Some Fejer and Hermite-Hadamard Type Inequalities Considering $\epsilon$-Convex and $(\sigma, \epsilon)$-Convex Functions

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Abstract. In current paper, new Hermite-Hadamard and Fejér type inequalities are proved by using the $\epsilon$-convexity and $(\sigma, \epsilon)$-convexity of differentiable functions and a positive function symmetric with respect to $\frac{\epsilon_j+\epsilon_k}{2}$. The results of the paper have been proved to contain previously established results related to differentiable convex functions.

1. Introduction

A function $\eta : U \subseteq \mathbb{R} \rightarrow \mathbb{R}$ forenamed as convex function, let

$$\eta(t \theta + (1-t) y) \leq t \eta(\theta) + (1-t) \eta(y)$$

holds for every $\theta, y \in I$ and $t \in [0, 1]$.

The subsequent double integral inequality

$$\eta \left( \frac{j + k}{2} \right) \leq \frac{1}{k - j} \int_j^k \eta(\theta) d\theta \leq \frac{\eta(j) + \eta(k)}{2}. \quad (1.1)$$

holds for convex functions and is notable in literature as the Hermite-Hadamard inequality. The inequalities in (1.1) holds in reversed order as $\eta$ is concave function.

The inequality (1.1) has been a likely of extensive study insomuch as discovery. A number of papers have been written which provide noteworthy extensions, generalizations and refinements for the inequalities (1.1), see for example [1]-[19].
Dragonm and Agarwal [2], proved subsequent inequalities for differentiable functions which estimate the difference between the middle and rightmost terms in (1.1).

Theorem 1.1. [2] Suppose \( \eta : U \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping at \( U^c \), and \( j, k \in U \) with \( j < k \), also \( \eta' \in L([j,k]) \). If \( |\eta|^\frac{1}{q} \) is convex function on \([j,k]\), so subsequent inequality holds:

\[
\left| \frac{\eta(j) + \eta(k)}{2} - \frac{1}{k-j} \int_j^k \eta(\theta)d\theta \right| \leq \frac{k-j}{8} \left[ \left| \frac{\eta'}{p} \right| + \left| \frac{\eta'}{q} \right| \right]. \quad (1.2)
\]

Theorem 1.2. [2] Let \( \eta : U \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is a differentiable mapping against \( I^a \), and \( j, k \in U \) with \( j < k \), including \( \eta' \in L([j,k]) \). Whenever \( \left| \eta' \right| \frac{1}{r} \) is a convex function supported \([j,k]\), the coming inequality holds:

\[
\left| \frac{\eta(j) + \eta(k)}{2} - \frac{1}{k-j} \int_j^k \eta(\theta)d\theta \right| \leq \frac{k-j}{2(p+1)} \left[ \left| \frac{\eta'}{p} \right| + \left| \frac{\eta'}{q} \right| \right]. \quad (1.3)
\]

Consider that \( p > 1 \) furthermore \( \frac{1}{p} + \frac{1}{q} = 1 \).

In [17], Pearce attained enhanced and resolution of constant in Theorem 1.2 wherever strengthen this consequence by proving the successive theorem.

Theorem 1.3. [17] Consider \( \eta : U \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is a differentiable mapping at \( I^a \), with \( j, k \in U \) and \( j < k \), together \( \eta' \in L([j,k]) \). If \( \left| \eta' \right| q \) is a convex function on \([j,k]\), also \( q \geq 1 \), then the subsequent inequality exists:

\[
\left| \frac{\eta(j) + \eta(k)}{2} - \frac{1}{k-j} \int_j^k \eta(\theta)d\theta \right| \leq \frac{k-j}{4} \left[ \left| \frac{\eta'}{q} \right| + \left| \frac{\eta'}{q} \right| \right]. \quad (1.4)
\]

If \( \left| \eta' \right| q \) is concave on \([j,k]\), a bit \( q \geq 1 \). Formerly

\[
\left| \frac{\eta(j) + \eta(k)}{2} - \frac{1}{k-j} \int_j^k \eta(\theta)d\theta \right| \leq \frac{k-j}{4} \left| \eta' \left( \frac{j+k}{2} \right) \right|. \quad (1.5)
\]

In [6], Dah-Yan Hwang established the following results for convex which affords weighted consolation of results inclined in Theorem 1.1, Theorem 1.2 and the inequality (1.4) of Theorem1.3.

