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Weighted Average Rating (War) Method for Solving Group Decision Making Problem Using Triangular Cubic Fuzzy Hybrid Aggregation (Tcfha) Operator

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Abstract The motivation behind this artifact is to inspect the strategy to various multiple attribute group decision making with triangular cubic fuzzy numbers, part of operational laws of triangular cubic fuzzy numbers are connected. We concentrate the group decision making problems in which everything the data gave over the chiefs is conveyed as choice structure anywhere the greater part of the components remain described by triangular cubic fuzzy numbers and the data roughly property weights are known. We first utilize the triangular cubic fuzzy hybrid aggregation (TCFHA) administrator to total all individual fuzzy choice structure provide by the decision makers into the aggregate cubic fuzzy decision matrix. Besides, we expend weighted normal rating technique and score function to give a way to deal with positioning the certain choices and choosing the furthermost appealing unique. At last we offer an expressive example.

AMS (MOS) Subject Classification Codes:0000-0003-1049-430X
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Triangular Cubic Fuzzy Hybrid Aggregation Operator, Weighted Average Rating, Score Function.

1. Introduction

The concept of fuzzy set was first proposed by Zadeh [12]. Atanassov [3, 4] initiated the notion of intuitionistic fuzzy set. Intuitionistic fuzzy set has three main parts: membership function, non-membership function and hesitancy function. Li [6] investigated multiple attribute decision making (MADM) with intuitionistic fuzzy information and Lin [7] presented a new method for handling multiple attribute fuzzy decision making problems, where the characteristics of the alternatives are represented by intuitionistic fuzzy sets. Furthermore, the proposed method allows the decision maker to assign the degree of membership and the degree of non-membership of the attribute to the fuzzy concept 'importance'. Wang [10] gave the definition of intuitionistic trapezoidal fuzzy number and interval valued intuitionistic trapezoidal fuzzy number. Wang and Zhang [11] gave the definition of expected values of intuitionistic trapezoidal fuzzy number and proposed the programming method of multi-criteria decision making based on intuitionistic trapezoidal fuzzy number incomplete certain information. Abbas et al. [1] define on the Upper and Lower Contra-Continuous Fuzzy Multifunctions. Akram et al. [2] define the Certain Characterization of m-Polar Fuzzy Graphs by Level Graphs. Hakeem et al. [9] define ON the Fuzzy Infra-Semiopen Sets. T. Mahmood et al. [8] generalized Aggregation Operators for Cubic Hesitant Fuzzy Sets and Their Applications to Multi Criteria Decision.

In [13], exactly ten years later the concept of a fuzzy set, Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set, i.e., a fuzzy set with an interval-valued membership function instead of a real number. Jun et al. [5], initiated the notion of cubic set and investigated some of its properties.

In this paper we focus on the issue of multi attribute decision making under triangular cubic fuzzy environment where all the information provided by the decision makers is characterized by triangular cubic fuzzy numbers and the information about the attribute weights are known. We first use the triangular cubic fuzzy hybrid aggregation (TCFHA) operator to aggregate all individual fuzzy decision matrices provided by the decision makers into the collective cubic fuzzy decision matrix. Next we calculate the weighted average rating by using the aggregated matrix and the given criteria weights. Finally we find the best alternative by using the score function.

This paper is organized as follows: Section 2, we give a review of basic concepts and related properties. Section 3, we presents triangular cubic fuzzy numbers and properties of operational rules. Section 4, presents an algorithm for weighted average rating method for solving group decision making problem using the triangular cubic fuzzy hybrid aggregation operator. Section 5, provides a practical example to illustrate the developed approach and finally, we conclude the paper in Section 6.

2. Basic Concepts

Definition 2.1. [?] *Let H be a universe of discourse. Then the fuzzy set can be defined as: $J = \{h, \tilde{\Omega}_J(h) | h \in H\}$. A fuzzy set in a set H is denoted by $\tilde{\Omega}_J : H \to I$, is a membership function, $\tilde{\Omega}_J(h)$ denoted the degree of membership of the element h to the set H, where*

$I = [0, 1]$. *The gathering of every single fuzzy subset of* H *is indicated by* I^H. *Characterize a connection on* I^H *as follows:* $(\forall \tilde{\Omega}, \eta \in I^H)(\tilde{\Omega} \leq \eta \Leftrightarrow (\forall h \in H)(\tilde{\Omega}(h) \leq \eta(h)))$.

Definition 2.2. [?] *By an interval number we mean a closed subinterval* $\tilde{h} = [\tilde{h}^-, \tilde{h}^+]$ *of* I, *where* $0 \leq \tilde{h}^- \leq \tilde{h}^+ \leq 1$. *The interval number* $\tilde{h} = [\tilde{h}^-, \tilde{h}^+]$ *including* $\tilde{h}^- = \tilde{h}^+$ *is meant by* $\tilde{h}$. *symbolize by* $[I]$ *the set of all interval numbers. Let us define what is known as elaborate minimum (briefly,* $rmin$*), the symbols "* $\succeq$ *","* $\preceq$ *and "=" in case of two elements in* $[I]$. *Consider two interval numbers* $\tilde{h}_1 = [\tilde{h}_1^-, \tilde{h}_1^+]$ *and* $\tilde{h}_2 = [\tilde{h}_2^-, \tilde{h}_2^+]$. *Then*

$$rmin\{\tilde{h}_1, \tilde{h}_2\} = [min\{\tilde{h}_1^-, \tilde{h}_2^-\}, min\{\tilde{h}_1^+, \tilde{h}_2^+\}],$$

