Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. (51)(9)2019) pp. 1-15

Some New Discrete Gronwall-Bihari type inequalities and their Applications

Madeeha
Department of Mathematics,
University of Sialkot, Sialkot.
Email: madeehabaloch@yahoo.com
Sabir Hussain
Department of Mathematics,
University of Engineering and Technology, Lahore.
Email: sabirhus@gmail.com

Received: 05 December, 2018 / Accepted: 26 March, 2019 / Published online: 01 August, 2019

Abstract. Some new Gronwall-Bihari type inequalities have been discussed. Consequently, we drive the bounded coefficients and uniqueness of solution of a nonlinear partial difference equation.

AMS (MOS) Subject Classification Codes: 30D05; 26D10
Key Words: Gronwall inequality; Bihari inequality; Boundedness; Uniqueness.

1. Introduction

For the development of many branches of mathematics inequalities have always been of inevitable importance. In fact, during the last century this significance has enhanced a lot. In both theory and applications this branch is multifaceted and experiencing a huge growth. While checking out the requirement of a system of differential equations regarding a parameter, out of many one of the inequality is due to Gronwall [6]. This inequality has attracted many researchers along with huge amount of papers which studies countless extensions, standard forms and numerous variants [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12]. The nonlinear generalization of Gronwall's inequality is by reason of Bihari [3] which has basic role in the study of nonlinear problems, while discrete inequalities which provide non-implicit bounds are thought of hand tools in the study of quantitative and qualitative analysis of solutions of difference equations. The main idea behind this paper is to highlight some new discrete Gronwall-Bihari type discrete inequalities with application as well as the boundedness and uniqueness of the solution of a certain partial difference equation is concerned. In this paper, we investigate more general Gronwall-Bihari inequalities as follows:

$$\begin{aligned} u(k_1,l_1) \le\ & a(k_1,l_1)+b(k_1,l_1)\sum_{k_1+1\le s\le\beta} c(s,l_1)u(s,l_1) \\ & +\sum_{k_1+1\le s<\infty}\ \sum_{l_1+1\le t<\infty}[f(k_1,l_1,s,t)w_1(u(s,t)) \\ & +\sum_{s\le\xi<\infty}\ \sum_{t\le\eta<\infty} g(k_1,l_1,\xi,\eta)w_2(u(s,t))], \end{aligned} \tag{1. 1}$$

$$\begin{aligned} u(k_1,l_1)\ \ \le\ & a(k_1,l_1)+b(k_1,l_1)\sum_{k_1+1\le s<\infty} c(s,l_1)u(s,l_1) \\ & +\sum_{0\le s\le k_1-1}\ \sum_{l_1+1\le t<\infty}[f(k_1,l_1,s,t)w_1(u(s,t)) \\ & +\sum_{0\le\xi\le s}\ \sum_{t\le\eta<\infty} g(k_1,l_1,\xi,\eta)w_2(u(s,t))] \end{aligned} \tag{1. 2}$$

$$\begin{aligned} u(k_1,l_1) \le\ & a(k_1,l_1)+b(k_1,l_1)\sum_{k_1+1\le s\le\beta} c(s,l_1)u(s,l_1) \\ & +\sum_{k_1+1\le s<\infty}\ \sum_{l_1+1\le t<\infty}[f(k_1,l_1,s,t)(w_1 o\psi_1)(u(s,t)) \\ & +\sum_{s\le\xi<\infty}\ \sum_{t\le\eta<\infty} g(k_1,l_1,\xi,\eta)(w_2 o\psi_2)(\log(u(s,t)))] \end{aligned} \tag{1. 3}$$

The format of paper is as below. After the Introduction, in Section 2 we have discussed our main results and in Section 3 we give applications to partial difference equation.

2. RESULTS

We assume some assumptions on functions and use symbolic representations to make the presentation easier and precise.

A1: $a(k_1,l_1), b(k_1,l_1)$ and $c(k_1,l_1)$ are bounded and non-negative for $k_1,l_1 \in N_0$ and $a(\infty,\infty)>0$.

A2: $f(k_1,l_1,s,t)$ and $g(k_1,l_1,\xi,\eta)$ are non-negative for $k_1,l_1,s,t,\xi,\eta\in N_0$.

A3: ψ_k and w_k, $1\le k\le 2$, are nondecreasing functions on $[0,\infty)$ and continuous and positive on $(0,\infty)$ such that $w_1 \propto w_2$, that is, $\phi_2(u) := \frac{w_2(u)}{w_1(u)}$ is nondecreasing on $(0,\infty)$.

A4: $\tilde{a}(k_1,l_1) := \sup_{\substack{k_1\le\tau \\ l_1\le\rho}} a(\tau,\rho)$; $\tilde{b}(k_1,l_1) := \sup_{\substack{k_1\le\tau \\ l_1\le\rho}} b(\tau,\rho)$.

A5: $\tilde{f}(k_1,l_1,s,t) := \sup_{\substack{k_1\le\tau \\ l_1\le\rho}} f(\tau,\rho,s,t)$; $\tilde{g}(k_1,l_1,s,t) := \sup_{\substack{k_1\le\tau \\ l_1\le\rho}} g(\tau,\rho,s,t)$.

A6: For $u\ge u_k>0$

$$W_k(u) := \int_{u_k}^{u}\frac{dz}{w_k(z)},\ \ 1\le k\le 2.$$

By $\mathbf{A3}$, W_k^{-1} is increasing, well defined and continuous in its corresponding domain because W_k is strictly increasing.

Obviously $\tilde{a}(k_1,l_1), \tilde{b}(k_1,l_1), \tilde{f}(k_1,l_1,s,t)$ and $\tilde{g}(k_1,l_1,\xi,\eta)$ are non-increasing and non-negative in k_1 and l_1 such that $\tilde{a}(k_1,l_1) \geq a(k_1,l_1)$; $\tilde{b}(k_1,l_1) \geq b(k_1,l_1)$; $\tilde{f}(k_1,l_1,s,t) \geq f(k_1,l_1,s,t)$; $\tilde{g}(k_1,l_1,\xi,\eta) \geq g(k_1,l_1,\xi,\eta)$ and $a(\infty,\infty) > 0$ implies $\tilde{a}(k_1,l_1) > 0$ for all $k_1 \geq L$ and $l_1 \geq L$ for some natural numbers K, L. As usual, $\Delta_1 f(k_1,l_1,s,t)$ and $\Delta_3 f(k_1,l_1,s,t)$ denote the first order forward difference in first and third components of f respectively.

