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Some Properties on Lifting of Frenet Formulas on Tangent Space TR^3

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Abstract. In this paper, we study the vertical, horizontal and complete lifts of Frenet formulas given by (1. 1), the first acceleration pool centers and the Darboux vector defined on space R^3 to its tangent space $TR^3 = R^6$. In addition, we include all special cases of the curvature κ and torsion τ_0 of the Frenet formulas with respect to the vertical, horizontal and complete lifts on space R^3 to its tangent space TR^3. As a result of this transformation on space R^3 to its tangent space TR^3 , we can speak about the features of Frenet formulas on space TR^3 by looking at the lifting of characteristics $\{T, N, B, \kappa, \tau_0\}$ of the first curve on space R^3. Each curve transformation supported by examples.

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1. INTRODUCTION

In differentiable geometry, the lift method has an important role. Because, it is possible to generalize it from the differentiable structures from any space (for example R^3) to extended spaces (TR^3) using the lift function [11, 12, 16, 17, 18, 20]. Also the Riemannian manifolds and the tangent bundles studyed a lot of authors [1, 2, 3, 8, 9, 10, 11, 14, 15] too. Thus, the Theorem 1.1 may be extended on space R^3 to its tangent space TR^3.

Theorem 1.1. *For a unit speed curve $\alpha_0(t)$ with curvatures $\kappa\rangle 0$ on R^3, the derivatives of Frenet frame $\{T, N, B\}$ are given by* [7, 18]

$$T^{'} = \kappa N,\ B' = -\tau_0 N,\ N' = -\kappa T + \tau_0 B \tag{1. 1}$$

where κ, T, N, B, τ_0 is the curvature, tangent vector, normal vector, binormal vector, torsion of the curve $\alpha_0(t)$, respectively.

Definition 1.2. *Let $\alpha_0(t)$ be a unit speed curve with curvatures $\kappa\rangle 0$ (the curve is a line for $\kappa = 0$,thus we will accept $\kappa\rangle 0$) on R^3, and suppose that T, B, N be respectively tangent, binormal, normal vectors of Frenet frame on any point of $\alpha_0(t)$. Then, we call that triple $\{T, N, B\}$ is Frenet frame such that* [5, 7, 18]

$$\begin{aligned} T.N &= B.N = B.T = 0, \\ T.T &= B.B = N.N = 1, \end{aligned} \tag{1. 2}$$

where "." is a dot (scalar) product.

The paper is structured as follows. In section 2, the vertical, horizontal and complete lifts of a vector field defined on any manifold M of dimension m and their lift properties will be extended to space TR^3. In section 3, vertical lift of the Theorem 1.1 will be obtained. Then, smilar to vertical, horizontal and complete lifts analogues of the related theorem are given. Later, we get the first acceleration pool centers according to vertical, complete and horizontal lifts of the Frenet formulas on TR^3. Finally, the Darboux vector with recpect to vertical,complete and horizontal lifts on TR^3 are defined.

In this study, all geometric objects will be assumed to be of class C^∞ and the sum is taken over repeated indices. Also, v, H and c denote the vertical, horizontal and complete lifts of any differentiable geometric structures defined on R^3 to its tangent space TR^3.

2. Lift of vector field

The vertical lift of a vector field ξ on the space R^3 to the extended $TR^3(= R^6)$ is the vector field $\xi^v \in \chi(TR^3)$ given by [11, 20]:

$$\xi^v(f^c) = (\xi f)^v$$

where $f^c \in F(TR^3)$ is the complete lift of the $f \in F(R^3)$.

The vector field $\xi^c \in \chi(TR^3)$ defined by

$$\xi^c(f^c) = (\xi f)^c, \ \forall f \in F(R^3)$$

is called the complete lift of a vector field ξ on R^3 to its tangent space TR^3.

