e-05	1.8791e-04	8.1424e-04	5.6205e-03	2.4103e-03
1	3.8983e-05	3.3162e-04	1.3972e-03	9.1458e-03	4.0247e-03

TABLE 7. Exact values for X_2 in example 5.2

t	$\underline{X_2}^0$	$\underline{X_2}^{0.5}$	${X_2}^1$	$\overline{X_2}^0$	$\overline{X_2}^{0.5}$
0	0	2.5000e-01	5.0000e-01	1.0000e+00	7.5000e-01
0.1	1.6656e-01	5.4097e-01	9.5166e-01	1.8928e+00	1.4013e+00
0.2	3.7325e-01	9.3270e-01	1.5968e+00	3.3032e+00	2.3811e+00
0.3	6.3249e-01	1.4599e+00	2.5123e+00	5.5028e+00	3.8397e+00
0.4	9.5977e-01	2.1679e+00	3.8033e+00	8.9010e+00	5.9934e+00
0.5	1.3745e+00	3.1163e+00	5.6144e+00	1.4115e+01	9.1525e+00
0.6	1.9008e+00	4.3826e+00	8.1429e+00	2.2072e+01	1.3762e+01
0.7	2.5690e+00	6.0685e+00	1.1659e+01	3.4171e+01	2.0462e+01
0.8	3.4169e+00	8.3068e+00	1.6534e+01	5.2515e+01	3.0168e+01
0.9	4.4918e+00	1.1271e+01	2.3272e+01	8.0269e+01	4.4195e+01
1	5.8528e+00	1.5187e+01	3.2565e+01	1.2219e+02	6.4426e+01

TABLE 8. Approximate values by VIM (m=5) for X_2 in example 5.2

t	$\underline{X_2}^0$	$\underline{X_2}^{0.5}$	${X_2}^1$	$\overline{X_2}^0$	$\overline{X_2}^{0.5}$
0	0	2.5000e-01	5.0000e-01	1.0000e+00	7.5000e-01
0.1	1.6656e-01	5.4097e-01	9.5165e-01	1.8928e+00	1.4013e+00
0.2	3.7325e-01	9.3270e-01	1.5968e+00	3.3032e+00	2.3811e+00
0.3	6.3249e-01	1.4599e+00	2.5122e+00	5.5027e+00	3.8397e+00
0.4	9.5977e-01	2.1679e+00	3.8033e+00	8.9001e+00	5.9932e+00
0.5	1.3745e+00	3.1162e+00	5.6141e+00	1.4110e+01	9.1514e+00
0.6	1.9008e+00	4.3825e+00	8.1420e+00	2.2056e+01	1.3758e+01
0.7	2.5690e+00	6.0681e+00	1.1657e+01	3.4119e+01	2.0449e+01
0.8	3.4169e+00	8.3058e+00	1.6526e+01	5.2367e+01	3.0131e+01
0.9	4.4917e+00	1.1268e+01	2.3253e+01	7.9881e+01	4.4101e+01
1	5.8524e+00	1.5182e+01	3.2523e+01	1.2125e+02	6.4203e+01

TABLE 9. Relative error by VIM (m=5) for X_2 in example 5.2

t	$\underline{X_2}^0$	$\underline{X_2}^{0.5}$	${X_2}^1$	$\overline{X_2}^0$	$\overline{X_2}^{0.5}$
0	0	0	0	0	0
0.1	6.0021e-10	2.1586e-09	7.5093e-09	5.3167e-08	2.1584e-08
0.2	1.9749e-08	9.4680e-08	3.4706e-07	2.4845e-06	1.0101e-06
0.3	1.5300e-07	8.1463e-07	3.0472e-06	2.1681e-05	8.8761e-06
0.4	6.5314e-07	3.6463e-06	1.3727e-05	9.6264e-05	3.9794e-05
0.5	2.0067e-06	1.1454e-05	4.3093e-05	2.9647e-04	1.2396e-04
0.6	4.9993e-06	2.8806e-05	1.0788e-04	7.2586e-04	3.0732e-04
0.7	1.0767e-05	6.2172e-05	2.3121e-04	1.5183e-03	6.5150e-04
0.8	2.0830e-05	1.2002e-04	4.4254e-04	2.8313e-03	1.2322e-03
0.9	3.7112e-05	2.1281e-04	7.7712e-04	4.8374e-03	2.1367e-03
1	6.1946e-05	3.5290e-04	1.2752e-03	7.7142e-03	3.4599e-03