$\tilde{h}_1 \succeq \tilde{h}_2$ *if and only if* $\tilde{h}_1^- \geq \tilde{h}_2^-$ *and* $\tilde{h}_1^+ \geq \tilde{h}_2^+$ *and correspondingly we may have* $\tilde{h}_1 \preceq \tilde{h}_2$ *and* $\tilde{h}_1 = \tilde{h}_2$. *To state* $\tilde{h}_1 \succ \tilde{h}_2$ *(resp.* $\tilde{h}_1 \prec \tilde{h}_2$*) and we stingy* $\tilde{h}_1 \succeq \tilde{h}_2$ *and* $\tilde{h}_1 \neq \tilde{h}_2$ *(resp.* $\tilde{h}_1 \preceq \tilde{h}_2$ *and* $\tilde{h}_1 \neq \tilde{h}_2$*). Let* $\tilde{h}_i \in [I]$ *where* $i \in \delta$, *then*

$$r\inf\left(\tilde{h}\right) = [inf^-_{i\in\Lambda}\tilde{h}_i, \overset{+}{\inf_{i\in\delta}}\tilde{h}_i],$$

$$r\sup(\tilde{h}) = [\overset{-}{\sup_{i\in\delta}}\tilde{h}_i, \overset{+}{\sup_{i\in\delta}}\tilde{h}_i].$$

Definition 2.3. [?] *Let* H *is a non-empty set. A function* $A : H \to [I]$ *is called an interval-valued fuzzy set (briefly, an IVF set) in* H. *Let* $[I]^H$ *stand for the set of all IVF sets in* H. *For every* $A \in [I]^H$ *and* $\tilde{h} \in H$, $A(\tilde{h}) = [A^-(\tilde{h}), A^+(\tilde{h})]$ *is called the degree of membership of an element* $\tilde{h}$ *to A, where* $A^- : H \to I$ *and* $A^+ : H \to I$ *are fuzzy sets in* H *which are called a lower fuzzy set and an upper fuzzy set in* H, *respectively. For simpleness, we denote* $A = [A^-, A^+]$. *For every* $A, B \in [I]^H$, *we define* $A \subseteq B \iff A(\tilde{h}) \leq B(\tilde{h})$ *for all* $h \in H$ *and* $A = B \iff A(\tilde{h}) = B(\tilde{h})$ $for\ all$ $\tilde{h} \in H$.

Definition 2.4. [?] *Let* H *is a non-empty set. By a cubic set in* H *we mean a structure* $F = \{h, \tilde{\Omega}(\tilde{h}), \tilde{\varkappa}(\tilde{h}) : h \in H\}$ *in which* $\tilde{\Omega}$ *is an IVF set in* H *and* $\varkappa$ *is a fuzzy set in* H. *A cubic set* $\tilde{F} = \{h, \tilde{\Omega}(\tilde{h}), \tilde{\varkappa}(\tilde{h}) : \tilde{h} \in H\}$ *is simply denoted by* $\tilde{F} = \langle\tilde{\Omega}, \tilde{\varkappa}\rangle$. *Denoted by* C^H *the collection of all cubic sets in* H. *A cubic set* $\tilde{F} = \langle\tilde{\Omega}, \tilde{\varkappa}\rangle$ *in which* $\tilde{\Omega}(\tilde{h}) = 0$ *And* $\tilde{\varkappa}(\tilde{h}) = 1$*(resp.*$\tilde{\Omega}(\tilde{h}) = 1$ *And* $\tilde{\varkappa}(\tilde{h}) = 0$ *for all* $\tilde{h} \in H$ *is denoted by 0 (resp. 1). A cubic set* $\tilde{D} = \langle\tilde{\lambda}, \tilde{\xi}\rangle$ *in which* $\tilde{\lambda}(\tilde{h}) = 0$ *and* $\tilde{\xi}(\tilde{h}) = 0$ *(resp.*$\tilde{\lambda}(\tilde{h}) = 1$ *and* $\tilde{\xi}(\tilde{h}) = 1$*) for all* $\tilde{h} \in H$ *is denoted by 0 (resp. 1).*

Definition 2.5. [?] *Let* H *is a non-empty set. A cubic set* $F = (\tilde{\vartheta}, \tilde{\lambda})$ *in* H *is said to be an internal cubic set if* $\tilde{\vartheta}^-(\tilde{h}) \leq \tilde{\lambda}(\tilde{h}) \leq \tilde{\vartheta}^+(\tilde{h})\ \forall\tilde{h} \in H$.

Definition 2.6. [?] *Let* H *is a non-empty set. A cubic set* $F = (\tilde{\vartheta}, \tilde{\lambda})$ *on* H *is said to be an external cubic set if* $\tilde{\lambda}(\tilde{h}) \notin (\tilde{\vartheta}^-(\tilde{h}), \tilde{\vartheta}^+(\tilde{h}))\ \forall\tilde{h} \in H$.