The horizontal lift of a vector field ξ on space R^3 to TR^3 is the vector field $\xi^H \in \chi(TR^3)$ determined by

$$\xi^H(f^v) = (\xi f)^v, \ \forall f \in F(R^3)$$

the general properties of vertical, horizontal and complete lifts of a vector field on R^3 as follows:

Proposition 2.1. [18, 19, 20]*Let be functions all* $f, g \in F(R^3)$ *and vector fields all* $\xi, \eta \in \chi(R^3)$. *Then, the following equalities are satisfied.*

$$\begin{aligned}
(\xi+\eta)^v &= \xi^v+\eta^v,\ (\xi+\eta)^c=\xi^c+\eta^c,\ (\xi+\eta)^H=\xi^H+\eta^H,\\
(f\xi)^v &= f^v+\xi^v,\ (f\xi)^c=f^c\xi^v+f^v\xi^c,\ \xi^v(f^v)=0,(fg)^H=0,\\
\xi^c(f^v) &= \xi^v(f^c)=(\xi f)^v,\xi^c(f^c)=(\xi f)^c,\xi^H(f^v)=(\xi f)^v,\\
\chi(U) &= Sp\left\{\frac{\partial}{\partial x^\alpha}\right\},\ \chi(TU)=Sp\left\{\frac{\partial}{\partial x^\alpha},\frac{\partial}{\partial y^\alpha}\right\},\\
\left(\frac{\partial}{\partial x^\alpha}\right)^c &= \frac{\partial}{\partial x^\alpha},\left(\frac{\partial}{\partial x^\alpha}\right)^v=\frac{\partial}{\partial y^\alpha},\ \left(\frac{\partial}{\partial x^\alpha}\right)^H=\frac{\partial}{\partial x^\alpha}-\chi\Gamma^\alpha_\beta\frac{\partial}{\partial y^\alpha}.
\end{aligned}$$

where Γ^α_β are Christoffel symbols, U and TU are respectively topolgical open sets of R^3 and TR^3, $f^v, f^c \in F(TR^3)$, $\xi^v, \eta^v, \xi^c, \eta^c, \xi^H, \eta^H \in \chi(TR^3), 1 \le \alpha, \beta \le 3$.

3. Lifting Frenet formulas

In this section, we compute the vertical, complete and horizontal lifts of Frenet formulas given by means of T, N and B Frenet vectors on a unit speed curve $\alpha_0(t)$ with curvature $\kappa\rangle 0$ on space R^3.

3.1. **The vertical lifting Frenet formulas.**

Let T^v be vertical lift of tangent vector T on a unit speed curve $\alpha_0(t)$. Lenght of T^v is given as:

$$\|T^v\| = T^vT^v = (TT)^v = 1$$

with respect to product rule, it follows

$$(T^vT^v)' = 0 = (T^v)'T^v + T^v(T^v)' = 2T^v(T^v)'. \tag{3. 3}$$

From (3. 3), $(T^v)'$ is orthonormal to T^v. Similarly, from (1. 2), we have

$$T^v.N^v = B^v.T^v = B^v.N^v = 0. \tag{3. 4}$$

In this case T^v, N^v and B^v are three orthonormal Frenet vectors on $\alpha_1(t) = (\alpha_0(t))^v$ in the $6-$dimensional space TR^3.

Theorem 3.2. *For a unit speed curve* $\alpha_1(t)$ *with curvature* $\kappa^v\rangle 0$ *on* TR^3, *the derivative's vertical lifts of the Frenet vectors are given as follows:*

$$(T')^v = \kappa^vN^v, (B')^v = -(\tau_0)^vN^v,\ (N')^v = -\kappa^vT^v + (\tau_0)^vB^v$$

where $(\tau_0)^v = -N^v.(B')^v$ is the torsion of the curve $\alpha_1(t)$.