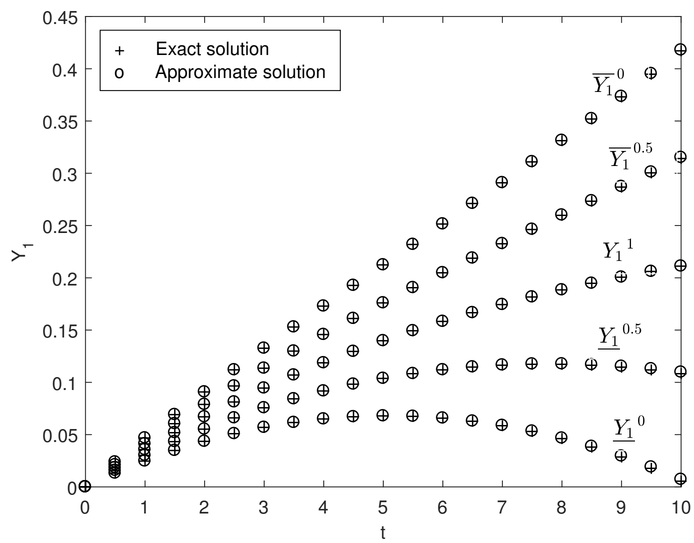

FIGURE 4. The analytical and approximate solutions of Y_1 by VIM (m=5).

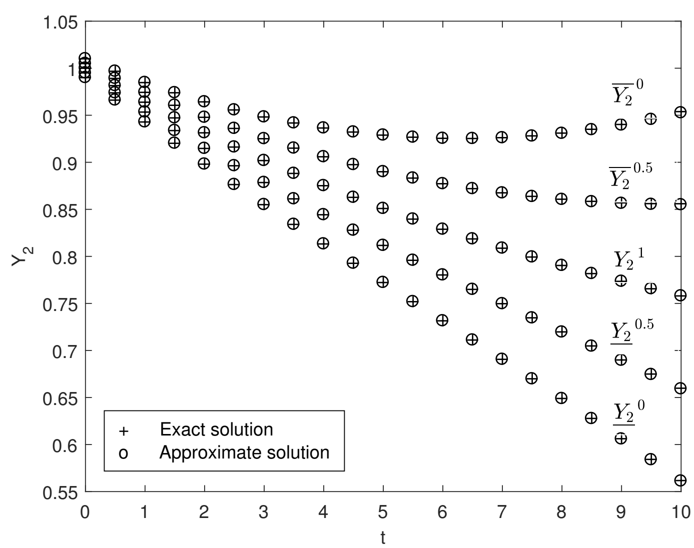

FIGURE 5. The analytical and approximate solutions of Y_2 by VIM (m=5).

TABLE 10. Exact values for Y_1 in example 5.3

t	$\underline{Y_1}^0$	$\underline{Y_1}^{0.5}$	Y_1^1	$\overline{Y_1}^0$	$\overline{Y_1}^{0.5}$
0	0	0	0	0	0
1	2.4686e-02	3.0151e-02	3.5663e-02	4.6827e-02	4.1222e-02
2	4.3551e-02	5.5238e-02	6.7011e-02	9.0817e-02	7.8871e-02
3	5.6902e-02	7.5659e-02	9.4535e-02	1.3264e-01	1.1353e-01
4	6.4965e-02	9.1747e-02	1.1867e-01	1.7295e-01	1.4574e-01
5	6.7890e-02	1.0377e-01	1.3981e-01	2.1235e-01	1.7600e-01
6	6.5751e-02	1.1194e-01	1.5828e-01	2.5146e-01	2.0479e-01
7	5.8549e-02	1.1640e-01	1.7440e-01	2.9088e-01	2.3256e-01
8	4.6211e-02	1.1725e-01	1.8844e-01	3.3121e-01	2.5976e-01
9	2.8588e-02	1.1455e-01	2.0062e-01	3.7308e-01	2.8680e-01
10	5.4550e-03	1.0829e-01	2.1117e-01	4.1712e-01	3.1412e-01