Lemma 2.1. [1] *Let $u_n, a_n, b_n, q_n \geq 0$ be sequences define for $n \in N_0$ such that: $u_n \leq a_n + q_n \sum_{n+1\leq s\leq m} b_s u_s$. Then*

$$u_n \leq a_n + q_n \sum_{n+1\leq s\leq m} b_s a_s \prod_{n+1\leq i\leq s-1} (1 + b_i q_i).$$

Lemma 2.2. [10] *Let $u_n, a_n, b_n, c_n \geq 0$ be sequences define for $n \in N_0$ such that: $u_n \leq a_n + b_n \sum_{n+1\leq s<\infty} c_s u_s$. Then*

$$u_n \leq a_n + b_n \sum_{n+1\leq s<\infty} c_s a_s \prod_{n+1\leq i<\infty} (1 + c_i b_i).$$

Theorem 2.3. *Let the conditions **A1-A3** be satisfied; if $\Delta_1 p(k_1,l_1)\tilde{a}(k_1,l_1)$ is non-positive for k_1, l_1 and non-decreasing in l_1 and $u(k_1,l_1)$ is non-negative function for $k_1, l_1 \in N_0$, satisfying (3.1), then*

$$\begin{aligned} u(k_1,l_1) \leq\; & W_2^{-1}[W_2(\tilde{a}(\infty,l_1)) + p(k_1,l_1) \\ & \times \sum_{k_1+1\leq s<\infty} \sum_{l_1+1\leq t<\infty} (\sum_{s\leq\xi<\infty} \sum_{t\leq\eta<\infty} \tilde{g}(k_1,l_1,\xi,\eta)) \\ & - \sum_{k_1\leq s<\infty} \frac{\Delta_3 g_2(k_1,l_1,s,l_1)}{\phi_2(W_1^{-1}(\tilde{z}(s+1,l_1) + g_2(k_1,l_1,s+1,l_1)))}], \end{aligned} \quad (2.\ 4)$$

for $k_1 \geq K$ and $l_1 \geq L$ provided that

$$\begin{aligned} & W_2(\tilde{a}(\infty,L)) + p(K,L) \sum_{K+1\leq s<\infty} \sum_{L+1\leq t<\infty} (\sum_{s\leq\xi<\infty} \sum_{t\leq\eta<\infty} \tilde{g}(K,L,\xi,\eta)] \\ & - \sum_{K\leq s<\infty} \frac{\Delta_3 g_2(K,L,s,L)}{\phi_2(W_1^{-1}(\tilde{z}(s+1,L) + g_2(K,L,s+1,L))))} \leq \int_{u_2}^{\infty} \frac{dz}{w_2(z)}. \end{aligned}$$

Proof. Under the conditions $\mathbf{A4}$ and $\mathbf{A5}$, inequality (1.1) is rewritten as:

$$\begin{aligned} u(k_1,l_1) \leq \tilde{a}(k_1,l_1) + \tilde{b}(k_1,l_1) & \sum_{k_1+1\leq s\leq\beta} c(s,l_1) u(s,l_1) \\ & + \sum_{k_1+1\leq s<\infty} \sum_{l_1+1\leq t<\infty} [\tilde{f}(k_1,l_1,s,t) w_1(u(s,t)) \\ & + \sum_{s\leq\xi<\infty} \sum_{t\leq\eta<\infty} \tilde{g}(k_1,l_1,\xi,\eta) w_2(u(s,t))] \end{aligned} \quad (2.\ 5)$$

Equivalently,

$$u(k_1,l_1) \leq Y(k_1,l_1) + \tilde{b}(k_1,l_1) \sum_{k_1+1\leq s\leq\beta} c(s,l_1) u(s,l_1), \quad (2.\ 6)$$

for

$$Y(k_1,l_1) := \tilde{a}(k_1,l_1) + \sum_{k_1+1\le s<\infty} \sum_{l_1+1\le t<\infty} [\tilde{f}(k_1,l_1,s,t)w_1(u(s,t)) + \sum_{s\le\xi<\infty} \sum_{t\le\eta<\infty} \tilde{g}(k_1,l_1,\xi,\eta)w_2(u(s,t))]$$

Obviously, $Y(k_1,l_1)$ is non-increasing in k_1.

Application of Lemma 2.1 for some fixed l_1, yields:

$$\begin{aligned} u(k_1,l_1) &\le Y(k_1,l_1) + \tilde{b}(k_1,l_1) \sum_{k_1+1\le s\le\beta} c(s,l_1)Y(s,l_1) \\ &\quad \times \prod_{k_1+1\le i\le s-1} (1 + c(i,l_1)\tilde{b}(i,l_1)) \\ &\le Y(k_1,l_1)[1 + \tilde{b}(k_1,l_1) \sum_{k_1+1\le s\le\beta} c(s,l_1) \\ &\quad \times \prod_{k_1+1\le i\le s-1} (1 + c(i,l_1)\tilde{b}(i,l_1))] \\ &= p(k_1,l_1)\tilde{a}(k_1,l_1) + \sum_{k_1+1\le s<\infty} \sum_{l_1+1\le t<\infty} [F(k_1,l_1,s,t)w_1(u(s,t)) \\ &\quad + \sum_{s\le\xi<\infty} \sum_{t\le\eta<\infty} G(k_1,l_1,\xi,\eta)w_2(u(s,t))], \end{aligned} \tag{2. 7}$$

for

$$p(k_1,l_1) := 1 + \tilde{b}(k_1,l_1) \sum_{k_1+1\le s\le\beta} c(s,l_1) \prod_{k_1+1\le i\le s-1} (1 + c(i,l_1)\tilde{b}(i,l_1)). \tag{2. 8}$$

Obviously, $p(k_1,l_1)$ is non-increasing in both variables.

provided that

$$\begin{cases} F(k_1,l_1,s,t) := p(k_1,l_1)\tilde{f}(k_1,l_1,s,t); \\ G(k_1,l_1,\xi,\eta) := p(k_1,l_1)\tilde{g}(k_1,l_1,\xi,\eta). \end{cases} \tag{2. 9}$$

