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Nonconvex Functions and Integral Inequalities

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Abstract. A new Hermite-Hadamard inequality for p-convex(nonconvex) functions is obtained, which can be viewed as a refinement of known results. We derive some new inequalities for functions whose derivatives in absolute value are nonconvex.Results obtained in this paper continue to hold for special cases. Techniques and ideas of this paper may stimulate further research in this field.

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1. Introduction

Inequalities are everywhere and play an important and significant role in almost all subjects of Mathematics including other areas of sciences, see [1-19]. Inequalities present a very active and attractive field of research. In recent years, much attention have given to develop various inequalities for several classes of convex functions and their generalizations using novel and new ideas, see [9, 6, 15]. Zhang et al. [19] investigated and studied a new class of convex functions which is called p-convex(nonconvex) functions. Motivated by this ongoing research, Noor et al. [13] have derived several inequalities for differentiable p-convex functions.

Inspired and motivated by the ongoing research in this field, we obtain some new Simpson type integral inequalities for nonconvex functions. Our results can be considered as a refinement of the previous results. For some recent investigations, see [19, 13, 14].

Let I be an interval. We say that a function $f : I = [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ is a p-convex function(nonconvex), if and only if,

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \qquad x \in [a,b]. \tag{1}$$

This double inequality is known as the Hermite-Hadamard inequality for nonconvex functions. The inequality 1 holds in reversed direction if f is a p-concave function.

If $p = 1$, then inequality 1 is known as Hermite-Hadamard inequality for convex functions which was introduced and studied by Hermite [8] and Hadamard [7] independently. If $p = -1$, then inequality (1.1) holds for the harmonic functions, [9].For recent developments, applications, generalizations and other aspects of the Hermite-Hadamard and Simpson type inequalities, see [1-19] and the references therein.

2. Preliminaries

Definition 0.1. *[19, 6]. A set $I = [a,b] \subseteq \mathbb{R}$ is said to be p-convex set, if*

$$\left[(1-t)x^p + ty^p\right]^{\frac{1}{p}} \in I, \quad \forall x,y \in I, t \in [0,1], \quad p \neq 0.$$

Definition 0.2. *[19, 6]. Let I be a p-convex set. A function $f : I = [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ is said to be p-convex function or belongs to the class $PC(I)$, if*

$$f\left[(1-t)x^p + ty^p\right]^{\frac{1}{p}} \leq (1-t)f(x) + tf(y), \quad \forall x,y \in I, t \in [0,1]. \tag{2}$$

If $t = \frac{1}{2}$, then

$$f\left(\left[\frac{x^p+y^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{f(x)+f(y)}{2}, \qquad \forall x,y \in I, \tag{3}$$

which is called Jensen p-convex function.

It is clear that the nonconvex functions include the convex functions and their variant forms as special cases, but the converse is not true, see [19, 11].

3. Main results

In this section, we obtain our main results.

Lemma 0.3. *Let $f : I = [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be a p-convex function, then*

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\leq \frac{1}{2}\left[f\left(\left[\frac{3a^p+b^p}{4}\right]^{\frac{1}{p}}\right) + f\left(\left[\frac{a^p+3b^p}{4}\right]^{\frac{1}{p}}\right)\right] \\ &\leq \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}\mathrm{d}x \leq \frac{1}{2}\left[f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) + \frac{f(a)+f(b)}{2}\right] \\ &\leq \frac{1}{2}[f(a)+f(b)]. \end{aligned} \tag{4}$$

Proof. By applying 1 on each of the interval $[a, [\frac{a^p+b^p}{2}]^{\frac{1}{p}}]$ and $[[\frac{a^p+b^p}{2}]^{\frac{1}{p}}, b]$, we have

$$f\left(\left[\frac{3a^p+b^p}{4}\right]^{\frac{1}{p}}\right) \leq \frac{2p}{b^p-a^p}\int_a^{\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}} \frac{f(x)}{x^{1-p}}\mathrm{d}x \leq \frac{1}{2}\left[f(a) + f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right], \tag{5}$$

and

$$f\left(\left[\frac{a^p+3b^p}{4}\right]^{\frac{1}{p}}\right) \le \frac{2p}{b^p-a^p}\int_{\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}}^{b}\frac{f(x)}{x^{1-p}}\mathrm{d}x \le \frac{1}{2}\left[f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+f(b)\right]. \tag{6}$$