3. Cubic fuzzy number

Definition 3.1. *The cubic fuzzy set on* H *is a set* $J = \langle\{h, [\Gamma^-(h), \Gamma^+(h)], \eta(h) : h \in H\}\rangle$. *The interval value fuzzy set and fuzzy set,* $[\Gamma_J^-, \Gamma_J^+]$ *and* η_j *are given by respectively* $\Gamma_J^-(h) : h \to [0, 1], h \in H \to\ \Gamma_J^-(h) \in [0, 1]; \Gamma_J^+(h) : h \to [0, 1], h \in H \to\ \Gamma_J^+(h) \in [0, 1]$ *and* $\eta_J(h) : h \to [0, 1], h \in H \to\ \eta_J(h) \in [0, 1]$.

Definition 3.2. *The cubic fuzzy set on H is a set $J = \langle\{h, [\Gamma^-(h), \Gamma^+(h)], \eta_J(h) : h \in H\}\rangle$ is called CF-normal, if there exist at least two points $h_0, h_1 \in H$ such that $\Gamma^-(h_0) = [1,1], \Gamma^+(h_0) = [1,1]$ and $\eta(h_1) = 1$. It is easily seen that given cubic fuzzy set J is CF-normal if there is at least one point that surely belongs to J and at least one point which does not belong to J.*

Definition 3.3. *The cubic fuzzy set $J = \langle\{h, [\Gamma^-(h), \Gamma^+(h)], \eta_J(h) : h \in H\}\rangle$ of the real line is called CF-convex, if $\forall h_1, h_2 \in H, \forall\lambda \in [0,1]$, $[\Gamma_J^-(h), \Gamma_J^+(h)](\lambda h_1 + (1-\lambda)h_2) \geq [\Gamma_J^-(h_1) \wedge \Gamma_J^-(h_2)][\Gamma_J^+(h_1) \wedge \Gamma_J^+(h_2)]$ and $\eta_J(h)(\lambda h_1 + (1-\lambda)h_2) \geq \eta_J(h_1) \wedge \eta_J(h_2)$. Thus J is CF-convex if its membership function is fuzzy convex and its nonmembership function is fuzzy concave.*

Definition 3.4. *The cubic fuzzy set $J = \langle\{h, [\Gamma_J^-(h), \Gamma_J^+(h)], \eta_J(h) : h \in H\}\rangle$ of the real line is called cubic fuzzy number (CFN) if*

(a) *J is CF-normal,*

(b) *J is CF-convex,*

(c) *$[\Gamma_J^-(h), \Gamma_J^+(h)]$ are fuzzy lower semi-continuous and fuzzy upper semi-continuous, $\eta_J(h)$ is fuzzy semi-continuous*

(d) $J = \{h \in H \,|\eta_J(h) < 1\}$ is bounded.

Definition 3.5. *Let $\tilde{b}$ be the triangular cubic fuzzy number on the set of real numbers, its IVTFS are defined as:*

$$\lambda_{\tilde{b}}(h) = \begin{cases} \frac{(h-r)}{(s-r)}[\omega_{b,}^-\omega_b^+] & r \leq h < s \\ \frac{(t-h)}{(t-s)}[\omega_b^-, \omega_b^+] & s \leq h < t \\ 0 & \textit{otherwise} \end{cases}$$

and its TFS is

$$\Gamma_{\tilde{b}}(h) = \begin{cases} \frac{(h-r)\eta_{\tilde{b}}}{(r-s)} & r \leq h < s \\ \frac{(r-h)\eta_{\tilde{b}}}{(t-s)} & s < h \leq t \\ 0 & \textit{otherwise} \end{cases}$$

where $0 \leq \lambda_{\tilde{b}}(h) \leq 1, 0 \leq \Gamma_{\tilde{b}}(h) \leq 1$ and r, s, t are real numbers. The values of $[\omega_{b,}^-\omega_b^+]$ consequently the maximum values of IVTFS and $\eta_{\tilde{b}}$ minimum TFS. Then the $TCFN$ $\widetilde{b}$ basically denoted by $\tilde{b} = [(r,s,t)]; [\langle[\omega_{b,}^-\omega_b^+], \eta_{\tilde{b}}\rangle$. Further, the $TCFN$ reduced to a $TCFN$. Moreover, if $\omega_b^- = 1, \omega_b^+ = 1$ and $\eta_{\tilde{b}} = 0$, if the $TCFN$ $\widetilde{b}$ is called a normal $TCFN$ denoted as $\tilde{b} = [(r,s,t)]; [\langle(1,1)], (0)]\rangle$. Therefore, the TCFN considered now can be regarded as generalized TCFN. Such numbers remand the doubt information in a more flexible approach than normal fuzzy numbers as the values $\omega_b^-, \omega_b^+, \eta_{\tilde{b}} \in [0,1]$ can be interpreted as the degree of confidence in the quantity characterized by r, s, t. Then $\tilde{b}$ is called triangular cubic fuzzy number (TCFN).