TABLE 11. Approximate values Y_1 by VIM (m=5) for Y_1 in example 5.3

t	$\underline{Y_1}^0$	$\underline{Y_1}^{0.5}$	Y_1^1	$\overline{Y_1}^0$	$\overline{Y_1}^{0.5}$
0	0	0	0	0	0
1	2.4686e-02	3.0151e-02	3.5663e-02	4.6827e-02	4.1222e-02
2	4.3551e-02	5.5238e-02	6.7011e-02	9.0817e-02	7.8871e-02
3	5.6903e-02	7.5660e-02	9.4535e-02	1.3264e-01	1.1353e-01
4	6.4972e-02	9.1753e-02	1.1867e-01	1.7295e-01	1.4574e-01
5	6.7917e-02	1.0379e-01	1.3981e-01	2.1237e-01	1.7602e-01
6	6.5832e-02	1.1201e-01	1.5828e-01	2.5150e-01	2.0484e-01
7	5.8751e-02	1.1657e-01	1.7440e-01	2.9098e-01	2.3269e-01
8	4.6658e-02	1.1764e-01	1.8844e-01	3.3142e-01	2.6003e-01
9	2.9488e-02	1.1533e-01	2.0063e-01	3.7349e-01	2.8733e-01
10	7.1377e-03	1.0973e-01	2.1118e-01	4.1787e-01	3.1510e-01

TABLE 12. Relative error by VIM (m=5) for Y_1 in example 5.3

t	$\underline{Y_1}^0$	$\underline{Y_1}^{0.5}$	Y_1^1	$\overline{Y_1}^0$	$\overline{Y_1}^{0.5}$
0	0	0	0	0	0
1	7.3025e-08	5.3113e-08	3.3175e-10	2.1575e-08	2.9240e-08
2	2.6245e-06	1.8321e-06	1.0599e-08	6.9079e-07	9.5634e-07
3	2.2676e-05	1.5050e-05	8.0297e-08	5.2247e-06	7.3995e-06
4	1.1065e-04	6.8903e-05	3.3728e-07	2.1823e-05	3.1666e-05
5	4.0066e-04	2.2971e-04	1.0251e-06	6.5684e-05	9.7793e-05
6	1.2259e-03	6.2873e-04	2.5385e-06	1.6036e-04	2.4534e-04
7	3.4464e-03	1.5081e-03	5.4555e-06	3.3821e-04	5.3253e-04
8	9.6636e-03	3.3007e-03	1.0568e-05	6.3983e-04	1.0384e-03
9	3.1467e-02	6.7796e-03	1.8905e-05	1.1124e-03	1.8633e-03
10	3.0848e-01	1.3363e-02	3.1758e-05	1.8067e-03	3.1279e-03

TABLE 13. Exact values for Y_2 in example 5.3

t	$\underline{Y_2}^0$	$\underline{Y_2}^{0.5}$	${Y_2}^1$	$\overline{Y_2}^0$	$\overline{Y_2}^{0.5}$
0	9.9000e-01	9.9500e-01	1.0000e+00	1.0100e+00	1.0050e+00
1	9.4286e-01	9.5341e-01	9.6390e-01	9.8475e-01	9.7435e-01
2	8.9804e-01	9.1472e-01	9.3131e-01	9.6422e-01	9.4781e-01
3	8.5500e-01	8.7848e-01	9.0184e-01	9.4819e-01	9.2508e-01
4	8.1322e-01	8.4426e-01	8.7514e-01	9.3646e-01	9.0587e-01
5	7.7222e-01	8.1164e-01	8.5090e-01	9.2891e-01	8.8999e-01
6	7.3149e-01	7.8026e-01	8.2884e-01	9.2548e-01	8.7725e-01
7	6.9057e-01	7.4974e-01	8.0872e-01	9.2617e-01	8.6754e-01
8	6.4898e-01	7.1973e-01	7.9033e-01	9.3105e-01	8.6077e-01
9	6.0620e-01	6.8990e-01	7.7347e-01	9.4022e-01	8.5691e-01
10	5.6174e-01	6.5990e-01	7.5798e-01	9.5388e-01	8.5597e-01

