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Adjacency spectral characterization of the graphs $\overline{K_w \bigtriangledown P_{17}}$ and $\overline{K_w \bigtriangledown S}$

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Abstract. In A.Z. Abdian, Two classes of multicone graphs determined by their spectra, J. Math. Ext., **10** (2016), 111–121, it was conjectured that the complement of the multicone graphs, the join of a clique and a regular graph, $K_w \bigtriangledown P_{17}$ and $K_w \bigtriangledown S$, are determined by their adjacency spectra, where P_{17} and S denote the Paley graph of order 17 and the Schläfli graph, respectively. In this article, we aim to answer to these conjectures.

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1. Introduction

In this paper, graphs are simple and undirected. For some terminology not given here see [12]. The join $G \bigtriangledown H$ of two disjoint graphs G and H is obtained by connecting every vertex of G to each vertex of H. We use $\overline{G}$ to show the complement of G. The adjacency matrix of G is denoted by $A(G)$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_t$ are the distinct eigenvalues of $A(G)$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively. The multi-set $\mathrm{Spec}_A(G) = \{[\lambda_1]^{m_1}, \ldots, [\lambda_n]^{m_t}\}$ is called the adjacency spectrum of G.

Until now, only some graphs with special structures are shown to be *determined by their spectra* (DS, for short) (see [5, 8, 9, 10, 11] and the references cited in them). Van Dam and Haemers [9] conjectured that almost all graphs are DS. About the background of the question "Which graphs are determined by their spectrum?", we refer to [9].

The join of a clique and a regular graph is called a multicone graph. In [1, 2, 3, 8, 9, 10, 11] some multicone graphs are shown to be determined by their adjacency spectra. In [1], it was conjectured that the complement of $K_w \bigtriangledown P_{17}$ and $K_w \bigtriangledown S$ are determined by their adjacency spectrum, where P_{17} and S denote the Paley graph of order 17 and the Schläfli graph, respectively. In this article, we prove these conjectures.

2. Some definitions and preliminaries

We present some essential lemmas that help us prove the main results.

Lemma 2.1 ([6, 7]). *The number of vertices, the number of edges and being regular can be extracted from the adjacency spectrum of a graph.*

Lemma 2.2 ([6]). *If K is a connected proper subgraph of G, then*

$$\lambda_1(G) = \lambda_{max}(G) > \lambda_1(K) = \lambda_{max}(K).$$

Lemma 2.3 ([6]). *If G is a k-regular graph on n vertices with*

$$\mathrm{Spec}_A(G) = \left\{[\lambda_t]^{m_t}, \ldots, [\lambda_2]^{m_2}, [k]^1\right\},$$

then $\mathrm{Spec}_A(\overline{G}) = \left\{[-1-\lambda_2]^{m_2}, [-1-\lambda_t]^{m_t}, \ldots, [n-k-1]^1\right\}$.

For further information about the adjacency spectrum of P_{17} and S see [7, 9].

3. The uniqueness of adjacency spectrum of the $\overline{K_w \bigtriangledown P_{17}}$

It is well-known that Paley graphs are self-complementary. Hence $\overline{K_w \bigtriangledown P_{17}} \cong wK_1 + P_{17}$ (disjoint unioun), and this gives us the following result.

Proposition 3.1. *The adjacency spectrum of the graph $\overline{K_w \bigtriangledown P_{17}}$ is $\left\{[8]^1, [r]^8, [0]^w, [r']^8\right\}$, where $r = \dfrac{-1+\sqrt{17}}{2}$ and $r' = \dfrac{-1-\sqrt{17}}{2}$.*

Lemma 3.2. *There exist no connected graph with the following spectrum*

$$\left\{[8]^1, [r]^8, [0]^w, [r']^8\right\},$$

where $r = \dfrac{-1+\sqrt{17}}{2}$ and $r' = \dfrac{-1-\sqrt{17}}{2}$.

Proof. Suppose by the contrary that G' is a **connected** graph with

$$\mathrm{Spec}_A(G') = \left\{[8]^1, [r]^\beta, [0]^\alpha, [r']^\theta\right\}.$$

We consider the following cases (in the following we always suppose that $1 \le \gamma < \beta$ and $1 \le \zeta < \alpha$ and $1 \le \tau < \theta$, unless the contrary is specified. In other words, we aim to consider the proper subgraphs of G' (if available)):

Case 1. Proper subgraph Γ of G' has four distinct eigenvalues. Hence, $\mathrm{Spec}_A(\Gamma) = \left\{[8]^1, [r]^\gamma, [0]^\zeta, [r']^\tau\right\}$. This contradicts Lemma 2.2, because $\lambda_1(G') = \lambda_1(\Gamma)$.

Case 2. Proper subgraph Γ of G' has three distinct eigenvalues. So the following four subcases hold: **Subcase 2.a**. $\left\{[0]^\zeta, [r]^\gamma, [r']^\tau\right\}$, **Subcase 2.b**. $\left\{[0]^\zeta, [8]^1, [r']^\tau\right\}$,

Subcase 2.c. $\left\{[0]^{\zeta},\, [8]^1,\, [r]^{\gamma}\right\}$, **Subcase 2.d**. $\left\{[8]^1,\, [r]^{\gamma},\, [r']^{\tau}\right\}$. Now we consider any of these subcases.