Proof. Let $(T')^v, (B')^v, (N')^v$ be vertical lifts of T', B', N' which are derivatives T, B, N, respectively. We already know

$$(T')^v = (\kappa)^vN^v$$

by definition of $(N)^v$, where the curvature κ^v describes variation in direction of T^v. Also, we shall find $(B')^v$ and $(N')^v$. In particular, given

$$(B')^v = a_1(T)^v + b_1(N)^v + c_1(B)^v.$$

If it can be identified a_1, b_1, c_1, B^v, T^v and N^v then it will be known $(B')^v$. Firstly, we have

$$\begin{aligned} T^v(B')^v &= a_1T^vT^v + b_1T^vN^v + c_1T^vB^v \\ &= a_1(TT)^v + b_1(TN)^v + c_1(TB)^v \\ &= a_1.1 + b_1.0 + c_1.0 \\ &= a_1. \end{aligned}$$

Similarly, $N^v.(B')^v = b_1$ and $(B)^v.(B')^v = c_1$. So, it follows

$$(B')^v = (T^v(B')^v)(T)^v + ((N)^v.(B')^v)(N)^v + ((B)^v.(B')^v)(B)^v.$$

Now let's identify $T^v(B')^v$. We know $T^v.(B)^v = 0 = (T.B)^v$, so that

$$(T^v.(B)^v)^{'} = 0 = (T^{'})^v(B)^v + T^v(B')^v$$

by vertical lift properties and the product rule.

$$\begin{aligned} T^v(B')^v &= -(T^{'})^v(B)^v \\ &= -(\kappa)^v(N)^v(B)^v \quad \text{(from (3. 4))} \\ a_1 &= 0. \end{aligned}$$

From $0 = ((N)^v.(B)^v)^{'} = (N^{'})^v.(B)^v + (N)^v.(B^{'})^v$, we get

$$\begin{aligned} (N)^v.(B^{'})^v &= -(N^{'})^v.(B)^v \\ &= -(-\kappa^vT^v + (\tau_0)^vB^v)(B)^v \\ &= \kappa^vT^v(B)^v - (\tau_0)^v(B^v)(B)^v \\ b_1 &= -(\tau_0)^v \end{aligned}$$

From $(B.B)^v = 1 = (B)^v(B)^v$, we have

$$\begin{aligned} 0 &= ((B^{'})^v.(B)^v)^{'} + (B)^v(B^{'})^v \\ &= 2(B)^v(B^{'})^v. \end{aligned}$$

Thus, we get $c_1 = (B)^v(B^{'})^v = 0$. From the above, $(B^{'})^v$ is calculated as:

$$(B')^v = -(\tau_0)^v(N)^v$$

Now it will be obtained $(N^{'})^v$ for $(B^{'})^v$. So, it follows

$$(N^{'})^v = (T^v(N')^v)(T)^v + ((N)^v.(N')^v)(N)^v + ((B)^v.(N')^v)(B)^v$$

From the same types of calculations, we get $(T.N)^v = T^vN^v = 0$,therefore $0 = (T^{'})^v.N^v + T^v(N')^v$ and $(T^{'})^v = (\kappa)^vN^v$ so it is obtained $T^v(N')^v = -(\kappa)^vN^vN^v = -(\kappa)^v$. Also $N^vN^v = 1$, so $(N)^v.(N')^v = 0$, $(B)^v.(N)^v = 0$, in this case $(B^{'})^v.N^v + B^v(N')^v = 0$. Thus, it is found to be $(B)^v.(N')^v = -(B^{'})^v.N^v = -N^v.(B^{'})^v = (\tau_0)^v$ from definition 1.2. Hence, $(N')^v$ is computed to be

$$(N^{'})^v = -(\kappa)^vT^v + (\tau_0)^v(B)^v.$$

Therfore, proof finished. □

Corollary 3.3. *The Frenet formulas on TR^3 are similar structure and apperance to R^3 with respect to vertical lifts.*

Example 3.4. *A circular helix curve $\alpha_0(t)$ on R^3 has similar appearance with the curve $\alpha_1(t) = (\alpha_0(t))^v$ on TR^3. Because of the curvature κ and torsion τ_0 of a circular helix curve is constant* [6]*, we write $\kappa^v = \kappa$ and $(\tau_0)^v = \tau_0$. So, the curve $\alpha_1(t) = (\alpha_0(t))^v$ on TR^3 has the same κ and τ_0.*

3.5. **The complete and horizontal lifting Frenet formulas.**

Theorem 3.6. *For a unit speed curve $\alpha_2(t) = (\alpha_0(t))^c$ with curvature $\kappa^c \rangle 0$ on tangent space TR^3, complete lifts of the derivatives of the Frenet frame are given by the following equalities:*

$$(T^{'})^c = \kappa^c N^c, \;\; (N')^c = -\kappa^c T^c + (\tau_0)^c B^c, \;\; (B')^c = -(\tau_0)^c N^c, \tag{3. 5}$$

where $(\tau_0)^c = -N^c.(B')^c$ is the torsion of curve $\alpha_2(t)$, respectively.