Theorem 1.4. [6] Authorize \( \eta : U \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is a differentiable mapping on \( I^a \), with \( j, k \in U^c \) along \( j < k \) and allow \( \rho : [j,k] \rightarrow [0,\infty) \) be continuous positive mapping also symmetric to \( \frac{j+k}{2} \). Assume \( \left| \eta' \right| \) is convex function at \([j,k]\), succeeding inequality holds:

\[
\left| \frac{\eta(j) + \eta(k)}{2} \right| \int_j^k \rho(\theta)d\theta - \int_j^k \eta(x)\rho(\theta)d\theta \leq \frac{k-j}{4} \left[ \left| \frac{\eta'}{\rho} \right| + \left| \frac{\eta'}{\rho} \right| \right] \int_0^1 \int_{L(j,k,t)} \rho(\theta)d\theta dt, \quad (1.6)
\]
where $U(j, k, t) = \frac{1+t}{2}j + \frac{1-t}{2}k$ and $L(j, k, t) = \frac{1+t}{2}j + \frac{1-t}{2}k$.

**Theorem 1.5.** [6] Confirming considerations of Theorem 1.4 are fulfilled along $q \geq 1$.

Assuming $\eta^q$ is convex function on $[j, k]$, pursuing inequality grips:

$$
\left| \eta(j) + \eta(k) \right| \frac{k-j}{2} \int_{j}^{k} \rho(\theta) d\theta - \int_{j}^{k} \eta(\theta) \rho(\theta) d\theta \leq k-j \left( \left| \eta(j) \right|^q + \left| \eta(k) \right|^q \right) \frac{q}{2} \int_{0}^{1} U(j, k, t) \rho(\theta) d\theta dt,
$$

(1.7)

site $U(j, k, t)$ with $L(j, k, t)$ are decided in Theorem 1.4.

The classical convexity that is stated above was generalized as $\epsilon$-convexity by G. Toader in [19] as follows:

**Definition 1.6.** Function $\eta : [0, k^*] \to \mathbb{R}$ named as $\epsilon$-convex if

$$
\eta(t(\theta + \epsilon(1-t))y) \leq t\eta(\theta) + \epsilon(1-t)\eta(y)
$$

for $\theta, y \in [0, k^*], \epsilon \in [0, 1]$ and $t \in (0, 1)$, where $k^* > 0$. A function $\eta : [0, k^*] \to \mathbb{R}$ named as $\epsilon$-concave if $-\eta$ is $\epsilon$-convex.

Obviously, for $\epsilon = 1$ the Interpretation 1.6 recaptures perception of standard convex functions which construed on $[0, k^*]$.

Assumption of $\epsilon$-convexity has been further generalized in [12] as declared in successive interpretation.

**Definition 1.7.** Function $\eta : [0, k^*] \to \mathbb{R}$ is known as $(\sigma, \epsilon)$-convex assuming

$$
\eta(t(\theta + \epsilon(1-t))y) \leq t^\sigma \eta(\theta) + \epsilon(1-t^\sigma)\eta(y)
$$

exists being $\theta, y \in [0, k^*], (\sigma, \epsilon) \in [0, 1]^2$ with $t \in (0, 1]$, as $k^* > 0$. Function $\eta : [0, k^*] \to \mathbb{R}$ renamed as $(\sigma, \epsilon)$-concave if $-\eta$ is $(\sigma, \epsilon)$-convex.

It can easily be seen that for $\sigma = 1$, the class of $\epsilon$-convex functions are derived from the above interpretation and for $\epsilon = \sigma = 1$ a class of convex functions are derived.

For several declarations concerning Hermite-Hadamard type inequalities for $\epsilon$-convex and $(\sigma, \epsilon)$-convex functions we specify the attentive reader to [1, 3, 4, 8, 13, 14, 15, 16, 10, 11, 18] and the references cited therein.