Summing up side by side, we obtain

$$\begin{aligned} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\le \frac{1}{2}\left[f\left(\left[\frac{3a^p+b^p}{4}\right]^{\frac{1}{p}}\right)+f\left(\left[\frac{a^p+3b^p}{4}\right]^{\frac{1}{p}}\right)\right] \\ &\le \frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}\mathrm{d}x \le \frac{1}{2}\left[f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+\frac{f(a)+f(b)}{2}\right] \\ &\le \frac{1}{2}[f(a)+f(b)]. \end{aligned}$$

□

We would like to mention that inequality (4) can be regarded as a refinement of the previous known inequalities.

Lemma 0.4. *Let $f : I = [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a,b]$, then*

$$\begin{aligned} &\frac{1}{6}\left[f(a)+4f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+f(b)\right]-\frac{p}{(b^p-a^p)}\int_a^b\frac{f(x)}{x^{1-p}}\mathrm{d}x \\ = &\frac{(b^p-a^p)}{p}\int_0^1\frac{\mu(t)}{\left[(1-t)a^p+tb^p\right]^{1-\frac{1}{p}}}f'\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\mathrm{d}t, \end{aligned} \tag{7}$$

where

$$\mu(t) = \begin{cases} t-\frac{1}{6}, & t \in [0,\frac{1}{2}), \\ t-\frac{5}{6}, & t \in [\frac{1}{2},1]. \end{cases}$$

Proof. Let

$$\begin{aligned} I &= \frac{(b^p-a^p)}{p}\int_0^1\frac{\mu(t)}{\left[(1-t)a^p+tb^p\right]^{1-\frac{1}{p}}}f'\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\mathrm{d}t \\ &= \frac{(b^p-a^p)}{p}\int_0^{\frac{1}{2}}\frac{t-\frac{1}{6}}{\left[(1-t)a^p+tb^p\right]^{1-\frac{1}{p}}}f'\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\mathrm{d}t \\ &\quad +\frac{(b^p-a^p)}{p}\int_{\frac{1}{2}}^1\frac{t-\frac{5}{6}}{\left[(1-t)a^p+tb^p\right]^{1-\frac{1}{p}}}f'\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\mathrm{d}t \\ &= I_1+I_2 \end{aligned}$$

Now

$$\begin{aligned} I_1 &= \frac{(b^p-a^p)}{p}\int_0^{\frac{1}{2}}\frac{t-\frac{1}{6}}{\left[(1-t)a^p+tb^p\right]^{1-\frac{1}{p}}}f'\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\mathrm{d}t \\ &= \left|\left(t-\frac{1}{6}\right)f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\right|_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\mathrm{d}t \\ &= \frac{1}{3}f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+\frac{f(a)}{6}-\frac{p}{(b^p-a^p)}\int_a^{\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}}\frac{f(x)}{x^{1-p}}\mathrm{d}x. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= \frac{(b^p-a^p)}{p}\int_{\frac{1}{2}}^{1}\frac{t-\frac{5}{6}}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}f'\big(\big[(1-t)a^p+tb^p\big]^{\frac{1}{p}}\big)\mathrm{d}t \\ &= \left|\left(t-\frac{5}{6}\right)f\big(\big[(1-t)a^p+tb^p\big]^{\frac{1}{p}}\big)\right|_{\frac{1}{2}}^{1}-\int_{\frac{1}{2}}^{1}f\big(\big[(1-t)a^p+tb^p\big]^{\frac{1}{p}}\big)\mathrm{d}t \\ &= \frac{f(b)}{6}+\frac{1}{3}f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)-\frac{p}{(b^p-a^p)}\int_{\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}}^{b}\frac{f(x)}{x^{1-p}}\mathrm{d}x. \end{aligned}$$