Definition 3.6. *Let $\widetilde{b}_1 = \begin{bmatrix} p_1, \\ q_1, \\ r_1 \end{bmatrix}; \left\langle \begin{matrix} [\omega_{b_1}^-, \\ \omega_{b_1}^+] \\ , \eta_{\tilde{b}_1} \end{matrix} \right\rangle$ and $\widetilde{b}_2 = \begin{bmatrix} p_2, \\ q_2, \\ r_2 \end{bmatrix}; \left\langle \begin{matrix} [\omega_{b_2}^-, \\ \omega_{b_2}^+], \\ \eta_{\tilde{b}_2} \end{matrix} \right\rangle$ be two $TCFNs$ and ξ be any real number. The operational rules over $TCFNs$ are solid as under:*

(1) : $\widetilde{b}_1 + \widetilde{b}_2 = [(p_1+p_2),(q_1+q_2),(r_1+r_2)];\langle[\omega^-_{b_1}+\omega^-_{b_2}-\omega^-_{b_1}\omega^-_{b_2}],[\omega^+_{b_1}+\omega^+_{b_2}-\omega^+_{b_1}\omega^+_{b_2}],\{\eta_{\tilde{b}_1},\eta_{\tilde{b}_2}\}\rangle$

(2) : $\widetilde{b}_1 - \widetilde{b}_2 = [(p_1-p_2),(q_1-q_2),(r_1-r_2)];\langle[(\omega^-_{b_1}-\omega^-_{b_2}+\omega^-_{b_1}\omega^-_{b_2}),(\omega^+_{b_1}-\omega^+_{b_2}+\omega^+_{b_1}\omega^+_{b_2})],\{\eta_{\tilde{b}_1},\eta_{\tilde{b}_2}\}\rangle$

(3) : $\widetilde{b}_1 . \widetilde{b}_2 = [(p_1p_2),(q_1q_2),(r_1r_2)];\langle[(\omega^-_{b_1}\omega^-_{b_2}),(\omega^+_{b_1}\omega^+_{b_2})],\{\eta_{\tilde{b}_1}+\eta_{\tilde{b}_2}-\eta_{\tilde{b}_1}\eta_{\tilde{b}_2}\}]\rangle$

(4) : $\xi\tilde{b}_1 = [\xi p_1,\xi q_1,\xi r_1],\langle[1-(1-\omega^-_{b_1})^{\xi}\},1-(1-\omega^+_{b_1})^{\xi}\}],\eta^{\xi}_{\tilde{b}_1}\rangle$ if $\xi > 0$

(5) : $\xi\tilde{b}_1 = [(p_1^{\xi},q_1^{\xi},r_1^{\xi})];\langle[(\omega^-_{b_1})^{\xi},(\omega^+_{b_1^+})^{\xi}],1-(1-\eta_{\tilde{b}_1})^{\xi}]\rangle$ if $\xi < 0$

Let $\widetilde{b}_1 = \begin{bmatrix} 0.2, \\ 0.4, \\ 0.6 \end{bmatrix}; \left\langle \begin{matrix} [0.8, \\ 0.10], \\ 0.9 \end{matrix} \right\rangle$ and $\widetilde{b}_2 = \begin{bmatrix} 0.1, \\ 0.3, \\ 0.5 \end{bmatrix}; \left\langle \begin{matrix} [0.7, \\ 0.9], \\ 0.8 \end{matrix} \right\rangle$ be two $TCFNs$

(1) : $\widetilde{b}_1 + \widetilde{b}_2 = [(0.2+0.1),(0.4+0.3),(0.6+0.5)];\langle[\{(0.8+0.7-(0.8)(0.7))\},\{(0.10+0.9-(0.10)(0.9))\}],\{(0.9)(0.8)\}]\rangle = [0.3,0.7,1.1],\langle[0.94,0.91],0.72\rangle$

(2) : $\widetilde{b}_1 - \widetilde{b}_2 = [(0.2-0.1),(0.4-0.3),(0.6-0.5)];\langle[\{(0.8-0.7+(0.8)(0.7)\},(0.10-0.9+(0.10)(0.9)\}],\{(0.9)(0.8)\}]\rangle$
$= [0.1,0.1,0.1],\langle[0.66,-0.71],0.72\rangle$

(3) : $\widetilde{b}_1 . \widetilde{b}_2 = [(0.2\times 0.1),(0.4\times 0.3),(0.6\times 0.5)];\langle[(0.8\times 0.7),(0.10\times 0.9)],\{0.9+0.8-(0.9)(0.8)\}]\rangle$
$= [0.02,0.12,0.3];\langle[0.56,0.09],0.98\rangle$

(4) : $\xi = 0.44, 0.23, 0.33$
$\xi\tilde{b}_1 = [0.088,0.092,0.198];\langle[0.5074,0.0239],0.9658\rangle$ if $\xi > 0$
(5)$\xi = 0.28, 0.34, 0.38$
$\xi\tilde{b}_1 = [0.6372,0.7323,0.8235];\langle[0.9394,0.4570],0.5831\rangle$ if $\xi < 0$

Definition 3.7. *The triangular cubic fuzzy hybrid aggregation (TCFHA) operator of dimension n is a mapping TCFHA:$\Lambda^n \to \Lambda$, that has an associated vector $\omega = (\omega_1,\omega_2,...,\omega_n)^T$ such that $\omega_j > 0$ and $\sum_{j=1}^{n}\omega_j = 1$. Furthermore TCFHA$(\tilde{b}_1,\tilde{b}_2,...\tilde{b}_n) = \sum_{i=1}^{n}\tilde{b}_{\sigma(j)}\omega_j =$*

$$\left[\sum_{i=1}^{n}(\tilde{p}_{\sigma(j)})\omega_j, \sum_{i=1}^{n}(\tilde{q}_{\sigma(j)})\omega_j, \sum_{i=1}^{n}(\tilde{r}_{\sigma(j)})\omega_j\right],$$