TABLE 14. Approximate values by VIM (m=5) for Y_2 in example 5.3

t	$\underline{Y_2}^0$	$\underline{Y_2}^{0.5}$	${Y_2}^1$	$\overline{Y_2}^0$	$\overline{Y_2}^{0.5}$
0	9.9000e-01	9.9500e-01	1.0000e+00	1.0100e+00	1.0050e+00
1	9.4286e-01	9.5341e-01	9.6390e-01	9.8475e-01	9.7435e-01
2	8.9804e-01	9.1472e-01	9.3131e-01	9.6422e-01	9.4781e-01
3	8.5500e-01	8.7848e-01	9.0184e-01	9.4819e-01	9.2508e-01
4	8.1322e-01	8.4425e-01	8.7514e-01	9.3645e-01	9.0587e-01
5	7.7220e-01	8.1163e-01	8.5090e-01	9.2888e-01	8.8997e-01
6	7.3146e-01	7.8022e-01	8.2884e-01	9.2541e-01	8.7719e-01
7	6.9050e-01	7.4964e-01	8.0872e-01	9.2601e-01	8.6740e-01
8	6.4881e-01	7.1952e-01	7.9033e-01	9.3069e-01	8.6046e-01
9	6.0587e-01	6.8947e-01	7.7347e-01	9.3950e-01	8.5629e-01
10	5.6114e-01	6.5911e-01	7.5796e-01	9.5253e-01	8.5481e-01

We can see in all Figures there are high agreement between approximate and exact solutions.

6. FUZZY BLOCH EQUATIONS

Bloch equations are widely used in physics and chemistry. We can illustrate Bloch equations as a system of ordinary differential equations as follows [34, 35]:

$$\begin{bmatrix} \frac{d}{dt}M_x(t) \\ \frac{d}{dt}M_y(t) \\ \frac{d}{dt}M_z(t) \end{bmatrix} = \begin{bmatrix} \frac{-1}{T_2} & \omega_0 & 0 \\ -\omega_0 & \frac{-1}{T_2} & 0 \\ 0 & 0 & \frac{-1}{T_1} \end{bmatrix} \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{M_0}{T_1} \end{bmatrix} \tag{6. 30}$$

$$M_x(0) = M_{x0},\ M_y(0) = M_{y0},\ M_z(0) = M_{z0},$$

TABLE 15. Relative error by VIM (m=5) for Y_2 in example 5.3

t	$\underline{Y_2}^0$	$\underline{Y_2}^{0.5}$	$Y_2{}^1$	$\overline{Y_2}^0$	$\overline{Y_2}^{0.5}$
0	0	0	0	0	0
1	8.5467e-10	1.0103e-09	2.1403e-11	1.4715e-09	1.3186e-09
2	5.5719e-08	6.5898e-08	1.3351e-09	9.5283e-08	8.5662e-08
3	6.4649e-07	7.6420e-07	1.4792e-08	1.0939e-06	9.8749e-07
4	3.7018e-06	4.3684e-06	8.0687e-08	6.1703e-06	5.5986e-06
5	1.4407e-05	1.6947e-05	2.9827e-07	2.3538e-05	2.1487e-05
6	4.3968e-05	5.1460e-05	8.6166e-07	7.0005e-05	6.4362e-05
7	1.1362e-04	1.3201e-04	2.0989e-06	1.7512e-04	1.6233e-04
8	2.6045e-04	2.9949e-04	4.5114e-06	3.8554e-04	3.6071e-04
9	5.4603e-04	6.1901e-04	8.8112e-06	7.6906e-04	7.2709e-04
10	1.0700e-03	1.1899e-03	1.5955e-05	1.4180e-03	1.3562e-03

where $M_x(t), M_y(t), M_z(t)$ stand for system magnetization (x, y, and z components); T_1, T_2 for relaxation times; M_0 for equilibrium magnetization and ω_0 for the resonant frequency.