Subcase 2.a. In the case, we have $\gamma r + \tau r' = \dfrac{-(\gamma+\tau)+(\gamma-\tau)\sqrt{17}}{2} = 0$. As a result, $-(\gamma+\tau)+(\gamma-\tau)\sqrt{17} = 0$. So $(\sqrt{17}-1)\gamma + (-1-\sqrt{17})\tau = 0$ and therefore $\dfrac{\gamma}{\tau} = \dfrac{1+\sqrt{17}}{\sqrt{17}-1} = \dfrac{18+2\sqrt{17}}{16}$, a contradiction, since $\dfrac{\gamma}{\tau}$ must be a rational number. In **Subcases 2.b, 2.c** and **2.d** we have $\lambda_1(G') = \lambda_1(\Gamma)$. This contradicts Lemma 2.2.

Case 3. Proper subgraph Γ of G' has only one eigenvalue. If G' has isolated vertices, then by **Subcase 2.d** we get a contradiction. So, let G' has χ $(1 \le \chi < \alpha \le w)$ isolated vertices. Therefore, $\mathrm{Spec}_A(H) = \left\{[8]^1,\, [r]^{\gamma},\, [0]^{\alpha-\chi},\, [r']^{\tau}\right\}$, where H is a subgraph of G'. Hereafter, by a similar argument to **Cases** 1 and 2, we receive a contradiction. □

Theorem 3.3. *For all w, the graph $\overline{K_w \bigtriangledown P_{17}}$ is DS with respect to its adjacency spectrum.*

Proof. Let $G \cong \overline{K_w \bigtriangledown P_{17}}$. It follows from Proposition 3.1 that $\mathrm{Spec}_A(G) = \mathrm{Spec}_A(\overline{K_w \bigtriangledown P_{17}}) = \left\{[8]^1,\, [r]^8,\, [0]^w,\, [r']^8\right\}$. We show that only the case

$$\left\{[8]^1,\, [r]^8,\, [r']^8\right\} \cup \{[0]^w\}$$

can happen. Note that only graphs with one or two distinct eigenvalue(s) are complete graphs (if a graph H has one distinct eigenvalue (this eigenvalue is obviously 0), then $H \cong mK_1$, where m denotes the multiplicity of 0. If a graph H has two distinct eigenvalues, then H is the disjoint union of complete graphs with the same number of vertices). It is obvious that if we discard the eigenvalue 0, then G does not have any subgraph with one or two distinct the adjacency eigenvalue(s) ($\mathrm{Spec}_A(K_w) = \left\{[-1]^{w-1}, [w-1]^1\right\}$). Now we consider two cases, one where a proper subgraph has spectrum $\left\{[8]^1,\, [r]^8,\, [r']^8\right\}$ and the other where a subgraph has some other spectrum with three distinct values. We show that the second case cannot hold.

To put that another way, we prove that only the spectrum of a graph that can be extracted by $\mathrm{Spec}_A(G)$ is $\left\{[8]^1,\, [r]^8,\, [r']^8\right\} \cup \{[0]^w\}$. If there is a graph K with three distinct eigenvalues, then one of the following cases holds: **Case 1.** $\left\{[0]^{\alpha},\, [r]^{\beta},\, [r']^{\theta}\right\}$, **Case 2.** $\left\{[0]^{\alpha},\, [8]^1,\, [r']^{\theta}\right\}$, **Case 3.** $\left\{[0]^{\alpha},\, [8]^1,\, [r]^{\theta}\right\}$, where $1 \le \alpha \le w$ and $1 \le \beta, \theta \le 8$. We show that there does not exist such a graph K using the fact that the sum of the adjacency eigenvalues of a graph is 0.

Case 1. In this case, we have $-(\beta+\theta)+(\beta-\theta)\sqrt{17} = 0$. So $(\sqrt{17}-1)\beta + (-1-\sqrt{17})\theta = 0$ and therefore $\dfrac{\beta}{\theta} = \dfrac{1+\sqrt{17}}{\sqrt{17}-1} = \dfrac{18+2\sqrt{17}}{16}$, a contradiction, since $\dfrac{\beta}{\theta}$ must be a rational number.

Case 2. In this case, we have $16+(-1+\sqrt{17})\theta = 0$. So $\theta = -1-\sqrt{17}$, a contradiction, since θ must be a rational number.

Case 3. In this case, we have $16+(-1-\sqrt{17})\theta = 0$. So $\theta = -1+\sqrt{17}$, a contradiction, since θ must be a rational number.