In Section 2, we prove some new Fejér and Harmine-Hadamard type inequalities by using the $\epsilon$- and $(\sigma, \epsilon)$-convexity of the differentiable mappings. The results of this paper contains some previously proved results for convex functions defined over the interval $[0, k^*]$ as special cases.

2. **FEJÈR TYPE INEQUALITIES FOR $\epsilon$-CONVEX AND $(\sigma, \epsilon)$-CONVEX FUNCTIONS**

**Lemma 2.1.** Consider $\eta : U \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping at $U^\circ$ with $\rho : [\epsilon j, k] \to [0, \infty)$ be continuous and symmetric considering $\frac{jk}{2}$ for settled $\epsilon \in (0, 1]$.
where $\epsilon_j, k \in U^\circ$ with $\epsilon_j < k$. If $\eta' \in L_1 [\epsilon_j, k]$, resulting expression exists

$$\frac{[\eta(\epsilon_j) + \eta(k)]}{2} \int_{\epsilon_j}^{k} \rho(\theta) d\theta - \int_{\epsilon_j}^{k} \rho(\theta) \eta(\theta) d\theta = k - \epsilon_j \frac{1}{4} \left[ \int_{0}^{\frac{1}{2}} \left[ \int_{U(t, \epsilon)}^{U(t, \epsilon)} \rho(\theta) d\theta \right] \left[ \eta' (U(t, \epsilon)) - \eta' (L(t, \epsilon)) \right] dt \right] \quad (2.8)$$

along

$$U(t, \epsilon) = \epsilon \left( \frac{1-t}{2} \right) j + \left( \frac{1+t}{2} \right) k$$

furthermore

$$L(t, \epsilon) = \epsilon \left( \frac{1+t}{2} \right) j + \left( \frac{1-t}{2} \right) k.$$

**Proof.** By the integration by parts, we get

$$W_1 = \int_{0}^{1} \left[ \int_{U(t, \epsilon)}^{U(t, \epsilon)} \rho(\theta) d\theta \right] \eta' (U(t, \epsilon)) dt$$

$$= \frac{2}{k - \epsilon_j} \left[ \int_{0}^{1} \left[ \int_{U(t, \epsilon)}^{U(t, \epsilon)} \rho(\theta) d\theta \right] \eta(U(t, \epsilon)) \right] - \int_{0}^{1} \left[ \frac{\rho(U(t, \epsilon)) + \rho(L(t, \epsilon))}{\eta(U(t, \epsilon))} \right] \eta(U(t, \epsilon)) dt$$

$$= \frac{2}{k - \epsilon_j} \eta(k) \int_{\epsilon_j}^{k} \rho(\theta) d\theta - 2 \int_{0}^{1} \rho(U(t, \epsilon)) \eta(U(t, \epsilon)) dt$$

$$= \frac{2}{k - \epsilon_j} \eta(k) \int_{\epsilon_j}^{k} \rho(\theta) d\theta - \frac{4}{k - \epsilon_j} \int_{\epsilon_j}^{k} \rho(\theta) \eta(\theta) d\theta.$$

Similarly, we can observe that

$$W_2 = -\frac{2}{k - \epsilon_j} \eta(\epsilon_j) \int_{\epsilon_j}^{k} \rho(\theta) d\theta + \frac{4}{k - \epsilon_j} \int_{\epsilon_j}^{\epsilon_j+\frac{1}{2}} \rho(\theta) \eta(\theta) d\theta.$$
Hence

\[ W_1 - W_2 = \frac{2}{k - \epsilon_j} [\eta(\epsilon_j) + \eta(k)] \int_{\epsilon_j}^{k} \rho(\theta) d\theta - \frac{4}{k - \epsilon_j} \int_{\epsilon_j}^{k} \rho(\theta) \eta(\theta) d\theta. \]

Multiplying the above result by \( \frac{k - \epsilon_j}{4} \), we get what is desired. \( \square \)