Now

$$I_1+I_2=\frac{1}{6}\left[f(a)+4f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+f(b)\right]-\frac{p}{(b^p-a^p)}\int_a^b\frac{f(x)}{x^{1-p}}\mathrm{d}x.$$

□

Theorem 0.5. *Let $f : I = [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a,b]$ and $|f'|^q$ is p-convex function on I for $q \geq 1$, then*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a)+4f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+f(b)\right]-\frac{p}{(b^p-a^p)}\int_a^b\frac{f(x)}{x^{1-p}}\mathrm{d}x\right| \\ \leq\ & \frac{(b^p-a^p)}{p}\big\{(C_1(p,a,b))^{1-\frac{1}{q}}[C_2(p,a,b)|f'(a)|^q+C_3(p,a,b)|f'(b)|^q]^{\frac{1}{q}} \\ &+(C_1(p,b,a))^{1-\frac{1}{q}}[C_3(p,b,a)|f'(a)|^q+C_2(p,b,a)|f'(b)|^q]^{\frac{1}{q}}\big\}, \end{aligned} \tag{8}$$

where

$$C_1(p,a,b) = \int_0^{\frac{1}{2}}\frac{\big|t-\frac{1}{6}\big|}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\mathrm{d}t \tag{9}$$

$$C_1(p,b,a) = \int_{\frac{1}{2}}^{1}\frac{\big|t-\frac{5}{6}\big|}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\mathrm{d}t \tag{10}$$

$$C_2(p,a,b) = \int_0^{\frac{1}{2}}\frac{\big|t-\frac{1}{6}\big|(1-t)}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\mathrm{d}t \tag{11}$$

$$C_2(p,b,a) = \int_{\frac{1}{2}}^{1}\frac{\big|t-\frac{5}{6}\big|t}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\mathrm{d}t \tag{12}$$

$$C_3(p,a,b) = \int_0^{\frac{1}{2}}\frac{\big|t-\frac{1}{6}\big|t}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\mathrm{d}t \tag{13}$$

$$C_3(p,b,a) = \int_{\frac{1}{2}}^{1}\frac{\big|t-\frac{5}{6}\big|(1-t)}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\mathrm{d}t. \tag{14}$$