Proof. Similarly to vertical lifts, the theorem easily proved with respect to complete lift. □

Corollary 3.7. *Let the curvature κ and torsion τ_0 of the curve $\alpha_0(t)$ on R^3 are non-constant functions (for example the general helix curve* [13]*). The Frenet formulas on TR^3 are similar structure and apperance to R^3 with respect to complete lifts (see the formulas (1. 1) and (3. 5)).*

Corollary 3.8. *Let the curvature κ and torsion τ_0 of the curve $\alpha_0(t)$ on R^3 be constant functions (for example circular helix curve* [6]*). Then the curve $\alpha_2(t) = (\alpha_0(t))^c$ on TR^3 is line with respect to complete lifts.*

Proof. Let the curvature κ and torsion τ_0 be constant, we get $\kappa^c = 0$ and $(\tau_0)^c = 0$. So, $(T^{'})^c = 0, \; (B')^c = 0, \; (N')^c = 0$. Then the curve $\alpha_2(t) = (\alpha_0(t))^c$ on TR^3 is line. □

Corollary 3.9. *Let the curvature κ and torsion τ_0 of the curve $\alpha_0(t)$ on R^3 be constant and non-constant functions, respectively (for example Salkowski curve* [4]*). Then the curve $\alpha_2(t) = (\alpha_0(t))^c$ on TR^3 is line with respect to complete lifts.*

Proof. Let the curvature κ be constant, we get $\kappa^c = 0$. So, $(T^{'})^c = 0, \; (N')^c = (\tau_0)^c B^c$, $(B')^c = -(\tau_0)^c N^c$. Then the curve $\alpha_2(t) = (\alpha_0(t))^c$ on TR^3 is line. □

Corollary 3.10. *Let the curvature κ and torsion τ_0 of the curve $\alpha_0(t)$ on R^3 be non-constant and constant functions, respectively (for example anti Salkowski curve* [4]*). Then $(T^{'})^c = 0$ and $(N')^c$ are on the same tangent plane with respect to complete lifts.*

Proof. Let the curvature τ_0 be constant, we get $(\tau_0)^c = 0$. So, $(T^{'})^c = \kappa^c N^c, \; (N')^c = -\kappa^c T^c, \; (B')^c = 0$. Then $(T^{'})^c = 0$ and $(N')^c$ are on the same tangent plane. □

Theorem 3.11. *All curves $\alpha_0(t)$ on R^3 is line on TR^3 with respect to horizontal lifts.*

Proof. Let the curvature κ and torsion τ_0 of the curve $\alpha_0(t)$ be constant or non-constant functions on R^3. For all functions on R^3, we write $f^H = 0$ with respect to horizontal lifts. So, $(\kappa)^H = (\tau_0)^H = 0$ and $(T^{'})^H = (B')^H = (N')^H = 0$ on TR^3.Consecuently, $\alpha_3(t) = (\alpha_0(t))^H$ on TR^3 is line. □

3.12. The first acceleration pool centers of the Frenet formulas on TR^3.

Definition 3.13. *The first acceleration pool centers of the Frenet formulas on R^3 are given by the following equalities* [7]*:*

$$\begin{aligned} T^{''} &= -\kappa^2 T + \kappa^{'} N + \kappa(\tau_0) B \\ N^{''} &= -\kappa^{'} T - (\kappa^2 + (\tau_0)^2) N - (\tau_0)^{'} B \\ B^{''} &= -\kappa(\tau_0) T - (\tau_0)^{'} N - (\tau_0)^2 B \end{aligned}$$

where κ, T, N, B, τ_0 is respectively curvature, tangent vector, normal vector, binormal vector, torsion of the curve $\alpha_0(t)$.