**Remark 2.2.** If we choose \( \epsilon = 1 \) in Lemma 2.1, we obtain the result proved in [3] [Lemma 2.1, page 9599].

**Remark 2.3.** If \( \rho(\theta) = \frac{1}{k - \epsilon_j}, \theta \in [\epsilon_j, k] \), then the subsequent equality holds

\[
\frac{\eta(\epsilon_j) + \eta(k)}{2} - \frac{1}{k - \epsilon_j} \int_{\epsilon_j}^{k} \eta(\theta) d\theta = \frac{k - \epsilon_j}{8} \int_{0}^{1} \left[ \eta' \left( \epsilon \left( \frac{1}{2} - t \right) \epsilon_j + \left( \frac{1 + t}{2} \right) k \right) - \eta' \left( \epsilon \left( \frac{1}{2} \right) \epsilon_j + \left( \frac{1 - t}{2} \right) k \right) \right] dt. 
\]

(2.9)

Now we present some Fejér type inequalities for \( \epsilon \)-convex functions.

**Theorem 2.4.** Let \( \eta : W \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( W^\circ \supseteq [0, \infty) \) and \( \rho : [\epsilon_j, k] \rightarrow [0, \infty) \) be continuous and symmetric considering \( \frac{\epsilon_j + k}{2} \) for settled \( \epsilon \in (0, 1] \), where \( \epsilon_j, k \in W^\circ \) with \( \epsilon_j < k \). Supposing \( \eta' \in L_1 [\epsilon_j, k] \) and \( \|\rho\|_1 \) is \( \epsilon \)-convex on \([0, k]\), ensuing inequality holds

\[
\left| \frac{\eta(\epsilon_j) + \eta(k)}{2} \right| \int_{\epsilon_j}^{k} \rho(\theta) d\theta - \int_{\epsilon_j}^{k} \rho(\theta) \eta(\theta) d\theta \leq \frac{k - \epsilon_j}{4} \left[ \epsilon \left| \eta'(\epsilon_j) \right| + \left| \eta'(k) \right| \right] \int_{0}^{1} \rho(\theta) d\theta dt. \quad (2.10)
\]
Proof. Taking absolute value on both sides of (2.8) and employing $\epsilon$-convexity on $[0, k]$, we have

$$
\left| \left[ \frac{\eta(\epsilon_j) + \eta(k)}{2} \right] \int_{\epsilon_j}^{k} \rho(\theta) \, d\theta - \int_{\epsilon_j}^{k} \rho(\theta) \eta(\theta) \, d\theta \right|
$$

\[ \leq \frac{k - \epsilon_j}{4} \int_{0}^{1} \left[ \int_{U(t, \epsilon)}^{L(t, \epsilon)} \rho(\theta) \, d\theta \right] \left[ \left| \eta'(U(t, \epsilon)) \right| + \left| \eta'(L(t, \epsilon)) \right| \right] \, dt
\]

\[ \leq \frac{k - \epsilon_j}{4} \int_{0}^{1} \left[ \int_{U(t, \epsilon)}^{L(t, \epsilon)} \rho(\theta) \, d\theta \right] \left[ \epsilon \left( \frac{1 - t}{2} \right) \left| \eta'(j) \right| + \frac{1 + t}{2} \left| \eta'(k) \right| \right] \, dt
\]

\[ + \epsilon \left( \frac{1 + t}{2} \right) \left| \eta'(j) \right| + \frac{1 - t}{2} \left| \eta'(k) \right| \, dt
\]

\[ = \frac{k - \epsilon_j}{4} \left[ \epsilon \left| \eta'(j) \right| + \left| \eta'(k) \right| \right] \int_{0}^{1} \int_{U(t, \epsilon)}^{L(t, \epsilon)} \rho(\theta) \, d\theta \, dt.
\]

Hence argument of theorem is concluded. \[ \square \]

Remark 2.5. The choice of $\epsilon = 1$, gives the result of Theorem 2.2 proved in [3] for convex functions defined on $[0, k]$.

Corollary 2.6. Under the assumptions of Theorem 2.4 and the choice of $\rho(\theta) = \frac{1}{\epsilon - \epsilon_j}$, subsequent inequality holds

$$
\left| \left[ \frac{\eta(\epsilon_j) + \eta(k)}{2} \right] \int_{\epsilon_j}^{k} \rho(\theta) \, d\theta - \int_{\epsilon_j}^{k} \rho(\theta) \eta(\theta) \, d\theta \right|
$$

\[ \leq \frac{k - \epsilon_j}{8} \left[ \epsilon \left| \eta'(j) \right| + \left| \eta'(k) \right| \right]. \tag{2.11}
\]

Remark 2.7. Assuming $\epsilon = 1$ in Corollary 2.6, we get the result proved in [2, Theorem 2.2] for convex functions rationale on $[0, k]$.