Obviously $p(k_1,l_1)\tilde{a}(k_1,l_1)$, $F(k_1,l_1,s,t)$ and $G(k_1,l_1,\xi,\eta)$ are non increasing in k_1 and l_1. Take any random positive integers K, L with Then, $K_1 \ge K$ and $L_1 \ge L$. From (2.7) yields auxiliary inequality as shown below:

$$\begin{aligned} u(k_1,l_1) &\le g_1(K,L,k_1,l_1) + \sum_{k_1+1\le s<\infty} \sum_{l_1+1\le t<\infty} [F(K,L,s,t)w_1(u(s,t)) \\ &\quad + \sum_{s\le\xi<\infty} \sum_{t\le\eta<\infty} G(K,L,\xi,\eta)w_2(u(s,t))], \end{aligned} \tag{2. 10}$$

for $k_1 \ge K$ and $l_1 \ge L$, where

$$g_1(k_1,l_1,s,t) := p(s,t)\tilde{a}(s,t) \tag{2. 11}$$

Consider,

$$z(k_1,l_1) := \sum_{k_1+1\le s<\infty}\sum_{l_1+1\le t<\infty}[F(K,L,s,t)w_1(u(s,t)) + \sum_{s\le\xi<\infty}\sum_{t\le\eta<\infty}G(K,L,\xi,\eta)w_2(u(s,t))],$$

then obviously $z(k_1,l_1)$ is nonnegative and non-increasing in both variables such that $z(\infty,l_1)=0$. In this case (2.10) has the form

$$u(k_1,l_1)\le z(k_1,l_1)+g_1(K,L,k_1,l_1). \tag{2. 12}$$

Moreover, by using (2.12)

$$\begin{aligned}-\Delta_1 z(k_1,l_1) &= \sum_{l_1+1\le t<\infty}[F(K,L,k_1+1,t)w_1(u(k_1+1,t)) \\ &\quad+\sum_{k_1+1\le\xi<\infty}\sum_{t\le\eta<\infty}G(K,L,\xi,\eta)w_2(u(k_1+1,t))] \\ &\le \sum_{l_1+1\le t<\infty}[F(K,L,k_1+1,t)w_1(z(k_1+1,t)+g_1(K,L,k_1+1,t)) \\ &\quad+\sum_{k_1+1\le\xi<\infty}\sum_{t\le\eta<\infty}G(K,L,\xi,\eta)w_2(z(k_1+1,t)+g_1(K,L,k_1+1,t))]\end{aligned} \tag{2. 13}$$

As, $\Delta_3 g_1(K,L,k_1,l_1)=\Delta_1 p(k_1,l_1)\tilde{a}(k_1,l_1)$ is non-positive k_1,l_1 and nondecreasing in l_1 therefore (2.13) is equivalent to

$$\begin{aligned}&-\frac{[\Delta_1 z(k_1,l_1)+\Delta_3 g_1(K,L,k_1,l_1)]}{w_1(z(k_1+1,l_1)+g_1(K,L,k_1+1,l_1))} \\ &\le \sum_{l_1+1\le t<\infty}[F(K,L,k_1+1,t)+\sum_{k_1+1\le\xi<\infty}\sum_{t\le\eta<\infty}G(K,L,\xi,\eta)\phi_2(z(k_1+1,t) \\ &\quad+g_1(K,L,k_1+1,t))]-\frac{\Delta_3 g_1(K,L,k_1,l_1)}{w_1(z(k_1+1,l_1)+g_1(K,L,k_1+1,l_1))}\end{aligned} \tag{2. 14}$$

By mean value Theorem for integral $\exists\,\tau$, such that:

$$z(k_1,l_1)+g_1(K,L,k_1,l_1)\le\tau\le z(k_1+1,l_1)+g_1(K,L,k_1+1,l_1)$$

Equivalently, by monotonicity of w_1

$$\frac{-1}{w_1(z(k_1,l_1)+g_1(K,L,k_1,l_1))}\le\frac{-1}{w_1(\tau)}\le\frac{-1}{w_1(z(k_1+1,l_1)+g_1(K,L,k_1+1,l_1))}$$

$$\begin{aligned}&\Rightarrow\ -\int_{z(k_1,l_1)+g_1(K,L,k_1,l_1)}^{z(k_1+1,l_1)+g_1(K,L,k_1+1,l_1)}\frac{d\tau}{w_1(\tau)} \\ &=\frac{[z(k_1,l_1)+g_1(K,L,k_1,l_1)]-[z(k_1+1,l_1)+g_1(K,L,k_1+1,l_1)]}{w_1(\tau)} \\ &\le-\frac{[z(k_1+1,l_1)+g_1(K,L,k_1+1,l_1)]-[z(k_1,l_1)+g_1(K,L,k_1,l_1)]}{w_1(z(k_1+1,l_1)+g_1(K,L,k_1+1,l_1))}.\end{aligned} \tag{2. 15}$$

A combination of (2.14) and (2.15) yields:

$$-\int_{z(k_1,l_1)+g_1(K,L,k_1,l_1)}^{z(k_1+1,l_1)+g_1(K,L,k_1+1,l_1)} \frac{d\tau}{w_1(\tau)}$$
$$\leq \sum_{l_1+1\leq t<\infty} [F(K,L,k_1+1,t) + \sum_{k_1+1\leq \xi<\infty} \sum_{t\leq \eta<\infty} G(K,L,\xi,\eta)\phi_2(z(k_1+1,t)$$
$$+g_1(K,L,k_1+1,t))] - \frac{\Delta_3 g_1(K,L,k_1,l_1)}{w_1(z(k_1+1,l_1)+g_1(K,L,k_1+1,l_1))}. \qquad (2.\ 16)$$

Setting k_1 by s in (2.16) and summing over s from k_1+1 to ∞ to get

$$-\int_{z(k_1,l_1)+g_1(K,L,k_1,l_1)}^{z(\infty,l_1)+g_1(K,L,\infty,l_1)} \frac{d\tau}{w_1(\tau)} \leq \sum_{k_1+1\leq s<\infty} \sum_{l_1+1\leq t<\infty} [F(K,L,s,t)$$
$$+ \sum_{s\leq \xi<\infty} \sum_{t\leq \eta<\infty} G(K,L,\xi,\eta)\phi_2(z(s,t)+g_1(K,L,s,t))]$$
$$- \sum_{k_1\leq s<\infty} \frac{\Delta_3 g_1(K,L,s,l_1)}{w_1(z(s+1,l_1)+g_1(K,L,s+1,l_1))}.$$

Equivalently,

$$W_1(z(k_1,l_1)+g_1(K,L,k_1,l_1)) \leq W_1(g_1(K,L,\infty,l_1))$$
$$+ \sum_{k_1+1\leq s<\infty} \sum_{l_1+1\leq t<\infty} [F(K,L,s,t) + \sum_{s\leq \xi<\infty} \sum_{t\leq \eta<\infty} G(K,L,\xi,\eta)$$
$$\times \phi_2(z(s,t)+g_1(K,L,s,t))]$$
$$- \sum_{k_1\leq s<\infty} \frac{\Delta_3 g_1(K,L,s,l_1)}{w_1(z(s+1,l_1)+g_1(K,L,s+1,l_1))} \qquad (2.\ 17)$$