It is possible to generalize to the first acceleration pool centers with respect to vertical lifts of the Frenet formulas on space R^3 to its tangent space TR^3 by using lift function [11, 12, 18, 20].

Theorem 3.14. *For a unit speed curve $\alpha_1(t)$ with curvatures $\kappa^v \rangle 0$ on TR^3, the first acceleration pool centers with respect to vertical lifts of the Frenet formulas on TR^3 are given as:*

$$\begin{aligned} (T^{''})^v &= -(\kappa^2)^v T^v + (\kappa^{'})^v N^v + \kappa^v (\tau_0)^v B^v \\ (N^{''})^v &= -(\kappa^{'})^v T^v - ((\kappa^2)^v + ((\tau_0)^2)^v) N^v + ((\tau_0)^{'})^v B^v \\ (B^{''})^v &= (\kappa)^v (\tau_0)^v T^v - ((\tau_0)^{'})^v N^v - ((\tau_0)^2)^v B^v \end{aligned}$$

where $(\kappa)^v, (\tau_0)^v$ is respectively curvature and torsion of the curve $\alpha_1(t)$ on TR^3.

Proof. From the derivatives of the Theorem 3.2, we get the following results

$$\begin{aligned} (T^{''})^v &= (\kappa^v)^{'} N^v + \kappa^v (N^v)^{'} \\ &= (\kappa^{'})^v N^v + \kappa^v (-\kappa^v T^v + (\tau_0)^v B^v) \\ &= -(\kappa^2)^v T^v + (\kappa^{'})^v N^v + \kappa^v (\tau_0)^v B^v. \end{aligned}$$

$$\begin{aligned} (N^{''})^v &= -(\kappa^v)^{'} T^v - \kappa^v (T^v)^{'} + ((\tau_0)^v)^{'} B^v + (\tau_0)^v (B^v)^{'} \\ &= -(\kappa^{'})^v T^v - \kappa^v (\kappa^v N^v) + ((\tau_0)^v)^{'} B^v + (\tau_0)^v (-(\tau_0)^v N^v) \\ &= -(\kappa^{'})^v T^v - ((\kappa^2)^v + ((\tau_0)^2)^v) N^v + ((\tau_0)^{'})^v B^v \end{aligned}$$

$$\begin{aligned} (B^{''})^v &= -((\tau_0)^v)^{'} N^v - (\tau_0)^v (N^v)^{'} \\ &= -((\tau_0)^v)^{'} N^v - (\tau_0)^v (-\kappa^v T^v + (\tau_0)^v B^v) \\ &= (\kappa)^v (\tau_0)^v T^v - ((\tau_0)^{'})^v N^v - ((\tau_0)^2)^v B^v \end{aligned}$$

Therfore, proof finished. □

Similarly, we can easily prove the following theorem of the first acceleration pool centers with respect to complete lifts of the Frenet formulas on TR^3 .

Theorem 3.15. *Let κ^c be the curvature of the curve $\alpha_2(t) = (\alpha_0(t))^c$ on TR^3. The first acceleration pool centers according to complete lifts of the Frenet formulas on TR^3 are given as:*

$$\begin{aligned} (T^{''})^c &= -(\kappa^2)^c T^c + (\kappa^{'})^c N^c + \kappa^c(\tau_0)^c B^c \\ (N^{''})^c &= -(\kappa^{'})^c T^c - ((\kappa^2)^c + ((\tau_0)^2)^c) N^c + ((\tau_0)^{'})^c B^c \\ (B^{''})^c &= (\kappa)^c(\tau_0)^c T^c - ((\tau_0)^{'})^c N^c - ((\tau_0)^2)^c B^c \end{aligned}$$

where $\alpha_2(t) = (\alpha_0(t))^c$ a unit speed curve with curvature $(\kappa)^c$ on TR^3.

