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Hyers–Ulam Stability of Linear Summation Equations

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Abstract. We prove that the homogeneous and non-homogeneous linear Volterra summation equations are Hyers–Ulam stable on $\mathcal{Z}_+$.

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Key Words: Hyers–Ulam stability, Linear operator, Volterra summation equation.

1. INTRODUCTION

Ulam in [23] posed a problem related with the stability of functional equations for homomorphism in 1940: *when an approximate homomorphism from group $\mathcal{G}_1$ to a metric group $\mathcal{G}_2$ can be approximated by an exact homomorphism?* Nearly, for the case where $\mathcal{G}_1$ and $\mathcal{G}_2$ are assumed to be Banach spaces, Hyers [9] brilliantly answered to the question by a direct approach. Aoki [2] and Rassias [19] latter improved the partial answer of Hyers. In fact, the most exciting result was of Rassias [19], who putted more general conditions on the bounds. Recently, Zada et al. studied Hyers–Ulam stability of different functional equations with different approaches [13, 14, 24, 25]. For more details about this area we recommend the book of Jung [10].

To find solutions of equations with continuous time like differential, integral and integro differential equations is a challenging task but Volterra equations provide us a powerful tool to handle such type of problems; e.g., the asymptotic behavior of Volterra equations are studied very well in [17, 21]. Furthermore, for Volterra summation equations the theory of stability via boundedness are studied with the approach of the direct Lyapunov methods [4, 6, 7]. About the solutions (existence and approximation) of Lyapunov summation equations we recommend [1]. While for Volterra summation equations with degenerate Kernels the stability criteria are derived in [5]. The stability problems and conditions in terms of the characteristic equations of some Volterra summation equations are investigated in [11]. On the other hand for the existence of unique solutions of Volterra summation equations weighted norms were utilized in [12, 15]. The problem of asymptotic equivalence in Volterra summation equations has been investigated in [18]. On the other hand the periodic solutions of linear and nonlinear Volterra summation equations of convolution or non-convolution types are studied in [3]. A detailed study on the oscillatory behavior, asymptotic behavior and properties of Volterra equations can be found in [8, 16, 17, 21, 22].

In this note, we study Hyers–Ulam stability of the homogeneous linear Volterra summation equation

$$w_m = \eta \sum_{s=0}^{m} K(m,s)w(s) \tag{1. 1}$$

and non-homogeneous linear Volterra summation equation

$$w_m = f_m + \eta \sum_{s=0}^{m} K(m,s)w(s), \tag{1. 2}$$

where the nucleus $K(m,s)$ of the summation equation and f_m are convergent sequences on the set $\mathcal{Z}_+$, the parameter η is a fixed real constant. Since $K(m,s)$ is convergent on $0 \leq s \leq m$, there exists a positive constant d such that $\|K(m,s)\| \leq d$.

2. Notation and Preliminaries

Here we list some definitions, notation and some tools which would be helpful in deriving our main results. Let $\mathcal{X}$ be a Banach space and $\mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$ denote the space of all bounded linear operators with norm $\|\cdot\|_\infty$ defined by

$$\|f\|_\infty = \max_{m \in \mathcal{Z}_+} \|f_m\|, \quad f \in \mathcal{B}(\mathcal{X}, \mathcal{Z}_+). \tag{2. 3}$$

Definition 2.1. *The summation equation (1.2) is said to have Hyers–Ulam stability on $\mathcal{Z}_+$ if and only if for every sequence $y \in \mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$ satisfying*

$$\left\| y_m - f_m - \eta \sum_{s=0}^{m} K(m,s)y(s) \right\| \leq \epsilon,$$

for all $m \in \mathcal{Z}_+$ and for some $\epsilon \geq 0$, there exists a solution $w \in \mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$ of (1.2) such that

$$\|y - w\|_\infty < M\epsilon,$$

where M is a non-negative constant.