$\langle[1-\prod_{j=1}^{n}(1-w^-_{b_{\sigma(j)}})^{\omega_j};1-\prod_{j=1}^{n}(1-w^+_{b_{\sigma(j)}})^{\omega_j}];[\prod_{j=1}^{n}(\eta_{b_{\sigma(j)}})^{\omega_j}]\rangle$*. Here $\tilde{b}_{\sigma(j)}$ is the jth largest of the weighted triangular fuzzy numbers, $\tilde{b}_{\sigma(j)}(b_j = b_j n\omega_j$, $j = 1,2,...n$, $w = (\Gamma_1,\Gamma_2,...,\Gamma_n)^T$ be the weight vector of b_j, $j = 1,2,...n, \Gamma_j \geq 0$, $\sum_{j=1}^{n}\Gamma_j = 1$ and n is the balancing coefficient. $(\sigma(1),\sigma(2),...,\sigma(n))$ is a permutation of $(1,2,\ldots,n)$, such that $\tilde{b}_{\sigma(j-1)} \geq \tilde{b}_{\sigma(j)}$ for all $j = 2,\ldots,n$.*

Let b_j be the triangular cubic fuzzy number and find the triangular cubic fuzzy hybrid aggregation (TCFHA) operator:

	Table
b_1	$[(0.1, 0.3, 0.5), \langle[(0.7, 0.9)], 0.8)]\rangle$
b_2	$[(0.3, 0.5, 0.7), \langle[(0.2, 0.4)], 0.3)]\rangle$
b_3	$[(0.2, 0.4, 0.6), \langle[(0.10, 0.12)], 0.11)]\rangle$
b_4	$[(0.5, 0.7, 0.9), \langle[(0.12, 0.14)], 0.13)]\rangle$

$w_j = 0.25, 0.25, 0.25, 0.25$

$\tilde{b}_j = [0.275, 0.475, 0.675]; \langle[0.3397, 0.5383], 0.2420\rangle.$

Definition 3.8. *Let $\tilde{b}_j$ is the triangular cubic fuzzy numbers. Then score function S and the accuracy function H of b are defined by follows:*$S(b) = \frac{[p+q+r],\langle\{[\omega_b^-,\omega_b^+]-\eta_b\}\rangle}{9}$ *and* $H(b) = \frac{[p+q+r],\langle\{[\omega_b^-,\omega_b^+]+\eta_b\}\rangle}{9}$ *respectively. It is obvious that* $S(b) \in [1, -1]$ *and* $H(b) \in [0; 1]$ *for any TCFS* b.

	Table
b_1	$[0.2, 0.4, 0.6], \langle[0.55, 0.57], 0.56\rangle$
b_2	$[0.10, 0.12, 0.14], \langle[0.22, 0.24], 0.23\rangle$
b_3	$[0.5, 0.10, 0.15], \langle[0.30, 0.32], 0.31\rangle$
b_4	$[0.20, 0.22, 0.24], \langle[0.60, 0.65], 0.62\rangle$
b_5	$[0.30, 0.40, 0.45], \langle[0.70, 0.80], 0.75\rangle$
b_6	$[0.30, 0.32, 0.34], \langle[0.74, 0.76], 0.75\rangle$

Score function $S(b_1) = \frac{1.2+\{[0.55+0.57]-0.56\}}{9} = \frac{1.2+0.56}{9} = \frac{1.76}{9} = 0.1955;$

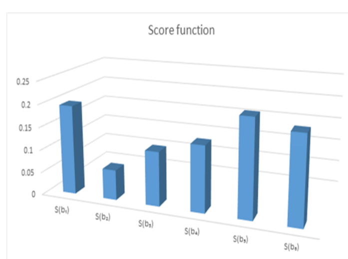
$S(b_2) = \frac{0.36+\{[0.22+0.24]-0.23\}}{9} = \frac{0.36+0.23}{9} = \frac{0.59}{9} = 0.0655;$

$S(b_3) = \frac{0.75+\{[0.30+0.32]-0.31\}}{9} = \frac{0.75+0.31}{9} = \frac{1.06}{9} = 0.1177;$

$S(b_4) = \frac{0.66+\{[0.60+0.65]-0.62\}}{9} = \frac{0.66+0.63}{9} = \frac{1.29}{9} = 0.1433;$

$S(b_5) = \frac{1.15+\{[0.70+0.80]-0.75\}}{9} = \frac{1.2+0.75}{9} = \frac{1.9}{9} = 0.2111;$

$S(b_6) = \frac{0.96+\{[0.74+0.76]-0.75\}}{9} = \frac{0.96+0.75}{9} = \frac{1.71}{9} = 0.19.$



Accuracy function

$H(b_1) = \frac{1.2+\{[0.55+0.57]+0.56\}}{9} = \frac{1.2+0.56}{9} = \frac{2.88}{9} = 0.32;$

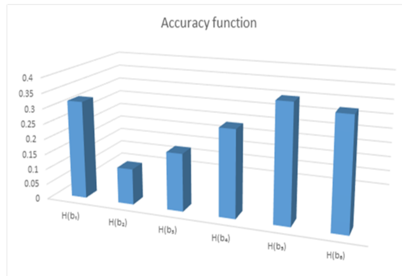
$H(b_2) = \frac{0.36+\{[0.22+0.24]+0.23\}}{9} = \frac{0.36+0.23}{9} = \frac{1.05}{9} = 0.1166;$