Proof. Using Lemma 0.4 and the power mean inequality, we have

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+f(b)\right]-\frac{p}{(b^p-a^p)}\int_a^b\frac{f(x)}{x^{1-p}}\mathrm{d}x\right|\\
\le\ &\frac{(b^p-a^p)}{p}\Bigg[\int_0^{\frac{1}{2}}\frac{|t-\frac{1}{6}|}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\left|f'\big(\big[(1-t)a^p+tb^p\big]^{\frac{1}{p}}\big)\right|\mathrm{d}t\\
&+\int_{\frac{1}{2}}^1\frac{|t-\frac{5}{6}|}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\left|f'\big(\big[(1-t)a^p+tb^p\big]^{\frac{1}{p}}\big)\right|\mathrm{d}t\Bigg]\\
\le\ &\frac{(b^p-a^p)}{p}\Bigg[\left(\int_0^{\frac{1}{2}}\frac{|t-\frac{1}{6}|}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\mathrm{d}t\right)^{1-\frac{1}{q}}\\
&\left(\int_0^{\frac{1}{2}}\frac{|t-\frac{1}{6}|}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\left|f'\big(\big[(1-t)a^p+tb^p\big]^{\frac{1}{p}}\big)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^1\frac{|t-\frac{5}{6}|}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\mathrm{d}t\right)^{1-\frac{1}{q}}\\
&\left(\int_{\frac{1}{2}}^1\frac{|t-\frac{5}{6}|}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\left|f'\big(\big[(1-t)a^p+tb^p\big]^{\frac{1}{p}}\big)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\Bigg]\\
\le\ &\frac{(b^p-a^p)}{p}\Bigg[\left(\int_0^{\frac{1}{2}}\frac{|t-\frac{1}{6}|}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\mathrm{d}t\right)^{1-\frac{1}{q}}\\
&\left(\int_0^{\frac{1}{2}}\frac{|t-\frac{1}{6}|\big[(1-t)|f'(a)|^q+t|f'(b)|^q\big]}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\mathrm{d}t\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^1\frac{|t-\frac{5}{6}|}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\frac{|t-\frac{5}{6}|\big[(1-t)|f'(a)|^q+t|f'(b)|^q\big]}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\mathrm{d}t\right)^{\frac{1}{q}}\Bigg]\\
=\ &\frac{(b^p-a^p)}{p}\Bigg[\left(\int_0^{\frac{1}{2}}\frac{|t-\frac{1}{6}|}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\frac{|t-\frac{1}{6}|(1-t)}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}|f'(a)|^q\mathrm{d}t\right.\\
&\left.+\int_0^{\frac{1}{2}}\frac{|t-\frac{1}{6}|t}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}|f'(b)|^q\mathrm{d}t\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\frac{|t-\frac{5}{6}|}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}\mathrm{d}t\right)^{1-\frac{1}{q}}\\
&\left(\int_{\frac{1}{2}}^1\frac{|t-\frac{5}{6}|(1-t)}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}|f'(a)|^q\mathrm{d}t+\int_{\frac{1}{2}}^1\frac{|t-\frac{5}{6}|t}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}|f'(b)|^q\mathrm{d}t\right)^{\frac{1}{q}}\Bigg]\\
=\ &\frac{(b^p-a^p)}{p}\big\{(C_1(p,a,b))^{1-\frac{1}{q}}[C_2(p,a,b)|f'(a)|^q+C_3(p,a,b)|f'(b)|^q]^{\frac{1}{q}}\\
&+(C_1(p,b,a))^{1-\frac{1}{q}}[C_3(p,b,a)|f'(a)|^q+C_2(p,b,a)|f'(b)|^q]^{\frac{1}{q}}\big\}.
\end{aligned}
$$

□

For appropriate and suitable choice of p and q, one can obtain several new and known results as special cases for various classes of convex functions and their variant forms.

Theorem 0.6. *Let $f : I = [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a,b]$ and $|f'|^q$ is p-convex function on I for $r, q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, then*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a) + 4f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) + f(b)\right] - \frac{p}{(b^p-a^p)}\int_a^b \frac{f(x)}{x^{1-p}}\mathrm{d}x\right| \\
\leq\ & \frac{(b^p-a^p)}{p}\left[(C_4(r,p;a,b))^{\frac{1}{r}}\left(\frac{|f'(a)|^q + |f'([\frac{a^p+b^p}{2}]^{\frac{1}{p}})|^q}{4}\right)^{\frac{1}{q}}\right. \\
&\left. +(C_4(r,p;b,a))^{\frac{1}{r}}\left(\frac{|f'([\frac{a^p+b^p}{2}]^{\frac{1}{p}})|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}}\right], \qquad (15)
\end{aligned}$$

where

$$C_4(r,p;a,b) = \int_0^{\frac{1}{2}} \frac{|t-\frac{1}{6}|^r}{\left[(1-t)a^p+tb^p\right]^{r-\frac{r}{p}}}\mathrm{d}t \qquad (16)$$

$$C_4(r,p;b,a) = \int_{\frac{1}{2}}^1 \frac{|t-\frac{5}{6}|^r}{\left[(1-t)a^p+tb^p\right]^{r-\frac{r}{p}}}\mathrm{d}t. \qquad (17)$$

