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The $(2n)^2$-Point Scheme Based on Bivariate Cubic Polynomial

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Abstract. In this paper, we first utilize the least squares strategy for the $(2n)^2$-observations to fit bivariate cubic polynomial for $n \geq 2$. At that point the $(2n)^2$-point approximating subdivision scheme is built. The proposed scheme can be utilized for displaying distinctive items as a part of three dimensional space. It can likewise be utilized for fitting information focuses as a part of 3-dimensional space. Visual exhibitions and applications of the plan have likewise been introduced to demonstrate the execution of the scheme.

AMS (MOS) Subject Classification Codes: 65D10; 65D17; 68W25
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1. Introduction

Computer-Aided Geometric Design (CAGD) is used for the analysis or optimization of design in most branches of industry and since the value of this field is rising every day and plays important role in design of cars, airplanes and electronic systems, as well as in many recent engineering processes. It allows mathematical results to represent curves/surfaces and is often used in computer graphics, engineering and computer algebra.

In computer graphics, a subdivision surface is a method of representing a smooth surface. In late 1970s, subdivision surfaces were introduced and in the fields of computer graphics, solid modeling and CAD, they have captivated much intention. There exists a variety of smoothing techniques in this field, some of which originated for the classical B-splines methods.

The two types of subdivision schemes are approximating and interpolating. No global system of equations demands to be solved so that these schemes are local. For the modeling of curves and surfaces in computer graphics field, subdivision field become a popular method.

A very well known technique, the least squares method is used for figuring of frameworks and to fix data. Nowadays this technique is commonly used for the numerical determination of parameters to estimate statistical characteristics of data and to fit a function to a set of the data. This technique has different types. The simplest type is called ordinary least squares and the advanced one is weighted least squares. Alternating least square and partial least square techniques are new additions to this method.

Fang et al. [2] proposed a generalized B-spline surface subdivision scheme of arbitrary order with a tension parameter. To generate various surfaces of revolution the tensor product subdivision scheme can be used and generated by classical analytic curves that can be exactly represented by generalized B-spline curves. To correct the shape of subdivision surfaces, the tension parameter can be used. Tan et al. [8] proposed a new binary 4-point subdivision scheme. They exposed that the limit curve is at least C^3 continuous by using the sign of subdivision scheme.

Mustafa et al. [3] proposed l_1-regression based subdivision schemes to hold on noisy data properly. This article was about l_1-regression linked with re-weighted least squares for data recovery. Mustafa et al. [4] also modeled the 3-dimensional objects by least square subdivision scheme. Mustafa and Bari [5] also worked on the wide-Ranging families of subdivision schemes for fitting data to subdivide models. By fitting local least squares polynomials, Dyn et al. [1] proposed univariate, stationary, and linear subdivision schemes in initial effort to design subdivision schemes for refining noisy data. Some numerical experiments that express the limit functions developed by these schemes from initial noisy data were also provided.

Shi and Liang [6] proposed a novel integration method to rebuild local surface exactly, the local weighted least squares (LWLS) surface was fitted in background neighborhood which contains sufficient information and the shifting operation was done in a parallel tiny neighborhood which contains corresponding overlapping points. Hence, a simple procedure was designed to describe and merge those corresponding overlapping points. Vilca et al. [7] presented bivariate Birnbaum-Saunders regression model. The dependence structure between observations derived from the bivariate normal distribution can be used to evaluate correlated log-lifetimes of two units by that bivariate regression model. We are going to construct the non-tensor product subdivision scheme by least square approach to design the surface models.

This paper is organized as follows: In Section 2, an algorithm is presented to generate $(2n)^2$-point non-tensor product subdivision scheme. In Section 3, some geometrical shapes have been produced by the proposed schemes. Conclusion and acknowledgement is presented in Section 4.

2. Least squares method to fit bivariate cubic polynomial to $(2n)^2$-observations

The bivariate cubic polynomial to the data $x_r = r$, $y_s = s$, $-n+1 \leqslant r, s \leqslant n$, $n \geqslant 2$ is

$$f(x,y) = \eta_1 + \eta_2 x + \eta_3 y + \eta_4 x^2 + \eta_5 xy + \eta_6 y^2 + \eta_7 x^3 + \eta_8 x^2 y + \eta_9 xy^2 + \eta_{10} y^3. \tag{2. 1}$$