Corollary 3.16. *Because of the Theorem 3.11, we get $(T^{''})^H = (N^{''})^H = (B^{''})^H = 0$.*

3.17. **The Darboux vector with recpect to vertical,horizontal and complete lifts on TR^3.**

Definition 3.18. *The Darboux vector ω on R^3 defined as* [7]*:*

$$\omega = (\tau_0, 0, \kappa) = \tau_0 T + \kappa B$$

ω is a vector in the plane (T, B) and perpendicular to the normal vector of the curve. ω vector field has the following properties:

$$\begin{aligned} \omega.T &= \tau_0,\ \omega.N = 0,\ \omega.B = \kappa \\ \omega\Lambda T &= T^{'},\ \omega\Lambda N = N^{'},\ \omega\Lambda B = B^{'}. \end{aligned}$$

Theorem 3.19. *Let $\alpha_1(t)$ be a unit speed curve with curvatures $(\kappa)^v\rangle 0$ on TR^3, The ω^v Darboux vector with respect to vertical lifts on TR^3 defined as:*

$$\begin{aligned} \omega^v &= (\tau_0)^v, 0, \kappa^v) \\ &= (\tau_0)^v T^v + (\kappa)^v B^v \end{aligned}$$

ω^v vector field has the following properties

$$\begin{aligned} \omega^v.T^v &= (\tau_0)^v,\ \omega^v.N^v = 0,\ \omega^v.B^v = (\kappa)^v \\ \omega^v\Lambda T^v &= (T^{'})^v,\ \omega^v\Lambda N^v = (N^{'})^v,\ \omega^v\Lambda B^v = (B^{'})^v. \end{aligned}$$

Proof. From Proposition 1 and Definition 3, we get the following results

$$\begin{aligned} \omega^v.T^v &= ((\tau_0)^v T^v + (\kappa)^v B^v).T^v \\ &= (\tau_0)^v (T.T)^v + (\kappa)^v (B.T)^v \\ &= (\tau_0)^v.1 + (\kappa)^v.0 \\ &= (\tau_0)^v \\ \omega^v.(N)^v &= ((\tau_0)^v T^v + (\kappa)^v B^v).(N)^v \\ &= (\tau_0)^v (T.N)^v + (\kappa)^v (B.N)^v \\ &= 0 \\ \omega^v.(B)^v &= ((\tau_0)^v T^v + (\kappa)^v B^v).(B)^v \\ &= (\tau_0)^v (T.B)^v + (\kappa)^v (B.B)^v \\ &= (\kappa)^v \end{aligned}$$

□

Theorem 3.20. *If we defined ω^c Darboux vector with respect to complete lifts on TR^3, then $\omega^c = ((\tau_0)^c, 0, (\kappa)^c) = (\tau_0)^c T^c + (\kappa)^c B^c$. we get*

$$\omega^c.T^c = \omega^c.(N)^c = \omega^c.(B)^c = 0$$

where κ and τ_0 non-constant functions.

Proof. The results get easily from (1. 2) and Proposition 1. □

Corollary 3.21. *Let the curvature κ and torsion τ_0 be constant, we get $\kappa^c = 0$ and $(\tau_0)^c = 0$. So, $\omega^c = 0$. Then the Darboux vector ω^c with respect to complete lifts on TR^3 is point.*

Corollary 3.22. *Let the curvature κ and torsion τ_0 of the curve $\alpha_0(t)$ on R^3 be non-constant and constant functions, respectively. Then we get $\omega^c = (\kappa)^c B^c$ (the Darboux vector ω^c linear dependency B^c on TR^3.*

Corollary 3.23. *Let the curvature κ and torsion τ_0 of the curve $\alpha_0(t)$ on R^3 be constant and non-constant functions, respectively. Then we get $\omega^c = (\tau_0)^c T^c$ (the Darboux vector ω^c linear dependency T^c on TR^3.*

Theorem 3.24. *Darboux vector ω^H with respect to horizontal lifts on TR^3 is a point everytime .*

Proof. From Theorem 3.11, we get $(\kappa)^H = (\tau_0)^H = 0$. So, $\omega^H = 0$ on TR^3 with respect to horizontal lifts. The theorem is proved. □

4. CONCLUSION

In this study, using lifting methods, we see that it may be generalized the Frenet formulas given by (1. 1), the first acceleration pool centers and the Darboux vector defined on space R^3 to its tangent space $TR^3 = R^6$.

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