Consider

$$\begin{cases} E(k_1,l_1) := W_1(z(k_1,l_1)+g_1(K,L,k_1,l_1)); \\ g_2(K,L,k_1,l_1) := W_1(g_1(K,L,\infty,l_1)) + \sum_{k_1+1\leq s<\infty} \sum_{l_1+1\leq t<\infty} F(K,L,s,t) \\ -\sum_{k_1\leq s<\infty} \frac{\Delta_3 g_1(K,L,s,l_1)}{w_1(z(s+1,l_1)+g_1(K,L,s+1,l_1))}; \\ \tilde{z}(k_1,l_1) := \sum_{k_1+1\leq s<\infty} \sum_{l_1+1\leq t<\infty} [\sum_{s\leq \xi<\infty} \sum_{t\leq \eta<\infty} G(K,L,\xi,\eta) \\ \times \phi_2(W_1^{-1}(E(s,t)))] \end{cases} \qquad (2.\ 18)$$

A combination of (2.17) and (2.18) yields:

$$E(k_1,l_1) \leq \tilde{z}(k_1,l_1) + g_2(K,L,k_1,l_1). \qquad (2.\ 19)$$

On the other hand

$$
\begin{aligned}
&-\int_{\tilde z(k_1,l_1)+g_2(K,L,k_1,l_1)}^{\tilde z(k_1+1,l_1)+g_2(K,L,k_1+1,l_1)} \frac{d\tau}{\phi_2(W_1^{-1}(\tau))}\\
&= W_2(W_1^{-1}(\tilde z(k_1,l_1)+g_2(K,L,k_1,l_1))) - W_2(W_1^{-1}(\tilde z(k_1+1,l_1)+g_2(K,L,k_1+1,l_1)))\\
&\le -\int_{\tilde z(k_1,l_1)+g_2(K,L,k_1,l_1)}^{\tilde z(k_1+1,l_1)+g_2(K,L,k_1+1,l_1)} \frac{d\tau}{\phi_2(W_1^{-1}(\tilde z(k_1+1,l_1)+g_2(K,L,k_1+1,l_1)))}\\
&\le \sum_{l_1+1\le t<\infty}[\sum_{k_1+1\le \xi<\infty}\sum_{t\le\eta<\infty} G(K,L,\xi,\eta)]\\
&-\frac{\Delta_3 g_2(K,L,k_1,l_1)}{\phi_2(W_1^{-1}(\tilde z(k_1+1,l_1)+g_2(K,L,k_1+1,l_1)))}. \qquad (2.\ 20)
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\tilde z(k_1,l_1)+g_2(K,L,k_1,l_1)}^{\tilde z(\infty,l_1)+g_2(K,L,\infty,l_1)} \frac{d\tau}{\phi_2(W_1^{-1}(\tau))}\\
&= -\sum_{k_1\le s<\infty}\int_{\tilde z(s,l_1)+g_2(K,L,s,l_1)}^{\tilde z(s+1,l_1)+g_2(K,L,s+1,l_1)} \frac{d\tau}{\phi_2(W_1^{-1}(\tau))}\\
&\le \sum_{k_1+1\le s<\infty}\sum_{l_1+1\le t<\infty}[\sum_{s\le \xi<\infty}\sum_{t\le\eta<\infty} G(K,L,\xi,\eta)]\\
&-\sum_{k_1\le s<\infty}\frac{\Delta_3 g_2(K,L,s,l_1)}{\phi_2(W_1^{-1}(\tilde z(s+1,l_1)+g_2(K,L,s+1,l_1)))}.
\end{aligned}
$$

Equivalently, by using g_2 as given in (2.18)

$$
\begin{aligned}
&W_2(W_1^{-1}(\tilde z(k_1,l_1)+g_2(K,L,k_1,l_1))) - W_2(g_1(K,L,\infty,l_1))\\
&\le \sum_{k_1+1\le s<\infty}\sum_{l_1+1\le t<\infty}[\sum_{s\le \xi<\infty}\sum_{t\le\eta<\infty} G(K,L,\xi,\eta)]\\
&-\sum_{k_1\le s<\infty}\frac{\Delta_3 g_2(K,L,s,l_1)}{\phi_2(W_1^{-1}(\tilde z(s+1,l_1)+g_2(K,L,s+1,l_1)))} \qquad (2.\ 21)
\end{aligned}
$$

A combination of (2.12), (2.19) and E in (2.18) yields:

$$
\begin{aligned}
u(k_1,l_1) &\le z(k_1,l_1)+g_1(K,L,k_1,l_1) = W_1^{-1}(E(k_1,l_1))\\
&\le W_1^{-1}(\tilde z(k_1,l_1)+g_2(K,L,k_1,l_1)) \qquad (2.\ 22)
\end{aligned}
$$

A combination of (2.21) and (2.22) yields:

$$
\begin{aligned}
u(k_1,l_1) \le\ & W_2^{-1}[W_2(g_1(K,L,\infty,l_1))\\
&+\sum_{k_1+1\le s<\infty}\sum_{l_1+1\le t<\infty}[\sum_{s\le \xi<\infty}\sum_{t\le\eta<\infty} G(K,L,\xi,\eta)]\\
&-\sum_{k_1\le s<\infty}\frac{\Delta_3 g_2(K,L,s,l_1)}{\phi_2(W_1^{-1}(\tilde z(s+1,l_1)+g_2(K,L,s+1,l_1)))}], \qquad (2.\ 23)
\end{aligned}
$$

for $k_1 \geq K$ and $l_1 \geq L$. By setting $k_1 \mapsto K$ and $l_1 \mapsto L$ in (2.23), we have

$$\begin{aligned} u(K,L) \ \leq \ & W_2^{-1}[W_2(g_1(K,L,\infty,L)) \\ & + \sum_{K+1\leq s<\infty} \sum_{L+1\leq t<\infty} [\sum_{s\leq\xi<\infty} \sum_{t\leq\eta<\infty} G(K,L,\xi,\eta)] \\ & - \sum_{K\leq s<\infty} \frac{\Delta_3 g_2(K,L,s,L)}{\phi_2(W_1^{-1}(\tilde{z}(s+1,L)+g_2(K,L,s+1,L)))}]. \end{aligned}$$

Since K and L are arbitrary therefore

$$\begin{aligned} u(k_1,l_1) \ \leq \ & W_2^{-1}[W_2(g_1(k_1,l_1,\infty,l_1)) \\ & + \sum_{k_1+1\leq s<\infty} \sum_{l_1+1\leq t<\infty} [\sum_{s\leq\xi<\infty} \sum_{t\leq\eta<\infty} G(k_1,l_1,\xi,\eta)] \\ & - \sum_{k_1\leq s<\infty} \frac{\Delta_3 g_2(k_1,l_1,s,l_1)}{\phi_2(W_1^{-1}(\tilde{z}(s+1,l_1)+g_2(K,L,s+1,l_1)))}]. \quad (2.\ 24) \end{aligned}$$