Proof. Using Lemma 0.4, inequalities (5),(6) and the Holder's integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a) + 4f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) + f(b)\right] - \frac{p}{(b^p-a^p)}\int_a^b \frac{f(x)}{x^{1-p}}\mathrm{d}x\right| \\
\leq\ & \frac{(b^p-a^p)}{p}\left[\int_0^{\frac{1}{2}}\left|\frac{t-\frac{1}{6}}{\left[(1-t)a^p+tb^p\right]^{1-\frac{1}{p}}}\right|\left|f'\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\right|\mathrm{d}t\right. \\
&\left. + \int_{\frac{1}{2}}^1\left|\frac{t-\frac{5}{6}}{\left[(1-t)a^p+tb^p\right]^{1-\frac{1}{p}}}\right|\left|f'\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\right|\mathrm{d}t\right] \\
\leq\ & \frac{(b^p-a^p)}{p}\left[\left(\int_0^{\frac{1}{2}}\left|\frac{t-\frac{1}{6}}{\left[(1-t)a^p+tb^p\right]^{1-\frac{1}{p}}}\right|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{2}}\left|f'\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&\left. +\left(\int_{\frac{1}{2}}^1\left|\frac{t-\frac{5}{6}}{\left[(1-t)a^p+tb^p\right]^{1-\frac{1}{p}}}\right|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_{\frac{1}{2}}^1\left|f'\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right] \\
=\ & \frac{(b^p-a^p)}{p}\left[\left(\int_0^{\frac{1}{2}}\frac{|t-\frac{1}{6}|^r}{\left[(1-t)a^p+tb^p\right]^{r-\frac{r}{p}}}\mathrm{d}t\right)^{\frac{1}{r}}\left(\frac{p}{(b^p-a^p)}\int_a^{\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}}\frac{|f'(x)|^q}{x^{1-p}}\mathrm{d}x\right)^{\frac{1}{q}}\right. \\
&\left. +\left(\int_{\frac{1}{2}}^1\frac{|t-\frac{5}{6}|^r}{\left[(1-t)a^p+tb^p\right]^{r-\frac{r}{p}}}\mathrm{d}t\right)^{\frac{1}{r}}\left(\frac{p}{(b^p-a^p)}\int_{\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}}^b\frac{|f'(x)|^q}{x^{1-p}}\mathrm{d}x\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{(b^p-a^p)}{p}\left[\left(\int_0^{\frac{1}{2}}\frac{|t-\frac{1}{6}|^r}{\left[(1-t)a^p+tb^p\right]^{r-\frac{r}{p}}}\mathrm{d}t\right)^{\frac{1}{r}}\left(\frac{|f'(a)|^q + |f'([\frac{a^p+b^p}{2}]^{\frac{1}{p}})|^q}{4}\right)^{\frac{1}{q}}\right. \\
&\left. +\left(\int_{\frac{1}{2}}^1\frac{|t-\frac{5}{6}|^r}{\left[(1-t)a^p+tb^p\right]^{r-\frac{r}{p}}}\mathrm{d}t\right)^{\frac{1}{r}}\left(\frac{|f'([\frac{a^p+b^p}{2}]^{\frac{1}{p}})|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}}\right] \\
=\ & \frac{(b^p-a^p)}{p}\left[(C_4(r,p;a,b))^{\frac{1}{r}}\left(\frac{|f'(a)|^q + |f'([\frac{a^p+b^p}{2}]^{\frac{1}{p}})|^q}{4}\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+(C_4(r,p;b,a))^{\frac{1}{r}}\left(\frac{|f'\big(\big[\frac{a^p+b^p}{2}\big]^{\frac{1}{p}}\big)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\Bigg].$$

□

If $p = 1$, then, from Theorem 0.6, we have

Corollary 0.7. *Let $f : I = [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, b]$ and $|f'|^q$ is harmonic convex function on I for $r, q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, then*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{(b-a)}\int_a^b f(x)\mathrm{d}x\right| \\
\leq\ & (b-a)\left(\frac{1+2^{r+1}}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}\left[\left(\frac{|f'(a)|^q+|f'\big(\frac{a+b}{2}\big)|^q}{4}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{|f'\big(\frac{a+b}{2}\big)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]. \qquad (18)
\end{aligned}$$