Now, suppose that the initial conditions and T_1, T_2, M_0, ω_0 in E.q (6. 30) are fuzzy numbers which lead to the fuzzy linear system of differential equations with fuzzy coefficients, i.e.

$$\begin{aligned}
&M_{x0}{}^r = [\underline{M_{x0}}^r,\ \overline{M_{x0}}^r],\ M_{y0}{}^r = [\underline{M_{y0}}^r,\ \overline{M_{y0}}^r],\\
&M_{z0}{}^r = [\underline{M_{z0}}^r,\ \overline{M_{z0}}^r],\ \{\frac{-1}{T_2}\}^r = [\underline{(\frac{-1}{T_2})},\ \overline{(\frac{-1}{T_2})}],\\
&\{\frac{-1}{T_1}\}^r = [\underline{(\frac{-1}{T_1})},\ \overline{(\frac{-1}{T_1})}],\ \{\frac{M_0}{T_1}\}^r = [\underline{(\frac{M_0}{T_1})},\ \overline{(\frac{M_0}{T_1})}],\\
&[0]^r = [\underline{0},\ \overline{0}],\ \omega_0{}^r = [\underline{\omega_0},\ \overline{\omega_0}].
\end{aligned} \tag{6. 31}$$

Because fuzzy Bloch equations have a condition of theorem (4.1), the VIM is convergent to the exact solution.

Example 6.1. *Suppose in Eq. (6. 30) we have*

$$\begin{aligned}
&M_x^r(0) = [0.5 \cdot 10^{-2}(1+r),\ 10^{-2}(2-r)],\ M_y^r(0) = [90+10r,\ 110-10r],\\
&M_z^r(0) = [0.5 \cdot 10^{-2}(1+r),\ 10^{-2}(2-r)],\ \{\frac{-1}{T_2}\}^r = [\frac{-1}{18+2r},\ \frac{-1}{22-2r}],\\
&\{\frac{-1}{T_1}\}^r = [\frac{-1}{0.9+0.1r},\ \frac{-1}{1.1-0.1r}],\ \{\frac{M_0}{T_1}\}^r = [\frac{0.9+0.1r}{1.1-0.1r},\ \frac{1.1-0.1r}{0.9+0.1r}],\\
&0^r = [0,\ 0],\ \omega_0^r = [0.9+.1r,\ 1.1-0.1r],\ 0 \le t \le 1.
\end{aligned}$$

In Figure 6, the approximate and exact solutions of M_z for different values of $r = 0, 0.5, 1$ by VIM ($m = 4$) are plotted. Figures 7 and 8 illustrate the approximate and exact solutions of M_x, M_y for $r = 1$ by VIM ($m = 4$). However, we cannot find exact solutions for $r = 0, 0.5$, Thus we rely on the approximate solutions. Tables 16, 17, 18 and 19 present the numerical results for $t = 0.1, 0.2, .., 1$.

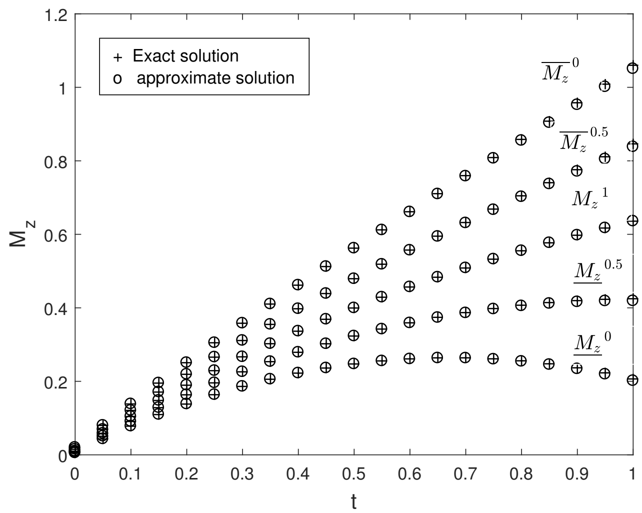

FIGURE 6. The analytical and approximate solutions of M_z by VIM (m=4).