A combination of (2.8), (2.9), (2.11) and (2.24) yields the required relation . □

Theorem 2.4. *Let the conditions **A1-A3** be satisfied; if $\Delta_1 p_1(k_1,l_1)\tilde{a}(k_1,l_1)$ is non-positive for k_1, l_1 and non-decreasing in l_1 and $u(k_1,l_1)$ is non-negative function for $k_1, l_1 \in N_0$, satisfying (1.2), then*

$$\begin{aligned} u(k_1,l_1) \leq & W_2^{-1}[W_2(p_1(0,l_1)\tilde{a}(0,l_1)) \\ & + p_1(k_1,l_1) \sum_{0\leq s\leq k_1-1} \sum_{l_1+1\leq t<\infty} [\sum_{0\leq\xi<s} \sum_{t\leq\eta<\infty} \tilde{g}(k_1,l_1,\xi,\eta)] \\ & + \sum_{0\leq s\leq k_1-1} \frac{\Delta_3 g_4(k_1,l_1,s,l_1)}{\phi_2(W_1^{-1}(\tilde{z}_1(s,l_1)+g_4(k_1,l_1,s,l_1)))}] \end{aligned}$$

for $k_1 \geq K$ and $l_1 \geq L$, provided that

$$\begin{aligned} & W_2(p_1(0,L)\tilde{a}(0,L)) + p_1(K,L) \sum_{0\leq s\leq K-1} \sum_{L+1\leq t<\infty} [\sum_{0\leq\xi\leq s} \sum_{t\leq\eta<\infty} \tilde{g}(K,L,\xi,\eta)] \\ & + \sum_{0\leq s\leq K-1} \frac{\Delta_3 g_4(K,L,s,L)}{\phi_2(W_1^{-1}(\tilde{z}_1(s,L)+g_4(K,L,s,L)))} \leq \int_{u_2}^{\infty} \frac{dz}{w_{2(z)}}, \end{aligned}$$

for

$$\begin{cases} p_1(k_1,l_1) := 1+\tilde{b}(k_1,l_1)\sum_{k_1+1\le s<\infty} c(s,l_1)\prod_{k_1+1\le i\le\infty}(1+c(i,l_1)\tilde{b}(i,l_1)); \\ \hat{F}(k_1,l_1,s,t) := p_1(k_1,l_1)\tilde{f}(k_1,l_1,s,t); \\ \hat{G}(k_1,l_1,s,t) := p_1(k_1,l_1)\tilde{g}(k_1,l_1,s,t); \\ E_1(k_1,l_1) := W_1(z_1(k_1,l_1)+g_1(K,L,k_1,l_1)); \\ z_1(k_1,l_1) := \sum_{0\le s\le k_1-1}\sum_{l_1+1\le t<\infty}[F(K,L,s,t)w_1(u(s,t)) \\ +\sum_{0\le\xi<s}\sum_{t\le\eta<\infty} G(K,L,\xi,\eta)w_2(u(s,t))]; \\ \tilde{z_1}(k_1,l_1) := \sum_{0\le s\le k_1-1}\sum_{l_1+1\le t<\infty}[\sum_{0\le\xi<s}\sum_{t\le\eta<\infty} G(K,L,\xi,\eta) \\ \times\phi_2(W_1^{-1}(E_1(s,t)))]; \\ g_3(K,L,k_1,l_1) := p_1(s,t)\tilde{a}(s,t); \\ g_4(K,L,k_1,l_1) := W_1(g_3(K,L,0,l_1)) \\ +p_1(k_1,l_1)\sum_{0\le\xi\le k_1-1}\sum_{l_1+1\le\eta<\infty}\tilde{f}(k_1,l_1,\xi,\eta) \\ +\sum_{0\le s\le k_1-1}\frac{\Delta_3 g_3(K,L,s,l_1)}{(w_1(z_1(s,l_1)+g_3(K,L,s,l_1)))}. \end{cases}$$

Proof. This tracked by Lemma 2.2 and the proof of Theorem 2.3. We omit the details. □

Theorem 2.5. *Let the conditions* ***A1-A3*** *be satisfied; if $\Delta_1 p(k_1,l_1)\tilde{a}(k_1,l_1)$ is non-positive for k_1,l_1 and non-decreasing in l_1 and $u(k_1,l_1)$ is non-negative function for $k_1,l_1\in N_0$, satisfying (1.3), then*

(C1): *If $\psi_1(u)\ge\psi_2(\log(u))$, then*

$$\begin{aligned} u(k_1,l_1) \le\ & V_{2,1}^{-1}[V_{2,1}(\tilde{a}(\infty,l_1)) \\ & +p(k_1,l_1)\sum_{k_1+1\le s<\infty}\sum_{l_1+1\le t<\infty}[\sum_{s\le\xi<\infty}\sum_{t\le\eta<\infty}\tilde{g}(k_1,l_1,\xi,\eta)] \\ & -\sum_{k_1\le s<\infty}\frac{\Delta_3 g_5(k_1,l_1,s,l_1)}{(\phi_2 o\psi_1)(V_{1,1}^{-1}(\tilde{z}_2(s+1,l_1)+g_5(k_1,l_1,s+1,l_1)))}] \end{aligned} \quad (2.\ 25)$$

(C2): *If $\psi_1(u)\le\psi_2(\log(u))$, then*

$$\begin{aligned} u(k_1,l_1) \le\ & V_{2,2}^{-1}[V_{2,2}(\tilde{a}(\infty,l_1)) \\ & +p(k_1,l_1)\sum_{k_1+1\le s<\infty}\sum_{l_1+1\le t<\infty}[\sum_{s\le\xi<\infty}\sum_{t\le\eta<\infty}\tilde{g}(k_1,l_1,\xi,\eta)] \\ & -\sum_{k_1\le s<\infty}\frac{\Delta_3 g_6(k_1,l_1,s,l_1)}{(\phi_2 o\psi_2)(V_{1,2}^{-1}(\tilde{z}_3(s+1,l_1)+g_6(k_1,l_1,s+1,l_1)))}] \end{aligned} \quad (2.\ 26)$$

for $k_1\ge K$ and $l_1\ge L$, provided that

$$\begin{aligned} V_{2,1}(\tilde{a}(\infty,L))+p(K,L) & \sum_{K+1\le s<\infty}\sum_{L+1\le t<\infty}[\sum_{s\le\xi<\infty}\sum_{t\le\eta<\infty}\tilde{g}(K,L,\xi,\eta)] \\ & -\sum_{K\le s<\infty}\frac{\Delta_3 g_5(K,L,s,L)}{(\phi_2 o\psi_1)(V_{1,1}^{-1}(\tilde{z}_2(s+1,L)+g_5(K,L,s+1,L)))} \\ & \le\int_{u_2}^{\infty}\frac{dz}{(w_2 o\psi_1)(z)}. \end{aligned}$$