A general bivariate polynomial function with respect to the observations $(x_r = r, y_s = s, f_{r,s})$ is given by

$$f_{r,s} \approx f(r,s) = \eta_1 + \eta_2 r + \eta_3 s + \eta_4 r^2 + \eta_5 rs + \eta_6 s^2 + \eta_7 r^3 + \eta_8 r^2 s + \eta_9 rs^2 + \eta_{10} s^3.$$

Where $f(r,s)$ is observed value and $f_{r,s}$ is exact value. Now we will call least squares approach for the selection of polynomial that minimizes R, i.e. the sum of the squares of differences between $f(r,s)$ and $f_{r,s}$. Differentiating

$$R = \sum_{r=-n+1}^{n} \sum_{s=-n+1}^{n} [f_{r,s} - (\eta_1 + \eta_2 r + \eta_3 s + \eta_4 r^2 + \eta_5 rs + \eta_6 s^2 + \eta_7 r^3 + \eta_8 r^2 s + \eta_9 rs^2 + \eta_{10} s^3]^2,$$

with respect to η_1, η_2,..., η_{10}, setting them to 0, we obtain ten normal equations. After simplifying and solving these system of linear equations, we get the values of unknowns

$$\eta_1 = \sum_{r=-n+1}^{n} \sum_{s=-n+1}^{n} \frac{1}{4n^2(16\,n^4 - 8n^2 + 1)(4n^2-9)}(224n^6 - 240r^2n^4 - 240s^2n^4 - 480rn^4 - 480sn^4 + 840r^3n^2 + 360r^2sn^2 + 360rs^2n^2 + 840s^3n^2 - 16n^4 - 840r^2n^2 - 576rsn^2 - 840s^2n^2 + 1188rn^2 + 1188sn^2 - 210r^3 - 810r^2s - 810rs^2 - 210s^3 - 658n^2 + 585r^2 + 1296rs + 585s^2 - 663r - 663s + 270)f_{r,s},$$

$$\eta_2 = \sum_{r=-n+1}^{n} \sum_{s=-n+1}^{n} \frac{-1}{4(16n^6 - 56n^4 + 49n^2 - 9)n^2(4n^2-1)}(-1440rn^6 + 1680r^3n^4 + 720rs^2n^4 + 480n^6 - 1800r^2n^4 - 432rsn^4 - 360s^2n^4 + 4776rn^4 + 576sn^4 - 3500r^3n^2 - 1080r^2sn^2 - 2340rs^2n^2 - 1668n^4 + 3990r^2n^2 + 2484rsn^2 + 1170s^2n^2 - 4492rn^2 - 1872sn^2 + 770r^3 + 2430r^2s + 1620rs^2 + 1851n^2 - 1965r^2 - 3402rs - 810s^2 + 1981r + 1296s - 663)f_{r,s},$$

$$\eta_3 = \sum_{r=-n+1}^{n}\sum_{s=-n+1}^{n} \frac{-1}{4(16\,n^6-56n^4+49n^2-9)n^2(4n^2-1)}(-1440n^6s+720 r^2sn^4+1680s^3n^4+480n^6-360r^2n^4-432rsn^4-1800s^2n^4+576rn^4+4776sn^4-2340r^2sn^2-1080rs^2n^2-3500s^3n^2-1668n^4+1170r^2n^2+2484rsn^2+3990s^2n^2-1872rn^2-4492sn^2+1620r^2s+2430rs^2+770s^3+1851n^2-810r^2-3402rs-1965s^2+1296r+1981s-663)f_{r,s},$$

$$\eta_4 = \sum_{r=-n+1}^{n}\sum_{s=-n+1}^{n} \frac{-15}{4(16n^6-56n^4+49n^2-9)n^2(4n^2-1)}(16n^6-48r^2n^4-120rn^4-24sn^4+280r^3n^2+72r^2sn^2+40n^4-336r^2n^2-72rsn^2+266 rn^2+78sn^2-70r^3-162r^2s-95n^2+159r^2+162rs-131r-54s+39)f_{r,s},$$

$$\eta_5 = \sum_{r=-n+1}^{n}\sum_{s=-n+1}^{n} \frac{-9}{2(n^2-1)n^2(16n^4-8n^2+1)}(-8rsn^2-6rn^2-6sn^2+30r^2s+30rs^2+8n^2-15r^2-52rs-15s^2+21r+21s-8)f_{r,s},$$