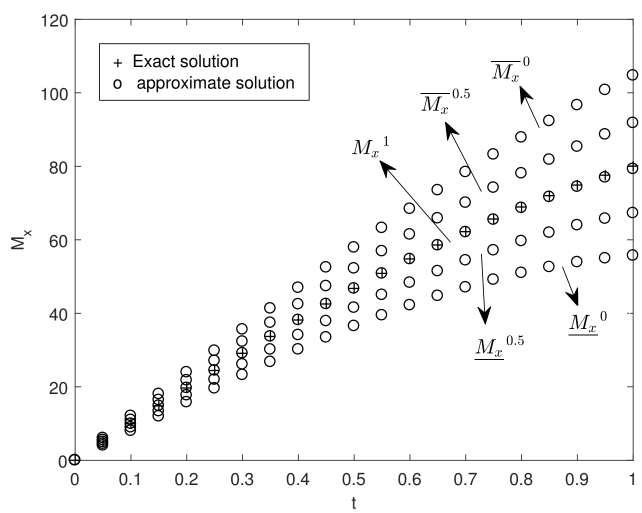

FIGURE 7. The analytical solution of M_x for $r = 1$, and approximate solution by VIM (m=4).

7. CONCLUSION

This study addresseses the variational iteration method for the approximate solution for a linear FSODEs in which Laplace transform is used to find Lagrange multipliers. It is shown

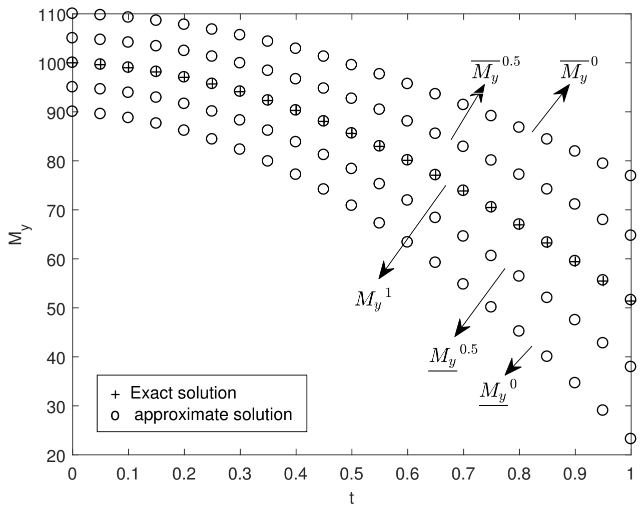

FIGURE 8. The analytical solution of M_y for $r = 1$, and approximate solution by VIM(m=4).

TABLE 16. Exact values for M_z in example 6.1

t	$\underline{M_z}^0$	$\underline{M_z}^{0.5}$	$M_z{}^1$	$\overline{M_z}^0$	$\overline{M_z}^{0.5}$
0	5.0000e-03	7.5000e-03	1.0000e-02	2.0000e-02	1.5000e-02
0.1	7.7959e-02	9.0761e-02	1.0421e-01	1.3835e-01	1.2076e-01
0.2	1.3812e-01	1.6329e-01	1.8946e-01	2.5066e-01	2.1911e-01
0.3	1.8608e-01	2.2581e-01	2.6659e-01	3.5805e-01	3.1103e-01
0.4	2.2233e-01	2.7896e-01	3.3638e-01	4.6163e-01	3.9744e-01
0.5	2.4723e-01	3.2325e-01	3.9953e-01	5.6242e-01	4.7923e-01
0.6	2.6105e-01	3.5915e-01	4.5668e-01	6.6145e-01	5.5719e-01
0.7	2.6391e-01	3.8701e-01	5.0838e-01	7.5973e-01	6.3212e-01
0.8	2.5584e-01	4.0711e-01	5.5516e-01	8.5825e-01	7.0477e-01
0.9	2.3678e-01	4.1964e-01	5.9750e-01	9.5799e-01	7.7587e-01
1	2.0651e-01	4.2475e-01	6.3580e-01	1.0600e+00	8.4613e-01

that the proposed method is convergent. Comparing between the exact and approximate solutions in the examples shows the accuracy and efficiency of this numerical method . In addition, this method was applied to solving fuzzy Bloch equations. The presented method can be used for various types of FDEs.