And

$$V_{2,2}(\tilde{a}(\infty,L)) + p(K,L)\sum_{K+1\le s<\infty}\sum_{L+1\le t<\infty}[\sum_{s\le\xi<\infty}\sum_{t\le\eta<\infty}\tilde{g}(K,L,\xi,\eta)]$$
$$-\sum_{K\le s<\infty}\frac{\Delta_3 g_6(K,L,s,L)}{(\phi_2 o\psi_2)(V_{1,2}^{-1}(\tilde{z}_3(s+1,L)g_6(K,L,s+1,L)))}$$
$$\le \int_{u_2}^{\infty}\frac{dz}{(w_2 o\psi_2)(z)},$$

provided that

$$V_{k,j}(u) := \int_{u_k}^{u}\frac{dz}{(w_k o\psi_j)(z)};\;\; u\ge u_k>0,\;\; 1\le k,j\le 2.$$

Proof. Under the conditions $\mathbf{A4}$, $\mathbf{A5}$, and $\mathbf{C1}$, inequality (1.3) is rewritten as:

$$u(k_1,l_1)\le Y_1(k_1,l_1)+\tilde{b}(k_1,l_1)\sum_{k_1+1\le s\le\beta}c(s,l_1)u(s,l_1), \tag{2. 27}$$

for

$$\begin{aligned} Y_1(k_1,l_1) \;\; := \;\; & \tilde{a}(k_1,l_1)+\sum_{k_1+1\le s<\infty}\sum_{l_1+1\le t<\infty}[\tilde{f}(k_1,l_1,s,t)w_1(\psi_1(u(s,t))) \\ & +\sum_{s\le\xi<\infty}\sum_{t\le\eta<\infty}\tilde{g}(k_1,l_1,\xi,\eta)w_2(\psi_2(\log(u(s,t))))]. \end{aligned}$$

Obviously, $Y_1(k_1,l_1)$ is non-increasing in k_1.

Using the similar process as in Theorem 2.3 and (2.6)-(2.13), yields

$$\begin{aligned} -\Delta_1 z_2(k_1,l_1) = & \sum_{l_1+1\le t<\infty}[F(K,L,k_1+1,t)(w_1 o\psi_1)(u(k_1+1,t)) \\ & +\sum_{k_1+1\le\xi<\infty}\sum_{t\le\eta<\infty}G(K,L,\xi,\eta)(w_2 o\psi_1)(u(k_1+1,t))] \\ \le & \sum_{l_1+1\le t<\infty}[F(K,L,k_1+1,t)(w_1 o\psi_1)(z_2(k_1+1,t) \\ & +g_1(K,L,k_1+1,t)) \\ & +\sum_{k_1+1\le\xi<\infty}\sum_{t\le\eta<\infty}G(K,L,\xi,\eta)(w_2 o\psi_1)(z_2(k_1+1,t) \\ & +g_1(K,L,k_1+1,t))], \end{aligned} \tag{2. 28}$$

provided that

$$\begin{aligned} z_2(k_1,l_1) := & \sum_{k_1+1\le s<\infty}\sum_{l_1+1\le t<\infty}[F(K,L,s,t)(w_1 o\psi_1)(u(s,t)) \\ & +\sum_{s\le\xi<\infty}\sum_{t\le\eta<\infty}G(K,L,\xi,\eta)(w_2 o\psi_2)(\log(u(s,t)))] \end{aligned}$$

Equivalently,

$$\begin{aligned}
&-\frac{\Delta_1 z_2(k_1,l_1)+\Delta_3 g_1(K,L,k_1,l_1)}{(w_1 o\psi_1)(z_2(k_1+1,l_1)+g_1(K,L,k_1+1,l_1))}\\
&\quad\leq \sum_{l_1+1\leq t<\infty}[F(K,L,k_1+1,t)+\sum_{k_1+1\leq \xi<\infty}\sum_{t\leq\eta<\infty}G(K,L,\xi,\eta)\\
&\qquad\times(\phi_2 o\psi_1)(z_2(k_1+1,t)+g_1(K,L,k_1+1,t))]\\
&\qquad-\frac{\Delta_3 g_1(K,L,k_1,l_1)}{(w_1 o\psi_1)(z_2(k_1+1,l_1)+g_1(K,L,k_1+1,l_1))}
\end{aligned} \quad (2.\ 29)$$

Again, using the similar process as in Theorem 2.3 from (2.14)-(2.17), yields

$$\begin{aligned}
&V_{1,1}(z_2(k_1,l_1)+g_1(K,L,k_1,l_1))\leq V_{1,1}(g_1(K,L,\infty,l_1))\\
&\quad+\sum_{k_1+1\leq s<\infty}\sum_{l_1+1\leq t<\infty}[F(K,L,s,t)+\sum_{s\leq\xi<\infty}\sum_{t\leq\eta<\infty}G(K,L,\xi,\eta)\\
&\qquad\times(\phi_2 o\psi_1)(z_2(s,t)+g_1(K,L,s,t))]\\
&\qquad-\sum_{k_1\leq s<\infty}\frac{\Delta_3 g_1(K,L,s,l_1)}{(w_1 o\psi_1)(z_2(s+1,l_1)+g_1(K,L,s+1,l_1))}
\end{aligned} \quad (2.\ 30)$$

Consider

$$\begin{cases}
E_2(k_1,l_1):=V_{1,1}(z_2(k_1,l_1)+g_1(K,L,k_1,l_1));\\
g_5(K,L,k_1,l_1):=V_{1,1}(g_1(K,L,\infty,l_1))+\sum_{k_1+1\leq s<\infty}\sum_{l_1+1\leq t<\infty}F(K,L,s,t)\\
-\sum_{k_1\leq s<\infty}\frac{\Delta_3 g_1(K,L,s,l_1)}{(w_1 o\psi_1)(z_2(s+1,l_1)+g_1(K,L,s+1,l_1))};\\
\tilde{z}_2(k_1,l_1):=\sum_{k_1+1\leq s<\infty}\sum_{l_1+1\leq t<\infty}[\sum_{s\leq\xi<\infty}\sum_{t\leq\eta<\infty}G(K,L,\xi,\eta)\\
\times(\phi_2 o\psi_1)(V_{1,1}^{-1}(E_2(s,t)))]
\end{cases} \quad (2.\ 31)$$