$$\eta_6 = \sum_{r=-n+1}^{n}\sum_{s=-n+1}^{n} \frac{-15}{4(16n^6-24n^4+9n^2-1)n^2(4n^2-9)}(16n^6-48s^2n^4-24rn^4-120sn^4+72rs^2n^2+280s^3n^2+40n^4-72rsn^2-336s^2n^2+78 rn^2+266sn^2-162rs^2-70s^3-95n^2+162rs+159s^2-54r-131s+39)f_{r,s},$$

$$\eta_7 = \sum_{r=-n+1}^{n}\sum_{s=-n+1}^{n} \frac{35}{2(4n^4-5n^2+1)(4n^2-9)n^2}(-6rn^2+10r^3+3n^2-15r^2+11r-3)f_{r,s},$$

$$\eta_8 = \sum_{r=-n+1}^{n}\sum_{s=-n+1}^{n} \frac{45}{2(16n^6-24n^4+9n^2-1)n^2}(-2sn^2+6r^2s+n^2-3r^2-6rs+3r+2\,s-1)f_{r,s},$$

$$\eta_9 = \sum_{r=-n+1}^{n}\sum_{s=-n+1}^{n} \frac{45}{2(4n^4-5n^2+1)n^2(4n^2-1)}(-2rn^2+6rs^2+n^2-6rs-3s^2+2\,r+3\,s-1)f_{r,s},$$

$$\eta_{10} = \sum_{r=-n+1}^{n}\sum_{s=-n+1}^{n} \frac{35}{2n^2(16\,n^6-56\,n^4+49\,n^2-9)}(-6\,sn^2+10\,s^3+3\,n^2-15s^2+11\,s-3)f_{r,s}.$$

Now by substituting the values of η_1, η_2, ..., η_{10} in (2. 1) and after some simplifications, we get the following unique bivariate cubic polynomial.

$$\begin{aligned} f(x,y) &= \frac{1}{\Phi_n}\sum_{r=-n+1}^{n}\sum_{s=-n+1}^{n}\{c_1+(c_2+c_5y+c_9y^2)x+(c_4+c_8y)x^2+c_7x^3+ \\ &\quad c_3y+c_6y^2+c_{10}y^3\}\, f_{r,s}, \end{aligned} \tag{2. 2}$$

where

$$\Phi_n = 256n^{10}-960n^8+1008n^6-340n^4+36n^2,$$

$$\begin{aligned} c_1 &= 224n^8-240n^6r^2-240n^6s^2-480n^6r-480n^6s+840n^4r^3+360n^4r^2s \\ &\quad +360n^4rs^2+840n^4s^3-240n^6-600n^4r^2-576n^4rs-600n^4s^2+1668 \\ &\quad n^4r+1668n^4s-1050n^2r^3-1170n^2r^2s-1170n^2rs^2-1050n^2s^3-642 \\ &\quad n^4+1425n^2r^2+1872n^2rs+1425n^2s^2-1851n^2r-1851n^2s+210r^3+ \\ &\quad 810r^2s+810rs^2+210s^3+928n^2-585r^2-1296rs-585s^2+663r+ \\ &\quad 663s-270, \end{aligned}$$

$$\begin{aligned} c_2 &= 1440n^6r-1680n^4r^3-720n^4rs^2-480n^6+1800n^4r^2+432n^4rs+360n^4 \\ &\quad s^2-4776n^4r-576n^4s+3500n^2r^3+1080n^2r^2s+2340n^2rs^2+1668n^4- \\ &\quad 3990n^2r^2-2484n^2rs-1170n^2s^2+4492n^2r+1872n^2s-770r^3-2430sr^2 \\ &\quad -1620rs^2-1851n^2+1965r^2+3402rs+810s^2-1981r-1296s+663, \end{aligned}$$

$$\begin{aligned} c_3 &= 1440\,n^6s-720n^4r^2s-1680n^4s^3-480n^6+360n^4r^2+432n^4rs+1800n^4 \\ &\quad s^2-576n^4r-4776n^4s+2340n^2r^2s+1080n^2rs^2+3500n^2s^3+1668n^4- \\ &\quad 1170n^2r^2-2484n^2rs-3990n^2s^2+1872n^2r+4492n^2s-1620r^2s-2430 \\ &\quad rs^2-770s^3-1851n^2+810r^2+3402rs+1965s^2-1296r-1981s+663, \end{aligned}$$

$$\begin{aligned} c_4 &= -240n^6+720n^4r^2+1800n^4r+360n^4s-4200n^2r^3-1080n^2r^2s-600n^4 \\ &\quad +5040n^2r^2+1080n^2rs-3990n^2r-1170n^2s+1050r^3+2430r^2s+1425 \\ &\quad n^2-2385r^2-2430rs+1965r+810s-585, \end{aligned}$$