$H(b_3) = \frac{0.75+\{[0.30+0.32]+0.31\}}{9} = \frac{0.75+0.31}{9} = \frac{1.68}{9} = 0.1866;$

$H(b_4) = \frac{0.66+\{[0.60+0.65]+0.62\}}{9} = \frac{0.66+0.63}{9} = \frac{2.53}{9} = 0.2811;$

$H(b_5) = \frac{1.15+\{[0.70+0.80]+0.75\}}{9} = \frac{1.15+2.25}{9} = \frac{3.4}{9} = 0.3777;$

$H(b_6) = \frac{0.96+\{[0.74+0.76]+0.75\}}{9} = \frac{0.96+2.25}{9} = \frac{3.21}{9} = 0.3566.$



4. Weighted Average Rating (WAR) Algorithm

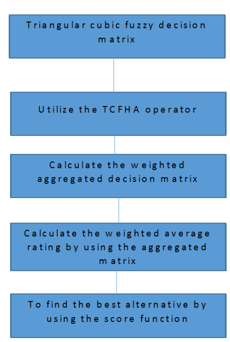
Step 1: Form a triangular cubic fuzzy decision matrix of R_k decision makers.

Step 2: Utilize the TCFHA operator $\tilde{b}_i = [p_i, q_i, r_i, s_i], \langle[\omega^-_{b_i}, \omega^+_{b_i}], \eta_{\tilde{b}_i}\rangle =$ TCFHA$_{v,w}(r^1, r^2, ..., r^n), i = 1, 2, ..., m$ to derive the collective overall preference triangular cubic fuzzy values of $B_i (i = 1, 2, ..., m)$ of the alternative B_i, where $V = (v_1, v_2, \ldots, v_n)$ be the weighting vector of decision makers with v_k in $[0, 1]$, $\sum_{k=1}^{t} v_k = 1, w = (w_1, w_2, ..., w_n)$ is the associated weighting vector of the TCFHA operator, with $\sum_{j=1}^{n} w_k = 1$.

Step 3: Calculate the weighted aggregated decision matrix, using the multiplication formula, $\gamma\tilde{b}_i = [\gamma p_i, \gamma q_i, \gamma r_i], \langle[1-(1-\omega_{b_i^-})^\gamma, 1-(1-\omega_{b_i^+})^\gamma)], \eta^\gamma_{\tilde{b}_i}\rangle$ where b_i is the triangular cubic fuzzy number.

Step 4: Calculate the weighted average rating by using the aggregated matrix and the given criteria weights with help of the formula $D(B_i) = \dfrac{\sum_{j=1}^{n} w_{ij}}{\sum_{j=1}^{n} w_{ij}}, i = 1, 2, , .., m.$

Step 5: To find the best alternative by using the score function.



5. Numerical Example

Let us assume there is a peril deal firm which wants to invest a sum of money in the best option. There is a board with four possible alternatives to invest the money. The peril investment company must take a decision according to the following four attributes.

1. G_1 is the peril analysis,
2. G_2 is the progress analysis,
3. G_3 is the public political impact analysis,
4. G_4 is the Environmental impact analysis.

The four conceivable alternatives $A_i(i = 1, 2, 3, 4)$ are to be assessed using the triangular cubic fuzzy numbers by the three decision makers with their weighting vector $v = (0.30, 0.30, 0.20, 0.20)^T$ under the above four attributes weighting vector $w = (0.2, 0.3, 0.3, 0.2)^T$ and idea the decision matrices $R_k = (r_{ij}^{(k)})$,$(k = 1, 2, 3)$ as follows:

$R_1 =$

C_1	C_2
$\left\{ [0.5, 0.6, 0.7]\,; \langle[0.2, 0.4], 0.3\rangle \right\}$	$\left\{ [0.6, 0.7, 0.8]\,; \langle[0.4, 0.6], 0.5\rangle \right\}$
$\left\{ [0.6, 0.7, 0.8]\,; \langle[0.4, 0.6], 0.5\rangle \right\}$	$\left\{ [0.8, 0.9, 0.10]\,; \langle[0.1, 0.5], 0.3\rangle \right\}$
$\left\{ [0.8, 0.9, 0.10]\,; \langle[0.1, 0.3], 0.2\rangle \right\}$	$\left\{ [0.10, 0.11, 0.12]\,; \langle[0.5, 0.9], 0.7\rangle \right\}$
$\left\{ [0.7, 0.8, 0.9]\,; \langle[0.3, 0.5], 0.4\rangle \right\}$	$\left\{ [0.11, 0.12, 0.13]\,; \langle[0.7, 0.9], 0.8\rangle \right\}$

C_3	C_4
$\left\{ [0.6, 0.7, 0.8]\,; \langle[0.4, 0.6], 0.5\rangle \right\}$	$\left\{ [0.5, 0.6, 0.7]\,; \langle[0.4, 0.6], 0.5\rangle \right\}$
$\left\{ [0.10, 0.11, 0.12]\,; \langle[0.5, 0.8], 0.7\rangle \right\}$	$\left\{ [0.9, 0.10, 0.11]\,; \langle[0.8, 0.10], 0.9\rangle \right\}$
$\left\{ [0.11, 0.12, 0.13]\,; \langle[0.7, 0.10], 0.8\rangle \right\}$	$\left\{ [0.12, 0.13, 0.14]\,; \langle[0.10, 0.12], 0.11\rangle \right\}$
$\left\{ [0.8, 0.9, 0.10]\,; \langle[0.2, 0.4], 0.3\rangle \right\}$	$\left\{ [0.15, 0.16, 0.17]\,; \langle[0.11, 0.13], 0.12\rangle \right\}$