Theorem 0.8. *Let $f : I = [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, b]$ and $|f'|^q$ is p-convex function on I for $r, q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, then*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+f(b)\right]-\frac{p}{(b^p-a^p)}\int_a^b\frac{f(x)}{x^{1-p}}\mathrm{d}x\right| \\
\leq\ & \frac{(b^p-a^p)}{p}\times\left(\frac{1+2^{r+1}}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}\Big[\big(C_5(q,p;a,b)|f'(a)|^q \\
&+C_6(q,p;a,b)|f'(b)|^q\big)^{\frac{1}{q}}+\big(C_6(q,p;b,a)|f'(a)|^q+C_5(q,p;b,a)|f'(b)|^q\big)^{\frac{1}{q}}\Big],
\end{aligned}$$

where

$$\begin{aligned}
C_5(q,p;a,b) &= \int_0^{\frac{1}{2}}\frac{(1-t)}{\big[(1-t)a^p+tb^p\big]^{q-\frac{q}{p}}}\mathrm{d}t \\
C_5(q,p;b,a) &= \int_{\frac{1}{2}}^1\frac{t}{\big[(1-t)a^p+tb^p\big]^{q-\frac{q}{p}}}\mathrm{d}t \\
C_6(q,p;a,b) &= \int_0^{\frac{1}{2}}\frac{t}{\big[(1-t)a^p+tb^p\big]^{q-\frac{q}{p}}}\mathrm{d}t \\
C_6(q,p;b,a) &= \int_{\frac{1}{2}}^1\frac{(1-t)}{\big[(1-t)a^p+tb^p\big]^{q-\frac{q}{p}}}\mathrm{d}t.
\end{aligned}$$

