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On the Evaluation of a Subdivision of the Ladder Graph

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Abstract. Let G be a subdivision of a ladder graph. In this paper, we study magic evaluation with type $(1, 1, 1)$ for a given any general ladder graph G. We prove that subdivided ladder admits magic evaluation having type $(1, 1, 1)$. We also prove that such a subdivision admits consecutive magic evaluation having type $(1, 1, 0)$.

AMS (MOS) Subject Classification Codes: 05C78
Key Words: magic labeling of type (a, b, c), ladder, subdivided ladder.

1. INTRODUCTION

Let G be a ladder graph. Let G has set of vertices $V(G)$, set of edges $E(G)$ and set of faces $F(G)$. We took G to be a planar, simple, connected, undirected and finite graph. Let $|V(G)| = v$ counts number of vertices, $|E(G)| = e$ counts number of edges and $|F(G)| = f$ counts the faces of G.

A map sending graph elements to positive or non-negative numbers is called a *valuation* (or *labeling*) of a graph. In 1983, Lih [6] introduced a magic-type valuation of the faces, edges and vertices for graph G. The type of valuation is defined as a one-one function $f : \{1, 2, \dots, |F(G)| \cup |E(G)| \cup |V(G)|\}$ mapped to $F(G) \cup E(G) \cup V(G)$ in such way that for all inner face the sum of the labels of the vertices, edges and faces around that face having there fixed value. We denote it as a magic type evaluation of $(1, 1, 1)$ of a planar graph $G(F(G), E(G), V(G))$. The magic labeling of grid graphs and honeycomb graphs was discussed in [1] and [2], respectively. On the same way, in [6] a valuation having the planar faces of a planar graph is called consecutive if corresponding to every integer s the sum of the valuation of all faces with s sides are given by some set of consecutive positive integers. For more details about consecutive valuations see [3], [5].

A bijective map from set of edges $E(G)$ onto $\{1, 2, \cdots, |E(G)|\}$ (the subset of positive integers) is called a magic valuation having type $(0, 1, 0)$ of a given G which is a planar graph if for each inner face summing over all the values of the edges is fixed which are contained in that face. Similarly, a bijective map from the union of sets $E(G) \cup V(G)$ onto $\{1, 2, \cdots, |V(G) \cup E(G)|\}$ which are subset of the positive integers is called a magic valuation of type $(1, 1, 0)$ of $G(V, E)$ the planar graph if for each face which is internal the resultant of summing up the labels of the edges and the vertices contained in that face having a fixed value. Such magic valuation having type $(1,1,0)$ is known as super if $\mu(V(G)) = \{1,2,\cdots,|V(G)|\}$. For more details see [7]. There is a natural relation between magic labeling having type $(1,1,0)$ and H-magic labeling of a graph which is planar. The later one was also introduced by Lih [6]. It is defined as a total labeling μ from the union of two sets $E(G)\cup V(G)$ onto $\{1,2,\cdots,|V(G)\cup E(G)|\}\subset\mathbb{Z}$ the subset of positive integers such that for every subgraph A of G isomorphic to H there exist a positive integer c such that $\sum\limits_{v\in V(A)}\mu(v)+\sum\limits_{e\in E(A)}\mu(e)+\mu(y)=c$. A graph that admits such a labeling is called H-magic. An H-magic labeling μ is called an H-supermagic valuation subject to $\mu(V(G))=\{1,2,\cdots,|V(G)|\}$. A detailed survey [4] is available on Graph valuations.

In the following section, we present few magic valuation has the type $(1,1,1)$ of the subdivided ladder.