$R_2 =$

C_1	C_2
$\left\{ [0.5, 0.7, 0.9]\,;\ \langle[0.4, 0.8], 0.6\rangle \right\}$	$\left\{ [0.1, 0.2, 0.3]\,;\ \langle[0.4, 0.10], 0.6\rangle \right\}$
$\left\{ [0.7, 0.9, 0.11]\,;\ \langle[0.5, 0.7], 0.6\rangle \right\}$	$\left\{ [0.2, 0.3, 0.4]\,;\ \langle[0.5, 0.7], 0.6\rangle \right\}$
$\left\{ [0.5, 0.7, 0.9]\,;\ \langle[0.4, 0.8], 0.6\rangle \right\}$	$\left\{ [0.3, 0.4, 0.5]\,;\ \langle[0.1, 0.4], 0.2\rangle \right\}$
$\left\{ [0.7, 0.9, 0.11]\,;\ \langle[0.10, 0.12], 0.13\rangle \right\}$	$\left\{ [0.11, 0.12, 0.13]\,;\ \langle[0.3, 0.5], 0.2\rangle \right\}$

C_3	C_4
$\left\{ [0.7, 0.9, 0.11]\,;\ \langle[0.10, 0.12], 0.13\rangle \right\}$	$\left\{ [0.11, 0.12, 0.13]\,;\ \langle[0.3, 0.5], 0.4\rangle \right\}$
$\left\{ [0.2, 0.4, 0.6]\,;\ \langle[0.9, 0.11], 0.10\rangle \right\}$	$\left\{ [0.2, 0.4, 0.6]\,;\ \langle[0.14, 0.16], 0.15\rangle \right\}$
$\left\{ [0.3, 0.5, 0.7]\,;\ \langle[0.2, 0.8], 0.4\rangle \right\}$	$\left\{ [0.11, 0.12, 0.13]\,;\ \langle[0.5, 0.7], 0.6\rangle \right\}$
$\left\{ [0.5, 0.7, 0.9]\,;\ \langle[0.11, 0.14], 0.12\rangle \right\}$	$\left\{ [0.2, 0.4, 0.6]\,;\ \langle[0.15, 0.17], 0.16\rangle \right\}$

$R_3 =$

C_1	C_2
$\left\{ [0.2, 0.3, 0.4]\,;\ \langle[0.8, 0.10], 0.9\rangle \right\}$	$\left\{ [0.4, 0.8, 0.12]\,;\ \langle[0.2, 0.12], 0.8\rangle \right\}$
$\left\{ [0.3, 0.4, 0.5]\,;\ \langle[0.2, 0.4], 0.3\rangle \right\}$	$\left\{ [0.1, 0.2, 0.3]\,;\ \langle[0.4, 0.8], 0.6\rangle \right\}$
$\left\{ [0.4, 0.5, 0.6]\,;\ \langle[0.3, 0.5], 0.4\rangle \right\}$	$\left\{ [0.2, 0.3, 0.4]\,;\ \langle[0.3, 0.5], 0.4\rangle \right\}$
$\left\{ [0.6, 0.7, 0.8]\,;\ \langle[0.9, 0.11], 0.10\rangle \right\}$	$\left\{ [0.6, 0.7, 0.8]\,;\ \langle[0.9, 0.11], 0.10\rangle \right\}$

C_3	C_4
$\left\{ [0.1, 0.2, 0.3]\,;\ \langle 0.6, 0.8], 0.7\rangle \right\}$	$\left\{ [0.2, 0.3, 0.4]\,;\ \langle[0.6, 0.8], 0.7\rangle \right\}$
$\left\{ [0.1, 0.2, 0.3]\,;\ \langle[0.6, 0.8], 0.7\rangle \right\}$	$\left\{ [0.4, 0.6, 0.8]\,;\ \langle[0.8, 0.12], 0.10\rangle \right\}$
$\left\{ [0.4, 0.5, 0.6]\,;\ \langle[0.3, 0.5], 0.4\rangle \right\}$	$\left\{ [0.6, 0.8, 0.10]\,;\ \langle[0.2, 0.4], 0.3\rangle \right\}$
$\left\{ [0.4, 0.5, 0.6]\,;\ \langle[0.5, 0.7], 0.6\rangle \right\}$	$\left\{ [0.1, 0.3, 0.5]\,;\ \langle[0.8, 0.10], 0.9\rangle \right\}$