A combination of (2.30) and (2.31) yields:

$$E_2(k_1,l_1)\leq \tilde{z}_2(k_1,l_1)+g_5(K,L,k_1,l_1). \quad (2.\ 32)$$

And

$$u(k_1,l_1)\leq V_{1,1}^{-1}(\tilde{z}_2(k_1,l_1)+g_5(K,L,k_1,l_1)) \quad (2.\ 33)$$

Similar to the process from (2.19) - (2.21), we obtain

$$\begin{aligned}
&V_{2,1}(V_{1,1}^{-1}(\tilde{z}_2(k_1,l_1)+g_5(K,L,k_1,l_1)))-V_{2,1}(g_1(K,L,\infty,l_1))\\
&\leq\sum_{k_1+1\leq s<\infty}\sum_{l_1+1\leq t<\infty}[\sum_{s\leq\xi<\infty}\sum_{t\leq\eta<\infty}G(K,L,\xi,\eta)]\\
&\quad-\sum_{k_1\leq s<\infty}\frac{\Delta_3 g_5(K,L,s,l_1)}{(\phi_2 o\psi_1)(V_{1,1}^{-1}(\tilde{z}_2(s+1,l_1)+g_5(K,L,s+1,l_1)))}
\end{aligned} \quad (2.\ 34)$$

A combination of (2.33) and (2.34) yields:

$$\begin{aligned} u(k_1,l_1) \;\le\; & V_{2,1}^{-1}[V_{2,1}(g_1(K,L,\infty,l_1)) \\ & + \sum_{k_1+1\le s<\infty}\ \sum_{l_1+1\le t<\infty}[\sum_{s\le\xi<\infty}\ \sum_{t\le\eta<\infty} G(K,L,\xi,\eta)] \\ & - \sum_{k_1\le s<\infty} \frac{\Delta_3 g_5(K,L,s,l_1)}{(\phi_2 o\psi_1)(V_{1,1}^{-1}(\tilde{z}_2(s+1,l_1)+g_5(K,L,s+1,l_1)))}] \end{aligned} \quad (2.\ 35)$$

for $k_1 \ge K$ and $l_1 \ge L$. By setting $k_1 \mapsto K$ and $l_1 \mapsto L$ in (2.35), we have

$$\begin{aligned} u(K,L) \;\le\; & V_{2,1}^{-1}[V_{2,1}(g_1(K,L,\infty,L)) \\ & + \sum_{K+1\le s<\infty}\ \sum_{L+1\le t<\infty}[\sum_{s\le\xi<\infty}\ \sum_{t\le\eta<\infty} G(K,L,\xi,\eta)] \\ & - \sum_{K\le s<\infty} \frac{\Delta_3 g_5(K,L,s,L)}{(\phi_2 o\psi_1)(V_{1,1}^{-1}(\tilde{z}_2(s+1,L)+g_5(K,L,s+1,L)))}]. \end{aligned} \quad (2.\ 36)$$

Since K and L are arbitrary, therefor

$$\begin{aligned} u(k_1,l_1) \le & V_{2,1}^{-1}[V_{2,1}(g_1(k_1,l_1,\infty,l_1)) \\ & + \sum_{k_1+1\le s<\infty}\ \sum_{l_1+1\le t<\infty}[\sum_{s\le\xi<\infty}\ \sum_{t\le\eta<\infty} G(k_1,l_1,\xi,\eta)] \\ & - \sum_{k_1\le s<\infty} \frac{\Delta_3 g_5(k_1,l_1,s,l_1)}{(\phi_2 o\psi_1)(V_{1,1}^{-1}(\tilde{z}_2(s+1,l_1)+g_5(K,L,s+1,l_1)))}] \end{aligned} \quad (2.\ 37)$$

A combination of (2.8), (2.9), (2.11) and (2.37) yields the required relation.

For $\psi_1(u) \le \psi_2(\log(u))$, we use the similar process from (2.27)-(2.28) and inequality (1.3), to obtain

$$\begin{aligned} & -\Delta_1 z_2(k_1,l_1) \\ & \le \sum_{l_1+1\le t<\infty}[F(K,L,k_1+1,t)(w_1 o\psi_2)(z_2(k_1+1,t)+g_1(K,L,k_1+1,t)) \\ & + \sum_{k_1+1\le\xi<\infty}\ \sum_{t\le\eta<\infty} G(K,L,\xi,\eta)(w_2 o\psi_2)(z_2(k_1+1,t)+g_1(K,L,k_1+1,t))]. \end{aligned}$$

Equivalently,

$$\begin{aligned} & -\frac{\Delta_1 z_2(k_1,l_1)+\Delta_3 g_1(K,L,k_1,l_1)}{(w_1 o\psi_2)(z_2(k_1+1,l_1)+g_1(K,L,k_1+1,l_1))} \\ & \le \sum_{l_1+1\le t<\infty}[F(K,L,k_1+1,t)+\sum_{k_1+1\le\xi<\infty}\ \sum_{t\le\eta<\infty} G(K,L,\xi,\eta)(\phi_2 o\psi_2)(z_2(k_1+1,t) \\ & \quad + g_1(K,L,k_1+1,t))] - \frac{\Delta_3 g_1(K,L,k_1,l_1)}{(w_1 o\psi_2)(z_2(k_1+1,l_1)+g_1(K,L,k_1+1,l_1))} \end{aligned}$$

And

$$u(k_1,l_1) \le V_{2,2}^{-1}[V_{2,2}(g_1(K,L,\infty,l_1)) + \sum_{k_1+1\le s<\infty}\sum_{l_1+1\le t<\infty}[\sum_{s\le\xi\le\infty}\sum_{t\le\eta<\infty} G(K,L,\xi,\eta)] - \sum_{k_1\le s<\infty} \frac{\Delta_3 g_6(K,L,s,l_1)}{(\phi_2 o\psi_2)(V_{1,2}^{-1}(\tilde{z}_3(s+1,l_1)+g_6(K,L,s+1,l_1)))}] \quad (2.\ 38)$$

provided that

$$\begin{cases} E_3(k_1,l_1) := V_{1,2}(z_2(k_1,l_1)+g_1(K,L,k_1,l_1)); \\ g_6(K,L,k_1,l_1) := V_{1,2}(g_1(K,L,\infty,l_1)) + \sum_{k_1+1\le s<\infty}\sum_{l_1+1\le t<\infty} F(K,L,s,t) \\ -\sum_{k_1\le s<\infty}\frac{\Delta_3 g_1(K,L,s,l_1)}{(w_1 o\psi_2)(z_2(s+1,l_1)+g_1(K,L,s+1,l_1))}; \\ \tilde{z}_3(k_1,l_1) := \sum_{k_1+1\le s<\infty}\sum_{l_1+1\le t<\infty}[\sum_{s\le\xi<\infty}\sum_{t\le\eta<\infty} G(K,L,\xi,\eta) \\ \times(\phi_2 o\psi_2)(V_{1,2}^{-1}(E_3(s,t)))] \end{cases}$$