2. Subdivision of ladders

Let $L_m \cong P_m \times P_2$ be a ladder graph. We take $V(L_m)$ and $E(L_m)$ to be the set of edges and the set of vertices of L_m, respectively. We take

$$V(L_m)=\{\nu_i,\epsilon_i;1\le i\le m\}$$

and

$$E(L_m)=\{\nu_i\nu_{i+1},\epsilon_i\epsilon_{i+1};1\le i\le m-1\}\cup\{\nu_i\epsilon_i;1\le i\le m\}.$$

Let $S_1(L_m)$ be a graph, namely the subdivision of the ladder graph L_m, obtained by dividing each edge of L_m by exactly one vertex. Note that the total number of vertices, edges and faces of $S_1(L_m)$ are $5m-2$, $6m-4$ and $m-1$, respectively. We write $V(S_1)$ and $E(S_1)$ to be the set of vertices and edges of $S_1(L_m)$, respectively, such that

$$V(S_1)=\{\nu_i,\epsilon_i,w_i;1\le i\le m\}\cup\{\acute{\nu_i},\acute{\epsilon_i};1\le i\le m-1\},$$

$$E(S_1)=\{\nu_i\acute{\nu_i},\acute{\nu_i}\nu_{i+1},\epsilon_i\acute{\epsilon_i},\acute{\epsilon_i}\epsilon_{i+1};1\le i\le m-1\}\cup\{\nu_iw_i,w_i\epsilon_i;1\le i\le m\}.$$

Here $\acute{\nu_i},\acute{\epsilon_i}$ and w_i be the new vertices introduced between the edges $\nu_i\nu_{i+1},\epsilon_i\epsilon_{i+1}$ and $\nu_i\epsilon_i$, respectively.

Theorem 1. *For any integer $m\ge 3$ and $L_m\cong P_m\times P_2$, the subdivision graph $S_1(L_m)$ own up a magic valuation of type $(1,1,1)$.*

Proof. For $S_1(L_m)$, we have $|V(S)|+|E(S)|+|F(S)|=12m-7$. Let U denotes the set of all vertices, edges and faces of $S_1(L_m)$. Define a valuation function μ in a way defined by

$$\mu:U\to\{1,2,\ldots,12m-7\}.$$

For vertices:
$\mu(x)=$

$$\begin{cases} 4(i-1)+1, & \text{if } x=\nu_i \text{ for } 1\le i\le m,\\ 4(2m-i)-2, & \text{if } x=\epsilon_i \text{ for } 1\le i\le m,\\ 4(i-1)+3, & \text{if } x=\acute{\nu_i} \text{ for } 1\le i\le m-1,\\ 4(2m-i-1), & \text{if } x=\acute{\epsilon_i} \text{ for } 1\le i\le m-1,\end{cases}$$

for m to be even

$$\begin{cases} 5(2m-1)+i, & \text{if } x=w_{2i-1} \text{ for } 1\le i\le m/2,,\\ 5(2m-1)+i+m/2-1, & \text{if } x=w_{2i} \text{ for } 1\le i\le m/2,\end{cases}$$

for m to be odd

$$\begin{cases} 5(2m-1)+i, & \text{if } x=w_{2i-1} \text{ for } 1\le i\le (m+1)/2, \\ 5(2m-1)+i+(m+1)/2-1, & \text{if } x=w_{2i} \text{ for } 1\le i\le (m-1)/2. \end{cases}$$

We label the edges in the following way

$$\mu(x)=\begin{cases} 4(i-1)+2, & \text{if } x=\nu_i\acute{\nu_i} \text{ for } 1\le i\le m-1, \\ 4i, & \text{if } x=\acute{\nu_i}\nu_{i+1} \text{ for } 1\le i\le m-1, \\ 4(2m-i)-3, & \text{if } x=\epsilon_i\acute{\epsilon_i} \text{ for } 1\le i\le m-1, \\ 4(2m-i-1)-1, & \text{if } x=\acute{\epsilon_i}\epsilon_{i+1} \text{ for } 1\le i\le m-1, \\ 8m+i-6, & \text{if } x=\nu_i w_i \text{ for } 1\le i\le m, \\ 5(2m-1)-i, & \text{if } x=w_i\epsilon_i \text{ for } 1\le i\le m. \end{cases}$$

If we denote the faces containing $\acute{\nu_i},\acute{\epsilon_i}$ as f_i, for all $i=1,\dots,m-1$, then we label the faces in the following way

$$\mu(x)=\{\ 12m-6-i, \quad \text{if } x=f_i \text{ for } 1\le i\le m-1.$$

□

3. Main Result

After studying subdivided ladder obtained by dividing each edge by exactly one vertex. In this section, we generalize our result by giving magic type labeling having type (a,b,c) of the subdivided ladder obtained by dividing each edge by exactly n vertices.