Definition 2.2. *Let $ker\mathcal{W}$ denote the kernel of the bounded linear operator $\mathcal{W} : \Lambda \to \Pi$. We define the induced one to one operator $\hat{\mathcal{W}}$ is a subspaces of $\mathcal{W}$ from $\Lambda/ker(\mathcal{W})$ into Π by $\hat{\mathcal{W}}(w + ker\mathcal{W}) = \mathcal{W}(w)$ for all $w \in \Lambda$.*

Definition 2.3. *Let $\mathcal{W} : \Lambda \to \Pi$ be an operator from space Λ to another space Π. We say that $\mathcal{W}$ has Hyers–Ulam stability if and only if, for any $g \in \mathcal{W}(\Lambda)$ and $f \in \Lambda$ such that $\|\mathcal{W}f - g\|_\infty \le \epsilon$ for some $\epsilon \ge 0$, there exists an $f_0 \in \Lambda$ with $\mathcal{W}f_0 = g$ and $\|f - f_0\|_\infty \le M\epsilon$ where M is non- negative constant. The smallest such M is called the Hyers–Ulam constant.*

We will use the following theorem [20] for summation equation in deriving our main results.

Theorem 2.4. *Let $\mathcal{W}$ be a bounded linear operator from Λ into Π, i.e., $\mathcal{W} : \Lambda \to \Pi$, where Λ and Π are complex Banach spaces. For $\mathcal{W}$ we state the following equivalent statements:*
(1) *$\mathcal{W}$ has the Hyers–Ulam stability.*
(2) *$\mathcal{W}(\Lambda)$ is closed.*
(3) *$\hat{\mathcal{W}}^{-1}$ is a linear operator such that $\|\hat{\mathcal{W}}^{-1}\|_\infty < \infty$.*
Moreover if one of these conditions is true, then $\|\mathcal{W}^{-1}\|_\infty = M$ is the Hyers–Ulam stability constant of $\mathcal{W}$.

Proof. The equivalence of (2) and (3) is well–known. We have to show the equivalence of (1) and (3) by the fact that $\mathcal{W}$ has the Hyers–Ulam stability and by definition of Hyers–Ulam stability.

Another way of stating this definition is:

for any $y \in \Lambda$ we can find a $y_0 \in ker(\mathcal{W})$ $such\ that$ $\|y - y_0\|_\infty \le M\|\mathcal{W}y\|_\infty$. (H)

If this condition holds, then

$$\|y + ker(\mathcal{W})\|_\infty \le M\|\mathcal{W}y\|_\infty,$$

for all $y \in \Lambda$, and hence $\hat{\mathcal{W}}^{-1}$ is bounded and $\|\hat{\mathcal{W}}^{-1}\|_\infty \le M$ which shows that $(1) \Rightarrow (3)$.
Now we have to find $(3) \Rightarrow (1)$. Assume that $\hat{\mathcal{W}}^{-1}$ is bounded and $\|\hat{\mathcal{W}}^{-1}\|_\infty \le L$, for any $y \in \Lambda$ we have

$$\|y + ker(\mathcal{W})\|_\infty = \|\hat{\mathcal{W}}^{-1}(\mathcal{W}y)\|_\infty \le \|\hat{\mathcal{W}}^{-1}\|_\infty \|\mathcal{W}y\|_\infty < L\|\mathcal{W}y\|_\infty,$$

so we can find a $y_0 \in ker(\mathcal{W})$ such that (H) holds, and thus $\mathcal{W}$ has the Hyers–Ulam stability, hence $(3) \Rightarrow (1)$. □

3. Main Results

Now we state our first result, for some bounded positive sequences.

Theorem 3.1. *If the kernel $K(m, s)$ is convergent on $0 \le s \le m$, then (1.1) is Hyers–Ulam stable on $\mathcal{Z}_+$ for all η.*

Proof. Define the operator $\mathcal{W} : \mathcal{B}(\mathcal{X}, \mathcal{Z}_+) \to \mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$ by

$$(\mathcal{W}g)_m = g_m - \eta \sum_{s=0}^{m} K(m,s)g(s), \;\; m \in \mathcal{Z}_+.$$