Theorem 2.8. Let $\eta : W \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $W^\circ \supseteq [0, \infty)$ and $\rho : [\epsilon_j, k] \to [0, \infty)$ be continuous and symmetric regarding $\frac{\epsilon_j + k}{2}$ for settled $\epsilon \in (0, 1]$, where $\epsilon_j, k \in W^\circ$ with $\epsilon_j < k$. If $\eta \in L_1[\epsilon_j, k]$ and $\left| \eta \right|$ is $\epsilon$-convex on $[0, k]$ for $q \geq 1$, specified inequality is

$$
\left| \left[ \frac{\eta(\epsilon_j) + \eta(k)}{2} \right] \int_{\epsilon_j}^{k} \rho(\theta) \, d\theta - \int_{\epsilon_j}^{k} \rho(\theta) \eta(\theta) \, d\theta \right|
$$

\[ \leq \frac{k - \epsilon_j}{2} \left[ \epsilon \left| \eta'(j) \right|^q + \frac{1}{2} \left| \eta'(k) \right|^q \right] \int_{0}^{1} \int_{U(t, \epsilon)}^{L(t, \epsilon)} \rho(\theta) \, d\theta \, dt. \tag{2.12}
\]
**Proof.** Applying Lemma 2.1 and usage of Hölder inequality, gives

\[
\left| \left[ \frac{\eta(\epsilon_j) + \eta(k)}{2} \right] \int_{\epsilon_j}^{k} \rho(\theta) d\theta - \int_{\epsilon_j}^{k} \rho(\theta) \eta(\theta) d\theta \right| \leq \frac{k - \epsilon_j}{4}
\]

\[
\times \left\{ \left( \int_{0}^{1} \left[ \int_{L(t,\epsilon)}^{U(t,\epsilon)} \rho(\theta) d\theta \right] \right)^{1 - \frac{q}{2}} \left( \int_{0}^{1} \left[ \int_{L(t,\epsilon)}^{U(t,\epsilon)} \rho(\theta) d\theta \right] \eta' \left( U(t,\epsilon) \right) \right)^{q} dt \right\}^{\frac{1}{q}}
\]

\[
+ \left( \int_{0}^{1} \left[ \int_{L(t,\epsilon)}^{U(t,\epsilon)} \rho(\theta) d\theta \right] \right)^{1 - \frac{q}{2}} \left( \int_{0}^{1} \left[ \int_{L(t,\epsilon)}^{U(t,\epsilon)} \rho(\theta) d\theta \right] \eta' \left( U(t,\epsilon) \right) \right)^{q} dt \right\}^{\frac{1}{q}}.
\]

(2.13)

Employing power-mean inequality \( \theta^r + y^r \leq 2^{1-r} (\theta + y)^r \) for \( j, k > 0 \) with \( r < 1 \),

\[
\left( \int_{0}^{1} \left[ \int_{L(t,\epsilon)}^{U(t,\epsilon)} \rho(\theta) d\theta \right] \eta' \left( U(t,\epsilon) \right) \right)^{q} dt \right)\right)^{\frac{1}{q}}
\]

\[
+ \left( \int_{0}^{1} \left[ \int_{L(t,\epsilon)}^{U(t,\epsilon)} \rho(\theta) d\theta \right] \eta' \left( U(t,\epsilon) \right) \right)^{q} dt \right)\right)^{\frac{1}{q}}
\]

\[
\leq 2^{1 - \frac{q}{2}} \left( \int_{0}^{1} \left[ \int_{L(t,\epsilon)}^{U(t,\epsilon)} \rho(\theta) d\theta \right] \eta' \left( U(t,\epsilon) \right) \right)^{q} dt \right)\right)^{\frac{1}{q}}
\]

(2.14)