By step 2 using the TCFHA Operator to aggregate all the three decision matrices into single collective decision matrix with triangular cubic fuzzy ratings. Consider,

$$A=\left\{\begin{array}{l} C_1 \\ \left[0.24,0.32,0.4\right];\langle[0.3742,0.3592],0.6948\rangle \\ \left[0.32,0.4,0.282\right];\langle[0.2483,0.4091],0.6178\rangle \\ \left[0.34,0.42,0.32\right];\langle[0.1768,0.4120],0.5448\rangle \\ \left[0.4,0.48,0.362\right];\langle[0.4247,0.1709],0.3493\rangle \\ C_2 \\ \left[0.33,0.51,0.366\right];\langle[0.3116,0.2916],0.6517\rangle \\ \left[0.33,0.42,0.24\right];\langle[0.3248,0.6507],0.5128\rangle \\ \left[0.18,0.243,0.306\right];\langle[0.2928,0.6507],0.4211\rangle \\ \left[0.246,0.282,0.318\right];\langle[0.6861,0.6068],0.3560\rangle \\ C_3 \\ \left[0.42,0.54,0.363\right];\langle[0.3685,0.5489],0.3959\rangle \\ \left[0.12,0.142,0.306\right];\langle[0.6907,0.6323],0.4046\rangle \\ \left[0.243,0.336,0.429\right];\langle[0.4144,0.5144],0.5397\rangle \\ \left[0.51,0.63,0.48\right];\langle[0.2664,0.4286],0.3164\rangle \\ C_4 \\ \left[0.162,0.204,0.246\right];\langle[0.3000,0.4746],0.7937\rangle \\ \left[0.3,0.22,0.302\right];\langle[0.4903,0.0782],0.4227\rangle \\ \left[0.166,0.21,0.074\right];\langle[0.1848,0.3082],0.4563\rangle \\ \left[0.09,0.172,0.254\right];\langle[0.3134,0.0825],0.4441\rangle \end{array}\right\}$$

Next we calculate the weighted aggregated decision matrix by using step 3, we get $v=(0.30,0.30,0.20,0.20)$

$V =$

C_1	C_2
$[0.072, 0.096, 0.12]; \langle [0.1312, 0.1249], 0.8966 \rangle$	$[0.099, 0.153, 0.1098]; \langle [0.3116, 0.2916], 0.6517 \rangle$
$[0.096, 0.12, 0.0846]; \langle [0.0820, 0.1461], 0.8654 \rangle$	$[0.099, 0.126, 0.072]; \langle [0.1112, 0.2706], 0.8184 \rangle$
$[0.102, 0.126, 0.096]; \langle [0.0566, 0.1472], 0.8334 \rangle$	$[0.054, 0.0729, 0.0918]; \langle [0.0987, 0.2706], 0.7714 \rangle$
$[0.12, 0.144, 0.1086]; \langle 0.1528, 0.0546], 0.7293 \rangle$	$[0.0738, 0.0846, 0.0954]; \langle [0.2936, 0.2442], 0.7335 \rangle$

C_3	C_4
$[0.084, 0.108, 0.0726]; \langle [0.0878, 0.1472], 0.8308 \rangle$	$[0.0324, 0.0408, 0.0492]; \langle [0.0688, 0.1207], 0.9548 \rangle$
$[0.024, 0.0284, 0.0612]; \langle [0.2091, 0.1813], 0.8344 \rangle$	$[0.06, 0.044, 0.0604]; \langle [0.1260, 0.0162], 0.8417 \rangle$
$[0.0486, 0.0672, 0.0858]; \langle [0.1014, 0.1345], 0.8839 \rangle$	$[0.0332, 0.042, 0.0148]; \langle [0.0400, 0.0710], 0.8547 \rangle$
$[0.102, 0.126, 0.096]; \langle [0.9399, 0.1058], 0.7944 \rangle$	$[0.018, 0.0344, 0.0508]; \langle [0.0727, 0.0170], 0.8501 \rangle$

Calculate the weighted average rating for each alternative by using step 4, we get

$D(A_1) = [0.2874, 0.3978, 0.3516]; \langle [0.1470, 0.2283], 0.2196 \rangle$

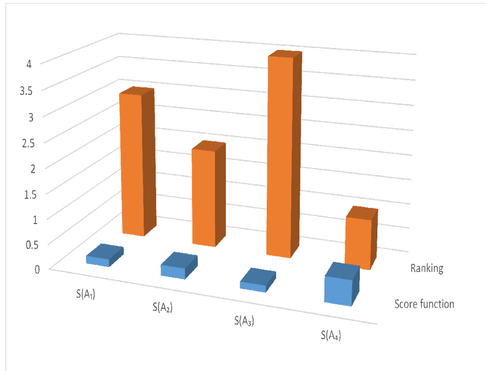
$D(A_2) = [0.279, 0.3184, 0.2782]; \langle [0.1972, 0.2856], 0.2672 \rangle$

$D(A_3) = [0.2378, 0.3081, 0.2884]; \langle [0.1108, 0.2898], 0.2659 \rangle$

$D(A_4) = [0.3138, 0.3890, 0.3508]; \langle [0.5448, 0.1960], 0.2472 \rangle$

To find the ranking order of the alternatives, use the score function,

$S(A_1) = 0.1557, S(A_2) = 0.2156, S(A_3) = 0.1347, S(A_4) = 0.4936.$



6. Conclusion

We have investigated the multi attribute decision making problems under cubic fuzzy environment and developed an approach to handling the situations where the attribute values are characterized by triangular cubic fuzzy numbers and the information about attribute weights are known. The approach first individual cubic fuzzy decision matrices into the collective cubic fuzzy decision matrix by using the triangular cubic fuzzy hybrid aggregation operator, then based on the collective cubic fuzzy decision matrix, we can utilize the weighted average rating method and the score function to get the best alternative.

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