Let $S_n(L_m)$ be a graph, namely the subdivision of ladder, obtained by dividing each edge of L_m by exactly n vertices. The total number of vertices, edges and faces of $S_n(L_m)$ are $m(3n+2)-2n$, $(3m-2)(n+1)$ and $m-1$, respectively. We write $V(S_n)$ and $E(S_n)$ to be the set of vertices and edges of $S_n(L_m)$, respectively, such that
$V(S_n)=\{\nu_{i0},\epsilon_{i0};1\le i\le m\}\cup\{\nu_{ij},\epsilon_{ij};1\le i\le m-1,1\le j\le n\}\cup\{w_{ij};1\le i\le m,1\le j\le n\}$. For our convenience, we use the notations $\nu_{i(n+1)}=\nu_{(i+1)0},\epsilon_{i(n+1)}=\epsilon_{(i+1)0}$ and $w_{i0}=\nu_{i0},w_{i(n+1)}=\epsilon_{i0}$. Thus
$E(S_n)=\{\nu_{ij}\nu_{i(j+1)},\epsilon_{ij}\epsilon_{i(j+1)}1\le i\le m-1,0\le j\le n\}\cup\{w_{ij},w_{i(j+1)};1\le i\le m,0\le j\le n\}$. Here $\acute{\nu_{ij}},\acute{\epsilon_{ij}}$ and w_{ij} be the new vertices introduced between the edges $\nu_{i0}\nu_{(i+1)0},\epsilon_{i0}\epsilon_{(i+1)0}$ and $\nu_{i0}\epsilon_{i0}$, respectively.

Theorem 2. *For any integers $n\ge 1,m\ge 3$, the graph $S_n(L_m)$ own up a super magic valuation having type $(1,1,1)$.*

Proof. Let $v=|V(S_n)|$, $e=|E(S_n)|$ and $f=|F(S_n)|$. We write U to be the set of all vertices, edges and faces of $S_1(L_m)$. Define a valuation function μ in a way defined as

$$\mu:U\to\{1,2,\dots,\}.$$

Case 1: when n-odd
Let $n=2k-1$, for some $k\ge 1$. For vertices and
for $1\le i\le m$

$$\mu(\nu_{i0})=i,$$
$$\mu(\epsilon_{i0})=2m+1-i,$$

for $1\le i\le m-1, 1\le j\le n$

$$\mu(\nu_{ij})=2jm-2(j-1)+i,$$
$$\mu(\epsilon_{ij})=2(j+1)m-2j+1-i,$$

for $1 \leq i \leq m, 1 \leq j \leq k-1, k > 1$

$$\mu(w_{i(2j-1)}) = 2m(n+j) - 2n + i,$$
$$\mu(w_{i(2j)}) = 2m(n+1+j) - 2n + 1 - i,$$

for m to be even and $1 \leq i \leq \frac{m}{2}$

$$\mu(w_{(2i-1)n}) = m(3n+1) - 2n + i,$$
$$\mu(w_{(2i)n}) = \frac{3m}{2}(2n+1) - 2n + i,$$

for m to be odd and $1 \leq i \leq \frac{m+1}{2}$

$$\mu(w_{(2i-1)n}) = m(3n+1) - 2n + i,$$

for m to be odd and $1 \leq i \leq \frac{m-1}{2}$

$$\mu(w_{(2i)n}) = \frac{3m(2n+1)+1}{2} - 2n + i.$$

Now for edges, we have
for $1 \leq i \leq m-1, 0 \leq j \leq n$

$$\mu(\nu_{ij}\nu_{i(j+1)}) = m(3n+2j+2) - 2(n+j) + i,$$
$$\mu(\epsilon_{ij}\epsilon_{i(j+1)}) = m(3n+2j+4) - 2(n+j) - 1 - i,$$