Since \( \eta' \) is \( \epsilon \)-convex on \([0, b]\) for settled \( \epsilon \in (0, 1] \) and \( q \geq 1 \), we attained

\[
\int_{0}^{1} \left| \eta' \left( U(t,\epsilon) \right) \right|^{q} dt + \int_{0}^{1} \left| \eta' \left( U(t,\epsilon) \right) \right|^{q} dt
\]

\[
\leq \epsilon \left( \frac{1 - t}{2} \right) \left| \eta' \left( a \right) \right|^{q} + \left( \frac{1 + t}{2} \right) \left| \eta' \left( b \right) \right|^{q}
\]

\[
+ \epsilon \left( \frac{1 + t}{2} \right) \left| \eta' \left( a \right) \right|^{q} + \left( \frac{1 - t}{2} \right) \left| \eta' \left( b \right) \right|^{q} = \epsilon \left| \eta' \left( a \right) \right|^{q} + \left| \eta' \left( b \right) \right|^{q}
\]

(2.15)

Using (2.15) in (2.14) and then resulting inequality in (2.13), we grab which was desired. \( \square \)

**Remark 2.9.** Assuming \( \epsilon = 1 \), we accomplished result of Theorem 2.4 proved in [3].
Corollary 2.10. Under the assumptions of Theorem 2.8 and the choice of $g(\theta) = \frac{1}{k-\epsilon j}$, $x \in [\epsilon j, k]$, subsequent result exists

$$\left| \eta(\epsilon j) + \eta(k) \right| \leq \frac{k - \epsilon j}{4} \left[ \epsilon \left( |\eta'(j)|^q + |\eta'(k)|^q \right)^{\frac{1}{q}} \right].$$

(2.16)

Remark 2.11. Consider $\epsilon = 1$ in Corollary 2.10, we draw the result proved in [17, Theorem 1].

Now we present some Fejér type inequalities for $$(\sigma, \epsilon)$$-convex functions.

Theorem 2.12. Endorse $\eta : W \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $W^0 \supset [0, \infty)$ and $\rho : [\epsilon j, k] \to [0, \infty)$ be continuous and symmetric by $k^2$ for established $\epsilon \in (0, 1]$, where $\epsilon j, k \in W^0$ with $\epsilon j < k$. Wherever $\eta' \in L_1[\epsilon j, k]$ and $|\eta'|$ is $$(\sigma, \epsilon)$$-convex on $[0, k]$ for $$(\sigma, \epsilon) \in (0, 1] \times (0, 1]$$, resulting inequality is

$$\left| \frac{\eta(\epsilon j) + \eta(k)}{2} \int_{\epsilon j}^{k} \rho(\theta) d\theta - \int_{\epsilon j}^{k} \rho(\theta) \eta(\theta) d\theta \right| \leq \frac{(k - \epsilon j)^2}{4} \|\rho\|_{\infty} \left[ \epsilon \chi(\sigma) \left| \eta'(j) \right| + (1 - \chi(\sigma)) \left| \eta'(k) \right| \right].$$

(2.17)

spot

$$\chi(\sigma) = \frac{2(2^{-\sigma} + \sigma)}{(\sigma + 2)(\sigma + 1)}$$ and $\|\rho\|_{\infty} = \sup_{\theta \in [\epsilon j, k]} |\rho(\theta)|$.

Proof. We observed the consequences of Lemma 2.1 can be drafted as

$$\left| \frac{\eta(\epsilon j) + \eta(k)}{2} \int_{\epsilon j}^{k} \rho(\theta) d\theta - \int_{\epsilon j}^{k} \rho(\theta) \eta(\theta) d\theta \right| = \frac{k - \epsilon j}{4} \int_{0}^{1} \left[ \int_{\epsilon j}^{t} \rho(t, \epsilon) \rho(\theta) d\theta \right] \left[ \eta'(U(t, \epsilon)) - \eta'(L(t, \epsilon)) \right] dt \leq \frac{(k - \epsilon j)^2}{4} \|\rho\|_{\infty} \int_{0}^{1} t \left[ \eta'(U(t, \epsilon)) - \eta'(L(t, \epsilon)) \right] dt.$$

(2.18)
Applying the inequality (2.20) in (2.19), we scored the result given by (2.17). □