Proof. Using Lemma 0.4 and the Holder's integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+f(b)\right]-\frac{p}{(b^p-a^p)}\int_a^b\frac{f(x)}{x^{1-p}}\mathrm{d}x\right| \\
\leq\ & \frac{(b^p-a^p)}{p}\left[\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|\frac{1}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}f'\Big(\big[(1-t)a^p+tb^p\big]^{\frac{1}{p}}\Big)\right|\mathrm{d}t\right. \\
&\left.+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|\frac{1}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}f'\Big(\big[(1-t)a^p+tb^p\big]^{\frac{1}{p}}\Big)\right|\mathrm{d}t\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(b^p-a^p)}{p}\Bigg[\Bigg(\int_0^{\frac{1}{2}}\big|t-\frac{1}{6}\big|^r\mathrm{d}t\Bigg)^{\frac{1}{r}}\Bigg(\int_0^{\frac{1}{2}}\Bigg|\frac{1}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}f'\big(\big[(1-t)a^p+tb^p\big]^{\frac{1}{p}}\big)\Bigg|^q\mathrm{d}t\Bigg)^{\frac{1}{q}}\\
&\quad+\Bigg(\int_{\frac{1}{2}}^1\big|t-\frac{5}{6}\big|^r\mathrm{d}t\Bigg)^{\frac{1}{r}}\Bigg(\int_{\frac{1}{2}}^1\Bigg|\frac{1}{\big[(1-t)a^p+tb^p\big]^{1-\frac{1}{p}}}f'\big(\big[(1-t)a^p+tb^p\big]^{\frac{1}{p}}\big)\Bigg|^q\mathrm{d}t\Bigg)^{\frac{1}{q}}\Bigg]\\
&= \frac{(b^p-a^p)}{p}\Bigg[\Bigg(\int_0^{\frac{1}{2}}\big|t-\frac{1}{6}\big|^r\mathrm{d}t\Bigg)^{\frac{1}{r}}\Bigg(\int_0^{\frac{1}{2}}\frac{1}{\big[(1-t)a^p+tb^p\big]^{q-\frac{q}{p}}}\Big|f'\big(\big[(1-t)a^p+tb^p\big]^{\frac{1}{p}}\big)\Big|^q\mathrm{d}t\Bigg)^{\frac{1}{q}}\\
&\quad+\Bigg(\int_{\frac{1}{2}}^1\big|t-\frac{5}{6}\big|^r\mathrm{d}t\Bigg)^{\frac{1}{r}}\Bigg(\int_{\frac{1}{2}}^1\frac{1}{\big[(1-t)a^p+tb^p\big]^{q-\frac{q}{p}}}\Big|f'\big(\big[(1-t)a^p+tb^p\big]^{\frac{1}{p}}\big)\Big|^q\mathrm{d}t\Bigg)^{\frac{1}{q}}\Bigg]\\
&\leq \frac{(b^p-a^p)}{p}\Bigg[\Bigg(\int_0^{\frac{1}{2}}\big|t-\frac{1}{6}\big|^r\mathrm{d}t\Bigg)^{\frac{1}{r}}\Bigg(\int_0^{\frac{1}{2}}\frac{1}{\big[(1-t)a^p+tb^p\big]^{q-\frac{q}{p}}}\big[(1-t)|f'(a)|^q+t|f'(b)|^q\big]\mathrm{d}t\Bigg)^{\frac{1}{q}}\\
&\quad+\Bigg(\int_{\frac{1}{2}}^1\big|t-\frac{5}{6}\big|^r\mathrm{d}t\Bigg)^{\frac{1}{r}}\Bigg(\int_{\frac{1}{2}}^1\frac{1}{\big[(1-t)a^p+tb^p\big]^{q-\frac{q}{p}}}\big[(1-t)|f'(a)|^q+t|f'(b)|^q\big]\mathrm{d}t\Bigg)^{\frac{1}{q}}\Bigg]\\
&= \frac{(b^p-a^p)}{p}\Bigg[\Bigg(\frac{1+2^{r+1}}{6^{r+1}(r+1)}\Bigg)^{\frac{1}{r}}\Bigg[\Bigg(\int_0^{\frac{1}{2}}\frac{(1-t)}{\big[(1-t)a^p+tb^p\big]^{q-\frac{q}{p}}}|f'(a)|^q\mathrm{d}t\\
&\quad+\int_0^{\frac{1}{2}}\frac{t}{\big[(1-t)a^p+tb^p\big]^{q-\frac{q}{p}}}|f'(b)|^q\big]\mathrm{d}t\Bigg)^{\frac{1}{q}}+\Bigg(\int_{\frac{1}{2}}^1\frac{(1-t)}{\big[(1-t)a^p+tb^p\big]^{q-\frac{q}{p}}}|f'(a)|^q\mathrm{d}t\\
&\quad+\int_{\frac{1}{2}}^1\frac{t}{\big[(1-t)a^p+tb^p\big]^{q-\frac{q}{p}}}|f'(b)|^q\mathrm{d}t\Bigg)^{\frac{1}{q}}\Bigg]\\
&= \frac{(b^p-a^p)}{p}\times\Bigg(\frac{1+2^{r+1}}{6^{r+1}(r+1)}\Bigg)^{\frac{1}{r}}\big[\big(C_5(q,p;a,b)|f'(a)|^q+C_6(q,p;a,b)|f'(b)|^q\big)^{\frac{1}{q}}\\
&\quad+\big(C_6(q,p;b,a)|f'(a)|^q+C_5(q,p;b,a)|f'(b)|^q\big)^{\frac{1}{q}}\big].
\end{aligned}$$

□

Remark. For appropriate and suitable choice of p and q, one can obtain several new and known results as special cases for various classes of convex functions and their variant forms. If $p=-1$, then our results continue to hold for harmonic convex functions, see [9]. We expect that the interested readers can obtain several results for the special cases of $p-$convex functions.

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