$$\begin{aligned} c_5 &= 576n^4rs+432n^4r+432n^4s-2160n^2r^2s-2160n^2rs^2-576n^4+1080n^2r^2 \\ &\quad +2448n^2rs+1080n^2s^2-2484n^2r-2484n^2s+4860r^2s+4860rs^2+1872 \\ &\quad n^2-2430r^2-8424rs-2430s^2+3402r+3402s-1296, \end{aligned}$$

$$\begin{aligned} c_6 &= -240n^6+720n^4s^2+360n^4r+1800n^4s-1080n^2rs^2-4200n^2s^3-600n^4 \\ &\quad +1080n^2rs+5040n^2s^2-1170n^2r-3990n^2s+2430rs^2+1050s^3+1425 \\ &\quad n^2-2430rs-2385s^2+810r+1965s-585, \end{aligned}$$

$$\begin{aligned} c_7 &= -1680n^4r+2800n^2r^3+840n^4-4200n^2r^2+3500n^2r-700r^3-1050n^2 \\ &\quad +1050r^2-770r+210, \end{aligned}$$

$$\begin{aligned} c_8 &= 360n^4(1-2s)+2160n^2r^2s-1080n^2r^2-2160n^2rs+1080n^2r+2340n^2s \\ &\quad -4860r^2s-1170n^2+2430r^2+4860\,rs-2430r-1620s+810, \end{aligned}$$

$$\begin{aligned} c_9 &= 360n^4(1-2r)+2160n^2rs^2-2160n^2rs-1080n^2s^2+2340n^2r+1080n^2s \\ &\quad -4860rs^2-1170n^2+4860rs+2430s^2-1620r-2430s+810, \end{aligned}$$

and

$$\begin{aligned} c_{10} &= -1680n^4s+2800n^2s^3+840n^4-4200n^2s^2+3500n^2s-700s^3- \\ &\quad 1050n^2+1050s^2-770s+210. \end{aligned}$$

2.1. **Subdivision surface scheme.** Here we will present $(2n)^2$-point approximating subdivision scheme to fit the data as well as for the modeling of distinct objects.
Taking $n=2$ and by evaluating equation (2. 2) at individual points (x,y) =$(\frac{1}{4},\frac{1}{4})$, $(\frac{3}{4},\frac{1}{4})$, $(\frac{1}{4},\frac{3}{4})$, $(\frac{3}{4},\frac{3}{4})$. Finally $f\left(\frac{1}{4},\frac{1}{4}\right)$=$f^{k+1}_{2i,2j}$, $f\left(\frac{3}{4},\frac{1}{4}\right)=f^{k+1}_{2i+1,2j}$, $f\left(\frac{1}{4},\frac{3}{4}\right)=f^{k+1}_{2i,2j+1}$, $f\left(\frac{3}{4},\frac{3}{4}\right)=f^{k+1}_{2i+1,2j+1}$, $f_{r,s}=f^{k}_{i+r,j+s}$, we get 16-point approximating subdivision scheme having four rules to fit a surface by an initial quad mesh.

$$\begin{aligned} f^{k+1}_{2i,2j} &= -\frac{103}{800}f^k_{i-1,j-1}+\frac{233}{1600}f^k_{i-1,j}+\frac{1}{50}f^k_{i-1,j+1}-\frac{293}{3200}f^k_{i-1,j+2}+\frac{233}{1600}f^k_{i,j-1} \\ &\quad +\frac{581}{1600}f^k_{i,j}+\frac{673}{3200}f^k_{i,j+1}+\frac{81}{800}f^k_{i,j+2}+\frac{1}{50}f^k_{i+1,j-1}+\frac{81}{800}f^k_{i+2,j}+\frac{3}{50} \\ &\quad f^k_{i+1,j+1}-\frac{27}{1600}f^k_{i+1,j+2}-\frac{293}{3200}f^k_{i+2,j-1}-\frac{27}{1600}f^k_{i+2,j+1}+\frac{673}{3200}f^k_{i+1,j} \\ &\quad -\frac{51}{1600}f^k_{i+2,j+2}, \end{aligned}$$