Taking the absolute value on both sides of (2.18), we gained

\[
\left| \frac{\eta(\epsilon j) + \eta(k)}{2} \right| \int_{\epsilon j}^{k} \rho(\theta) \, d\theta - \int_{\epsilon j}^{k} \rho(\theta) \eta(\theta) \, d\theta \leq \frac{(k - \epsilon j)^2}{4} \|\rho\|_{\infty} \int_{0}^{1} t \left[ \left| \eta'(U(t, \epsilon)) \right| + \left| \eta'(L(t, \epsilon)) \right| \right] \, dt. \tag{2.19}
\]

Adopting \((\sigma, \epsilon)\)-convexity of \(\eta'\) on \([0, k]\), we have

\[
\int_{0}^{1} t \left[ \left| \eta'(U(t, \epsilon)) \right| + \left| \eta'(L(t, \epsilon)) \right| \right] \, dt
\]

\[
\leq \int_{0}^{1} t \left\{ \left( \frac{1 + t}{2} \right)^{\sigma} \left| \eta'(b) \right| + \epsilon \left[ 1 - \left( \frac{1 + t}{2} \right)^{\sigma} \right] \left| \eta'(j) \right| + \left( \frac{1 - t}{2} \right)^{\sigma} \left| \eta'(k) \right| + \epsilon \left[ 1 - \left( \frac{1 - t}{2} \right)^{\sigma} \right] \left| \eta'(a) \right| \right\} \, dt
\]

\[
= \left| \eta'(k) \right| \int_{0}^{1} t \left[ \left( \frac{1 + t}{2} \right)^{\sigma} + \left( \frac{1 - t}{2} \right)^{\sigma} \right] \, dt
\]

\[
+ \epsilon \left| \eta'(j) \right| \int_{0}^{1} t \left[ 2 - \left( \frac{1 - t}{2} \right)^{\sigma} - \left( \frac{1 + t}{2} \right)^{\sigma} \right] \, dt
\]

\[
= \left\{ \frac{2(2^{-\sigma} + \sigma)}{(\sigma + 2)(\sigma + 1)} \right\} \left| \eta'(k) \right| + \epsilon \left\{ 1 - \frac{2(2^{-\sigma} + \sigma)}{(\sigma + 2)(\sigma + 1)} \right\} \left| \eta'(j) \right|. \tag{2.20}
\]

Applying the inequality (2.20) in (2.19), we scored the result given by (2.17). □

**Corollary 2.13.** Presume conditions of Theorem 2.12 are fulfilled and \(\rho(\theta) = \frac{1}{\epsilon^{\sigma} - 1}, \theta \in [\epsilon j, k]\), subsequent inequality holds

\[
\left| \frac{\eta(\epsilon j) + \eta(k)}{2} \right| - \frac{1}{k - \epsilon j} \int_{\epsilon j}^{k} \eta(x) \, dx \leq \frac{k - \epsilon j}{4} \epsilon \chi(\sigma) \left| \eta'(j) \right| + \left( 1 - \chi(\sigma) \right) \left| \eta'(k) \right|. \tag{2.21}
\]

**Remark 2.14.** If \(\sigma = \epsilon = 1\) in (2.21), we get the result proved in [2, Theorem 2.2] for convex functions defined on \([0, k]\).
Theorem 2.15. Let \( \eta : W \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( W^\circ \supset [0, \infty) \) and \( \rho : \epsilon j, k \to [0, \infty) \) be continuous and symmetric by \( \frac{\epsilon_j + k}{2} \), settle \( \epsilon \in (0, 1] \), where \( \epsilon j, k \in W^\circ \) with \( \epsilon j < k \). Granted \( \eta' \in L_1 \) [\( \epsilon j, k \)] and \( \left\| \eta' \right\|_q \) is \((\sigma, \epsilon)\)-convex on \( [0, k] \) for \( q \geq 1 \), \((\sigma, \epsilon) \in (0, 1] \times (0, 1] \), coming inequality grips

\[
\left| \frac{\eta(\epsilon j) + \eta(k)}{2} \right| \int_{\epsilon j}^k \rho(\theta) d\theta - \int_{\epsilon j}^k \rho(\theta) \eta(\theta) d\theta \right|
\leq \frac{(k - \epsilon j)^2}{4} \left\| \rho \right\|_\infty \left[ \epsilon \chi (\sigma) \left| \eta' (j) \right|^q + (1 - \chi (\sigma)) \left| \eta' (k) \right|^q \right]^{\frac{1}{q}}, \tag{2.22}
\]

where \( \chi (\sigma) \) and \( \left\| \rho \right\|_\infty \) are construe in Theorem 2.12.