A combination of (2.8), (2.9), (2.11) and (2.38) yields the required relation . □

3. APPLICATIONS

Consider a non-linear partial difference equation

$$v(k_1,l_1) = \alpha(k_1,l_1) + \gamma(k_1,l_1)\sum_{k_1+1\le s\le\beta} f_1(s,l_1,v(s,l_1)) + \sum_{k_1+1\le s<\infty}\sum_{l_1+1\le t<\infty}[f_2(k_1,l_1,s,t,v(s,t)) + \sum_{s\le\xi<\infty}\sum_{t\le\eta<\infty} f_3(k_1,l_1,\xi,\eta,v(s,t))], \quad (3.\ 39)$$

for $k_1,l_1\in N_0$, where $v(k_1,l_1)$ is an unknown function for $k_1,l_1\in N_0$.

The following two examples prove the boundedness and uniqueness of the solution of (3.39), respectively.

Example 3.1. *Suppose,* $|f_1(s,l_1,v(s,l_1))| \le e(s,l_1)|v(s,l_1)|$; $|f_2(k_1,l_1,s,t,v(s,t))| \le f(k_1,l_1,s,t)w_1(|v(s,t)|)$; $|f_3(k_1,l_1,\xi,\eta,v(s,t))| \le g(k_1,l_1,\xi,\eta)w_2(|v(s,t)|)$, *provided that* $w_1 \propto w_2$. *If* $v(k_1,l_1)$ *is a solution of* (3.39), *then*

$$u(k_1,l_1) \le a(k_1,l_1) + b(k_1,l_1)\sum_{k_1+1\le s\le\beta} e(s,l_1)u(s,l_1) + \sum_{k_1+1\le s<\infty}\sum_{l_1+1\le t<\infty}[f(k_1,l_1,s,t)w_1(u(s,t)) + \sum_{s\le\xi<\infty}\sum_{t\le\eta<\infty} g(k_1,l_1,\xi,\eta)w_2(u(s,t))], \quad (3.\ 40)$$

for $u(k_1,l_1) = |v(k_1,l_1)|;\ a(k_1,l_1) \ge |\alpha(k_1,l_1)|;\ b(k_1,l_1) \ge |\gamma(k_1,l_1)|$.

Now application of Theorem 2.3 yields:

$$u(k_1,l_1) \leq W_2^{-1}[W_2(\tilde{a}(\infty,l_1)) \\ +p_2(k_1,l_1)\sum_{k_1+1\leq s<\infty}\sum_{l_1+1\leq t<\infty}[\sum_{s\leq\xi<\infty}\sum_{t\leq\eta<\infty}\tilde{g}(k_1,l_1,\xi,\eta)] \\ -\sum_{k_1\leq s<\infty}\frac{\Delta_3 g_2(k_1,l_1,s,l_1)}{\phi_2(W_1^{-1}(\tilde{z}_1(s+1,l_1)+g_2(k_1,l_1,s+1,l_1)))}], \quad (3.\ 41)$$

provided that

$$p_2(k_1,l_1) := 1+\tilde{b}(k_1,l_1)\sum_{k_1+1\leq s\leq\beta}e(s,l_1)\prod_{k_1+1\leq i\leq s-1}(1+e(i,l_1)\tilde{b}(i,l_1)).$$

Clearly (3.41) *implies the boundedness of solutions of equation* (3.39).

Example 3.2. *Suppose,* $|f_1(s,k_1,l_1,v(s,l_1)) - f_1(s,l_1,\bar{v}(s,l_1))| \leq e(s,l_1)|v(s,l_1) - \bar{v}(s,l_1)|$; $|f_2(k_1,l_1,s,t,v(s,t)) - f_2(k_1,l_1,s,t,\bar{v}(s,t))| \leq f(k_1,l_1,s,t)w_1(|v(s,t)-\bar{v}(s,t)|)$; $|f_3(k_1,l_1,\xi,\eta,v(s,t)) - f_3(k_1,l_1,\xi,\eta,\bar{v}(s,t))| \leq g(k_1,l_1,\xi,\eta)w_2(|v(s,t)-\bar{v}(s,t)|)$, *provided that* $w_1 \propto w_2$. *Then, equation* (3.39) *has at most one solution.*

Solution : Suppose $v(k_1,l_1)$ and $\bar{v}(k_1,l_1)$ are two solutions of (3.39). Then

$$|v(k_1,l_1)-\bar{v}(k_1,l_1)| \\ \leq \gamma(k_1,l_1)\sum_{k_1+1\leq s\leq\beta}|f_1(s,l_1,v(s,l_1)) - f_1(s,l_1,\bar{v}(s,l_1))| \\ +\sum_{k_1+1\leq s<\infty}\sum_{l_1+1\leq t<\infty}[|f_2(k_1,l_1,s,t,v(s,t)) - f_2(k_1,l_1,s,t,\bar{v}(s,t))| \\ +\sum_{s\leq\xi<\infty}\sum_{t\leq\eta<\infty}|f_3(k_1,l_1,\xi,\eta,v(s,t)) - f_2(k_1,l_1,s,t,\bar{v}(s,t))|] \\ \leq \gamma(k_1,l_1)\sum_{k_1+1\leq s\leq\beta}e(k_1,l_1)|v(s,l_1)) - \bar{v}(s,l_1)| \\ +\sum_{k_1+1\leq s<\infty}\sum_{l_1+1\leq t<\infty}[f(k_1,l_1,s,t)w_1(|v(s,t)-\bar{v}(s,t)|) \\ +\sum_{s\leq\xi<\infty}\sum_{t\leq\eta<\infty}g(k_1,l_1,\xi,\eta)w_2(|v(s,t)-\bar{v}(s,t)|)]$$

Treating $|v(k_1,l_1)-\bar{v}(k_1,l_1)|$ as one variable, and possible applicability of Theorem 2.3 yields $|v(k_1,l_1)-\bar{v}(k_1,l_1)| \leq 0$, which implies that $|v(k_1,l_1)-\bar{v}(k_1,l_1)| = 0$, that is, $v(k_1,l_1) \equiv \bar{v}(k_1,l_1)$. This proves the uniqueness of the solution of (3.39).

4. Authors contributions

All authors contributed equally. All authors read and approved the final manuscript

5. Acknowledgments

We would like to express our thanks to the anonymous referees and the editor for their constructive comments and suggestions, which greatly improved this article.

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