It follows from Lemma 3.2 that there does not exist a connected graph G' such that $\mathrm{Spec}_A(G') = \left\{[k]^1,\ [r]^\beta,\ [0]^\alpha, [r']^\theta\right\}$. Therefore, we conclude that only the case $\left\{[8]^1,\ [r]^8,\ [r']^8\right\} \cup \{[0]^w\}$ can happen and so $G \cong wK_1 \cup P_{17}$ or $G \cong \overline{K_w \bigtriangledown P_{17}}$. This completes the proof. □

4. The uniqueness of the adjacency spectrum of $\overline{K_w \bigtriangledown S}$

Since $\mathrm{Spec}_A(S) = \{[10]^1, [1]^{20}, [-5]^6\}$, it follows from Lemma 2.3 that $\mathrm{Spec}_A(\overline{S}) = \{[16]^1, [-2]^{20}, [4]^6\}$. On the other hand, $\overline{K_w \bigtriangledown S} \cong wK_1 + \overline{S}$. Hence, $\mathrm{Spec}_A(\overline{K_w \bigtriangledown S}) = \mathrm{Spec}_A(wK_1) \cup \mathrm{Spec}_A(\overline{S})$. Therefore, we have the following result.

Proposition 4.1. *The adjacency spectrum of the graph $\overline{K_w \bigtriangledown S}$ is $\left\{[16]^1,\ [-2]^{20},\ [0]^w, [4]^6\right\}$.*

Theorem 4.2. *For all w, the graph $\overline{K_w \bigtriangledown S}$ is DS with respect to its adjacency spectrum.*

Proof. Let $\mathrm{Spec}_A(G_1) = \mathrm{Spec}_A(\overline{K_w \bigtriangledown S}) = \left\{[16]^1,\ [-2]^{20},\ [0]^w, [4]^6\right\}$. We use induction on w. We show that only the case $\left\{[16]^1,\ [-2]^{20},\ [4]^6\right\} \cup \{[0]^w\}$ can happen. Let $w = 1$. We prove the theorem in a more general case. First we prove that there is no **connected** graph G' with $\left\{[16]^1,\ [-2]^\alpha,\ [0]^1, [4]^\beta\right\}$, where $1 \leq \alpha \leq 20$ and $1 \leq \beta \leq 6$. We consider the following cases:

Case 1. Proper subgraph Γ of G' has four distinct eigenvalues. Therefore, $\mathrm{Spec}_A(\Gamma) = \left\{[16]^1,\ [-2]^\gamma,\ [0]^1, [4]^\iota\right\}$, where $1 \leq \gamma < \alpha$ and $1 \leq \iota < \beta$. This contradicts Lemma 2.2, because $\lambda_1(G') = \lambda_1(\Gamma)$.

Case 2. Proper subgraph Γ of G' has three distinct eigenvalues. Therefore, we have the following subcases: **Subcase 2.1**. $\mathrm{Spec}_A(\Gamma) = \left\{[16]^1,\ [-2]^\gamma,\ [0]^1\right\}$, **Subcase 2.2**. $\mathrm{Spec}_A(\Gamma) = \left\{[16]^1,\ [0]^1, [4]^\iota\right\}$, **Subcase 2.3**. $\mathrm{Spec}_A(\Gamma) = \left\{[16]^1,\ [-2]^\gamma,\ [4]^\iota\right\}$, **Subcase 2.4**. $\mathrm{Spec}_A(\Gamma) = \left\{[-2]^\gamma,\ [0]^1, [4]^\iota\right\}$. For the **Subcases 2.1**, **2.2** and **2.3**, by a similar way of **Case 1**, we have a contradiction. So, we consider **Case 2.4**. We show that **Case 2.4** cannot happen. On the contrary, suppose that **Case 2.4** happens. So, we have a graph Ω such that $\mathrm{Spec}_A(\Omega) = \left\{[16]^1,\ [-2]^{\alpha-\gamma},\ [4]^{\beta-\iota}\right\}$, a contradiction.

It is clear that no proper subgraph G' has two distinct the adjacency eigenvalues, since $\mathrm{Spec}_A(K_w) = \left\{[-1]^{w-1}, [w-1]^1\right\}$ (in the adjacency spectrum of G' there does not exist the eigenvalue -1). Hence proper subgraph G' must have only one eigenvalue. In other words, proper subgraph G' must be an isolated vertex. This means that G' is disconnected and by what has been proved one can easily conclude that only the case $\left\{[16]^1,\ [-2]^{20},\ [4]^6\right\} \cup \left\{[0]^1\right\}$ can happen and so $G' \cong K_1 \cup \overline{S}$ or $\overline{G'} \cong \overline{K_1 \bigtriangledown S}$. Now, let the theorem be true for w. In other words, one deduce that if $\mathrm{Spec}_A(G_1) = \mathrm{Spec}_A(\overline{K_w \bigtriangledown S})$, then $G_1 \cong \overline{K_w \bigtriangledown S}$, where G_1 denotes an arbitrary graph cospectral with some complements of the multicone graph $K_w \bigtriangledown S$. We show that it follows from

$\mathrm{Spec}_A(G) = \mathrm{Spec}_A(\overline{K_{w+1} \bigtriangledown S})$ that $G \cong \overline{K_{w+1} \bigtriangledown S}$. It is clear that G has one vertex more than G_1. On the other hand, the number of the edges G_1 and G are equal and $\mathrm{Spec}_A(K_1 \cup G_1) = \mathrm{Spec}_A(G)$. So, we must have $G = K_1 \cup G_1$. Now, the inductive hypothesis completes the proof. □

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