Proof. Continuing from (2.19) and employing Hölder inequality, we achieved

\[
\left| \frac{\eta(\epsilon j) + \eta(k)}{2} \right| \int_{\epsilon j}^k \rho(\theta) d\theta - \int_{\epsilon j}^k \rho(\theta) \eta(\theta) d\theta \right|
\leq \frac{(k - \epsilon j)^2}{4} \left\| \rho \right\|_\infty \left( \frac{1}{0} t^\frac{1}{r} \right)^{1 - \frac{1}{r}}
\times \left\{ \left( \frac{1}{0} t \left| \eta' (U (t, \epsilon)) \right|^q dt \right)^\frac{1}{q} + \left( \frac{1}{0} t \left| \eta' (U (t, \epsilon)) \right|^q dt \right)^\frac{1}{q} \right\}. \tag{2.23}
\]

Accepting power-mean inequality \( \theta^r + y^r \leq 2^{1-r} (\theta + y)^r \) for \( j, k > 0 \) and \( r < 1 \), we attain

\[
\left( \frac{1}{0} t \left| \eta' (U (t, \epsilon)) \right|^q dt \right)^\frac{1}{q} + \left( \frac{1}{0} t \left| \eta' (U (t, \epsilon)) \right|^q dt \right)^\frac{1}{q}
\leq 2^{1-r} \left( \frac{1}{0} t \left| \eta' (U (t, \epsilon)) \right|^q dt + \frac{1}{0} t \left| \eta' (U (t, \epsilon)) \right|^q dt \right)^\frac{1}{q}. \tag{2.24}
\]
Since \( \eta' \) is \((\sigma, \epsilon)\)-convex on \([0, k]\) for \(q \geq 1\), \((\sigma, \epsilon) \in (0, 1] \times (0, 1]\), we have
\[
\int_0^t \left| \eta' (U(t, \epsilon)) \right|^q dt + \int_0^t \left| \eta' (U(t, \epsilon)) \right|^q dt \\
\leq \int_0^t \left\{ \left( \frac{1 + t}{2} \right) \sigma \left| \eta' (k) \right|^q + \epsilon \left[ 1 - \left( \frac{1 + t}{2} \right) \sigma \right] \left| \eta' (j) \right|^q \right\} dt \\
+ \left\{ \frac{2 (2^{-\sigma} + \sigma)}{(\sigma + 2) (\sigma + 1)} \right\} \left| \eta' (k) \right|^q + \epsilon \left\{ 1 - \frac{2 (2^{-\sigma} + \sigma)}{(\sigma + 2) (\sigma + 1)} \right\} \left| \eta' (j) \right|^q .
\] (2. 25)

Using (2. 25) in (2. 24) and then the resulting inequality in (2. 23), we get the appropriate inequality. \(\square\)

**Corollary 2.16.** Expect the conditions of Theorem 2.15 are convinced and \(\rho (\theta) = \frac{1}{k - \epsilon j}, \theta \in [\epsilon j, k], \) ensuing inequality grips
\[
\left| \eta (\epsilon j) + \eta (k) \right|
\leq \frac{k}{2} - \int_{\epsilon j}^k \eta (x) dx \\
\leq \frac{k - \epsilon j}{4} \left[ \chi (\sigma) \left| \eta' (j) \right|^q + (1 - \chi (\sigma)) \left| \eta' (k) \right|^q \right]^{\frac{1}{q}} ,
\] (2. 26)

spot \(\chi (\alpha)\) is defined in Theorem 2.12.

**Remark 2.17.** Assuming \(\sigma = \epsilon = 1\) in (2. 26), we get the result craved in [17, Theorem 1] for convex functions decided on \([0, k].\)

**REFERENCES**