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TABLE 17. Approximate values by VIM (m=4) for M_z in example 6.1

t	$\underline{M_z}^0$	$\underline{M_z}^{0.5}$	$M_z{}^1$	$\overline{M_z}^0$	$\overline{M_z}^{0.5}$
0	5.0000e-03	7.5000e-03	1.0000e-02	2.0000e-02	1.5000e-02
0.1	7.7959e-02	9.0761e-02	1.0421e-01	1.3835e-01	1.2076e-01
0.2	1.3811e-01	1.6329e-01	1.8946e-01	2.5065e-01	2.1910e-01
0.3	1.8606e-01	2.2579e-01	2.6659e-01	3.5803e-01	3.1100e-01
0.4	2.2226e-01	2.7889e-01	3.3638e-01	4.6152e-01	3.9735e-01
0.5	2.4705e-01	3.2305e-01	3.9953e-01	5.6211e-01	4.7895e-01
0.6	2.6060e-01	3.5864e-01	4.5668e-01	6.6069e-01	5.5652e-01
0.7	2.6298e-01	3.8594e-01	5.0838e-01	7.5809e-01	6.3070e-01
0.8	2.5410e-01	4.0506e-01	5.5516e-01	8.5507e-01	7.0203e-01
0.9	2.3373e-01	4.1603e-01	5.9750e-01	9.5231e-01	7.7097e-01
1	2.0153e-01	4.1876e-01	6.3580e-01	1.0504e+00	8.3791e-01

TABLE 18. Relative error by VIM (m=4) for M_z in example 6.1

t	$\underline{M_z}^0$	$\underline{M_z}^{0.5}$	$M_z{}^1$	$\overline{M_z}^0$	$\overline{M_z}^{0.5}$
0	0	0	0	0	0
0.1	8.4371e-07	8.0273e-07	0	7.4110e-07	7.5199e-07
0.2	1.4806e-05	1.3972e-05	0	1.2968e-05	1.3099e-05
0.3	8.1048e-05	7.5075e-05	0	6.8326e-05	6.9234e-05
0.4	2.7749e-04	2.5059e-04	0	2.2142e-04	2.2565e-04
0.5	7.3887e-04	6.4573e-04	0	5.5012e-04	5.6461e-04
0.6	1.6886e-03	1.4149e-03	0	1.1549e-03	1.1950e-03
0.7	3.4988e-03	2.7766e-03	0	2.1574e-03	2.2522e-03
0.8	6.8152e-03	5.0341e-03	0	3.6975e-03	3.8973e-03
0.9	1.2844e-02	8.6080e-03	0	5.9301e-03	6.3150e-03
1	2.4117e-02	1.4086e-02	0	9.0207e-03	9.7105e-03

TABLE 19. Exact and approximate values $M_x{}^1, M_y{}^1$ and their relative errors in example 6.1

t	Exa. $M_x{}^1$	App. $M_x{}^1$	Rel. $M_x{}^1$	Exa. $M_y{}^1$	App. $M_y{}^1$	Rel. $M_y{}^1$
0	1.0000e-02	1.0000e-02	0	1.0000e+02	1.0000e+02	0
0.1	9.9434e+00	9.9434e+00	8.3370e-07	9.9003e+01	9.9003e+01	5.6040e-09
0.2	1.9679e+01	1.9679e+01	1.3403e-05	9.7030e+01	9.7030e+01	2.2716e-07
0.3	2.9121e+01	2.9119e+01	6.8354e-05	9.4108e+01	9.4109e+01	2.1228e-06
0.4	3.8180e+01	3.8171e+01	2.1824e-04	9.0278e+01	9.0279e+01	1.0816e-05
0.5	4.6767e+01	4.6742e+01	5.3986e-04	8.5587e+01	8.5590e+01	3.9549e-05
0.6	5.4803e+01	5.4741e+01	1.1377e-03	8.0089e+01	8.0098e+01	3.0409e-04
0.7	6.2213e+01	6.2080e+01	2.1486e-03	7.3847e+01	7.3870e+01	1.1755e-04
0.8	6.8930e+01	6.8671e+01	3.7486e-03	6.6932e+01	6.6980e+01	7.1440e-04
0.9	7.4892e+01	7.4430e+01	6.1614e-03	5.9418e+01	5.9512e+01	1.5702e-03
1	8.0048e+01	7.9274e+01	9.6699e-03	5.1387e+01	5.1557e+01	3.3048e-03

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