Clearly, $\mathcal{W}$ is well defined on space $\mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$. Next we have to show that $\mathcal{W}$ is bounded. For this consider

$$\begin{aligned}
\|\mathcal{W}\|_\infty &= \sup_{\|g\|=1} \|\mathcal{W}g\|_\infty \\
&= \sup_{\|g\|=1} \sup_{m\in\mathcal{Z}_+} \left\| g_m - \eta \sum_{s=0}^{m} K(m,s)g(s) \right\| \\
&\leq \sup_{\|g\|=1} \sup_{m\in\mathcal{Z}_+} \left(\|g_m\| + |\eta| \sum_{s=0}^{m} \|K(m,s)\| \|g(s)\| \right) \\
&\leq \sup_{\|g\|=1} \left(\sup_{m\in\mathcal{Z}_+} \|g_m\| + |\eta| \sup_{m\in\mathcal{Z}_+} \sum_{s=0}^{m} \|K(m,s)\| \|g(s)\| \right) \\
&\leq \sup_{\|g\|=1} \left(1 + |\eta| \sum_{s=0}^{m} \sup_{m\in\mathcal{Z}_+} \|K(m,s)\| \right) \|g\|_\infty \quad (\textit{using (2.3)}) \\
&\leq \sup_{\|g\|=1} \left(1 + |\eta| d \sum_{s=0}^{m} \right) \|g\|_\infty \\
&\leq \sup_{\|g\|=1} \big(1 + |\eta| dm \big) \|g\|_\infty \\
&\leq (1 + |\eta| dm) < \infty,
\end{aligned}$$

thus, we can write

$$\|\mathcal{W}\|_\infty < \infty,$$

this shows that $\mathcal{W}$ is bounded. Next we have to show that $\mathcal{W}(\mathcal{B}(\mathcal{X}, \mathcal{Z}_+))$ is closed. As for every sequence $y \in \mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$, there is a sequence $f \in \mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$ such that $\mathcal{W}f = y$. Moreover, $\mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$ is a complex Banach space from which it follows that $\mathcal{W}$ is closed. From Theorem 2.4, we can say that $\mathcal{W}$ has Hyers–Ulam stability, i.e., if for each sequence $g \in \mathcal{W}(\mathcal{B}(\mathcal{X}, \mathcal{Z}_+))$ and $y \in \mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$ we have

$$\|\mathcal{W}y - g\|_\infty \leq \epsilon,$$

for some $\epsilon \geq 0$, then there exists a $w \in \mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$ such that $\mathcal{W}w = g$ and

$$\|y - w\|_\infty \leq M\epsilon,$$

where we can call M by Hyers–Ulam constant of $\mathcal{W}w = g$. Since $0 \in \mathcal{W}(\mathcal{B}(\mathcal{X}, \mathcal{Z}_+))$, therefore, replacing g by 0, the above statement is then read as: if for any $y \in \mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$

$$\left\| y_m - \eta \sum_{s=0}^{m} K(m,s)y(s) \right\| \leq \epsilon,$$

for all $m \in \mathcal{Z}_+$ and for some $\epsilon \geq 0$, then there exists a $w \in \mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$ such that

$$w_m = \eta \sum_{s=0}^{m} K(m,s)w(s),$$

and $\|y - w\|_\infty \leq M\epsilon$ where we can call M as a Hyers–Ulam constant of (1.1). □

By repeating the above process in the same way, one can prove that:

Theorem 3.2. *If the kernel $K(m,s)$ is convergent on $0 \leq s \leq m$ and $f \in \mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$, then (1.2) is Hyers–Ulam stable on $\mathcal{Z}_+$ for all η.*

Proof. Since $f \in \mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$, from the stability of $\mathcal{W}$ it follows that if for any $y \in \mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$

$$\|\mathcal{W}y - f\|_\infty \leq \epsilon,$$

for some $\epsilon \geq 0$, then there exists a $w \in \mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$ such that $\mathcal{W}w = f$ and

$$\|y - w\|_\infty \leq M\epsilon,$$

which implies that if for $y \in \mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$ we have

$$\left\| y_m - f_m - \eta \sum_{s=0}^{m} K(m,s)y(s) \right\| \leq \epsilon,$$

for all $m \in \mathcal{Z}_+$ and for some $\epsilon \geq 0$, then there exists a $w \in \mathcal{B}(\mathcal{X}, \mathcal{Z}_+)$ such that

$$w_m = f_m + \eta \sum_{s=0}^{m} K(m,s)w(s)$$

and

$$\|y - w\|_\infty \leq M\epsilon$$

where we can call M by Hyers–Ulam constant of (1.2). □

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