$$\begin{aligned} f^{k+1}_{2i+1,2j} &= -\frac{293}{3200}f^k_{i-1,j-1}+\frac{81}{800}f^k_{i-1,j}-\frac{27}{1600}f^k_{i-1,j+1}-\frac{51}{1600}f^k_{i-1,j+2}+ \\ &\quad \frac{1}{50}f^k_{i,j-1}+\frac{673}{3200}f^k_{i,j}+\frac{3}{50}f^k_{i,j+1}-\frac{27}{1600}f^k_{i,j+2}+\frac{233}{1600}f^k_{i+1,j-1}+ \\ &\quad \frac{581}{1600}f^k_{i+1,j}+\frac{673}{3200}f^k_{i+1,j+1}+\frac{81}{800}f^k_{i+1,j+2}-\frac{103}{800}f^k_{i+2,j-1}+\frac{233}{1600} \\ &\quad f^k_{i+2,j}+\frac{1}{50}f^k_{i+2,j+1}-\frac{293}{3200}f^k_{i+2,j+2}. \end{aligned}$$

$$\begin{aligned} f^{k+1}_{2i,2j+1} &= -\frac{293}{3200}f^k_{i-1,j-1}+\frac{1}{50}f^k_{i-1,j}+\frac{233}{1600}f^k_{i-1,j+1}-\frac{103}{800}f^k_{i-1,j+2}+\frac{81}{800} \\ &\quad f^k_{i,j-1}+\frac{673}{3200}f^k_{i,j}+\frac{581}{1600}f^k_{i,j+1}+\frac{233}{1600}f^k_{i,j+2}-\frac{27}{1600}f^k_{i+1,j-1}+\frac{3}{50} \\ &\quad f^k_{i+1,j}+\frac{673}{3200}f^k_{i+1,j+1}+\frac{1}{50}f^k_{i+1,j+2}-\frac{51}{1600}f^k_{i+2,j-1}-\frac{27}{1600}f^k_{i+2,j} \\ &\quad +\frac{81}{800}f^k_{i+2,j+1}-\frac{293}{3200}f^k_{i+2,j+2}, \end{aligned}$$

$$\begin{aligned} f_{2i+1,2j+1}^{k+1} \quad = \quad & -\frac{51}{1600}f_{i-1,j-1}^{k} - \frac{27}{1600}f_{i-1,j}^{k} + \frac{81}{800}f_{i-1,j+1}^{k} - \frac{293}{3200}f_{i-1,j+2}^{k} \\ & -\frac{27}{1600}f_{i,j-1}^{k} + \frac{3}{50}f_{i,j}^{k} + \frac{673}{3200}f_{i,j+1}^{k} + \frac{1}{50}f_{i,j+2}^{k} + \frac{81}{800}f_{i+1,j-1}^{k} \\ & +\frac{673}{3200}f_{i+1,j}^{k} + \frac{581}{1600}f_{i+1,j+1}^{k} + \frac{233}{1600}f_{i+1,j+2}^{k} - \frac{293}{3200}f_{i+2,j-1}^{k} \\ & +\frac{1}{50}f_{i+2,j}^{k} + \frac{233}{1600}f_{i+2,j+1}^{k} - \frac{103}{800}f_{i+2,j+2}^{k}. \end{aligned}$$

where $f_{i,j}^{k+1} = (\frac{i}{2^{k+1}}, \frac{j}{2^{k+1}})$ and $f_{i,j}^{k} = (\frac{i}{2^{k}}, \frac{j}{2^{k}})$ are control points at level $k+1$ and k respectively.

Remark 2.2. *For different values of n (i.e. $n = 3, 4, 5, 6, \ldots$) in the bivariate polynomial (2. 2) and by adopting similar procedure, we get a family of non tensor product approximating subdivision schemes called $(2n)^2$-point scheme based on bivariate cubic polynomial.*

2.3. **Generalization and variants.** Here we offer the speculation and variations of the $(2n)^2$-point approximating subdivision scheme. In the past section, we have minimized the entirety of squares of the errors. A more broad methodology is to minimize the weighted aggregate of the squares of the error assuming control over all information focuses. On the off chance that this entirety is indicated by R and $f(x,y)$ is bivariate cubic polynomial then

$$R = \sum_{r=-n}^{n} \sum_{s=-n}^{n} w_r w_s (f_{r,s} - f(r,s))$$

where w_r and w_s are certain numbers and are called weights. The weights are endorsed by relative exactness of information focuses. On the off chance that all the information focuses are precise, we set $w_r = w_s = 1$ for all r and s.
We can get summed up subdivision plan by minimizing R and utilizing the same methodology received as a part of past section. The diverse varieties of the proposed scheme can likewise be determined by utilizing distinctive degree bivariate polynomials.
The generalization and variations of the proposed plans can likewise be gotten by utilizing different sorts of technique for least squares, for example, iterative weighted least squares, moving least squares, improved moving least square, adaptive moving least square, orthogonal distance fitting, iterative least square and robust moving least squares.