for $1 \leq i \leq m, 0 \leq j \leq \frac{n-1}{2}$

$$\mu(w_{i(2j)}w_{i(2j+1)}) = m(5n+2j+4) - 4n - 2 + i,$$
$$\mu(w_{i(2j+1)}w_{i2(j+1)}) = m(5n+2j+6) - 4n - 1 - i.$$

Finally, if f_i be a face containing ν_{ij} and ϵ_{ij} for all i, then for the faces we have the following map

$$\mu(x) = 6m(n+1) - 4n - 2 - i, \quad \text{if} \quad x = f_i \quad \text{for } 1 \leq i \leq m-1.$$

Case 2: for n-even, say $n = 2k$, for some $k \geq 1$.
For vertices and
for $1 \leq i \leq m$

$$\mu(\nu_{i0}) = i,$$
$$\mu(\epsilon_{i0}) = 2m + 1 - i,$$

for $1 \leq i \leq m-1, 1 \leq j \leq n$

$$\mu(\nu_{ij}) = 2jm - 2(j-1) + i,$$
$$\mu(\epsilon_{ij}) = 2(j+1)m - 2j + 1 - i,$$

for $1 \leq i \leq m, 1 \leq j \leq k$;

$$\mu(w_{i(2j-1)}) = 2m(n+j) - 2n + i,$$
$$\mu(w_{i(2j)}) = 2m(n+1+j) - 2n + 1 - i.$$

Now for edges, we have
for $1 \leq i \leq m-1, 0 \leq j \leq n$

$$\mu(\nu_{ij}\nu_{i(j+1)}) = m(3n+2j+2) - 2(n+j) + i,$$
$$\mu(\epsilon_{ij}\epsilon_{i(j+1)}) = m(3n+2j+4) - 2(n+j) - 1 - i,$$

for $1 \leq i \leq m, 0 \leq j \leq \frac{n}{2} - 1$

$$\mu(w_{i(2j)}w_{i(2j+1)}) = m(5n+2j+4) - 4n - 2 + i,$$
$$\mu(w_{i(2j+1)}w_{i2(j+1)}) = m(5n+2j+6) - 4n - 1 - i,$$

for m to be even and $1 \leq i \leq \frac{m}{2}$

$$\mu(w_{(2i-1)n}\epsilon_{(2i-1)0}) = 2m(3n+2) - 2(2n+1) + i,$$
$$\mu(w_{(2i)n}\epsilon_{(2n)0}) = \frac{3m}{2}(4n+3) - 2(2n+1) + i,$$

for m to be odd and $1 \leq i \leq \frac{m+1}{2}$

$$\mu(w_{(2i-1)n}\epsilon_{(2i-1)0}) = m(6n+4) - 2(2n+1) + i,$$
$$\mu(w_{(2i)n}\epsilon_{(2n)0}) = \frac{2m(6n+5)+1}{2} - 2(2n+1) + i.$$

Finally, if f_i be a face containing ν_{ij} and ϵ_{ij} for all i, then for the faces we have the following map

$$\mu(x) = 6m(n+1) - 4n - 2 - i, \quad \text{if} \quad x = f_i \quad \text{for } 1 \leq i \leq m-1.$$

□

It is easy to note that in the proof of the above theorem, if we do not include the labeling of the faces then $S_n(L_m)$ satisfy the consecutive super magic valuation with type $CM(1,1,0)$. Thus we state the following obvious corollary.

Corollary 3. *For any integers $n \geq 1, m \geq 3$, the graph $S_n(L_m)$ has a consecutive magic valuation which is of the type $(1,1,0)$.*

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