3. Numerical examples

We are going to present the applications of the $(2n)^2$-point approximating subdivision scheme. The scheme is only suitable for the modeling of 3D data lying on the rectangular polygonal mesh. The scheme recursively refines the rectangular polygonal mesh and in the limit it gives smooth and pleasant shape.
There are two major steps to refine the rectangular polygonal mesh.

- Subdivision rule
- Topological rule (i.e. connectivity rule)

The subdivision rules help to insert/introduce new points in the mesh by using the neighboring points. While the topological rules tell us how to connect the newly computed points.

These rules also help to make new faces then guide us to connect new faces to get refine mesh. These two rules are repeatedly apply on the mesh to get denser mesh. Then by applying shading and texturing on the refined mesh, we get smooth 3D model.
A sketch of an object in the form of quadrilateral mesh has been considered. Then this sketch has been iteratively refined by 16-point $(n = 2)$ scheme then finally a smooth object has been obtained. More explanation can be seen in Figure 1.

- The sketch of an object called initial mesh is shown in Figure 1(a). Here points joined by straight lines are called edges. Four edges make a quadrilateral face called regular face. These regular faces make a regular quadrilateral mesh.
- The result after the first iterative step of 16-point approximating subdivision scheme is shown in Figure 1(b). Here we observe that new mesh is denser than the initial mesh.
- The result after the second iterative step of 16-point approximating scheme is shown in Figure 1(c). Here we also observe that new mesh is even more denser than the mesh after first iterative level.
- The refined mesh/limit surface is obtained after sufficiently large number of iterative steps of 16-point scheme as shown in Figure 1(d).

4. Conclusion

The general procedure to develop the family of $(2n)^2$-point approximating subdivision scheme has been introduced. In this procedure, least squares technique has been used as an initial component then subdivision rules have been developed. The well known topological rules have been combined with subdivision rules to get approximating subdivision scheme. The standard procedure has been introduced to implement the subdivision scheme. These schemes are suitable for modeling of the 3D data lying on the rectangular grid/mesh. The validity of the scheme has been checked by an example.

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Authors contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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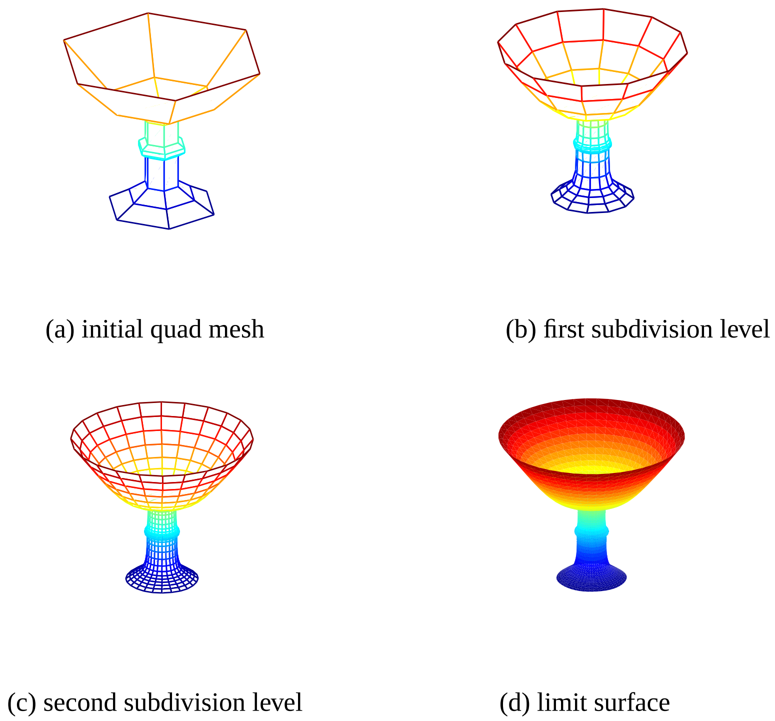

FIGURE 1. *(a) shows the initial mesh and (b)-(c) show the refinement after first and second level whereas (d) shows the limit surface.*

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