Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 50(2)(2018) pp. 21-32

Unsteady Rotational Flow of Fractional Maxwell Fluid in a Cylinder Subject to Shear Stress on the Boundary

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Received: 19 June, 2017 / Accepted: 04 October, 2017 / Published online: 18 January, 2018

Abstract. In this article the rotational flow of some fractional Maxwell fluid is studied. An infinite straight circular cylinder is filled with the fluid and its motion is generated by a time dependent torsion, applied to the surface of the cylinder. As novelty, the dimensionless governing equation related to the non-trivial shear tension is used and the first exact solutions analogous to a ramped shear stress on the surface are obtained using integral transforms. The obtained results allow us to provide solution for ordinary Maxwell fluid performing similar motion. In addition, the effect of non-integer order paerameter on shear stress and velocity profiles is analyzed by graphical interpretations using Mathcad software.

AMS (MOS) Subject Classification Codes: 58F15, 58F17; 53C35.
Key Words: Maxwell fluid, Caputo derivatives, Velocity field, Shear stress, Circular cylinder.

1. Introduction

Rotating flows due to the shear stress is one of the significant current topics in fluid dynamics because of its useful applications in meteorology, geophysics, turbo machinery and so on [9]. Rotating flows caused by system rotation or swirl flows caused by swirl generators etc. There are many examples of flow near rotating machines, for instance rotating-disc systems are used to model (experimentally and computationally) the flow and heat transfer associated with the internal-air systems of gas turbines, where discs rotate close to a rotating or a stationary surface [10]. Optimum design for the model requires an understanding about the principles of rotating flows, development of core concepts and appropriate solutions of the general understanding. The flow in rotating curved pipes with a constant circular cross-section have found wide applications in heat exchangers, piping systems, electric motors, chemical reactors and many other engineering systems. The flows in rotating pipes have been studied by numerous researchers. Miyazaki [33],[34] analyzed the laminar boundary layer flow and heat transfer in rotating curved pipes of circular and rectangular cross-sections for the case of positive rotation. Ito and Motai [18] studied the flow with negative rotation and found the reversal of the secondary flow for the first time.

During the last few years, various investigations have been made to study the different flow models of non-Newtonian fluids [49], [37], [20], [22], [19], [41], [50], [38], [2], [35], [26]. These fluids are generally used in industry and very much differ in their rheology. Some fluids such as glycerin, crude oil or some polymeric solutions exhibit both viscous and elastic behavior. Such viscoelastic fluid are reffered as Maxwell fluids, the constitutive relation can be recovered from the Jefferys-Oldroyd B fluid by setting retardation time to be zero. Maxwell fluid [6] model is a rate type model and by the reason of the ramification of its governing equations, researchers have given significant consideration and discussed its flow in diverse geometries. Petrov and Cherepanov [39], discussed the flow of viscoplastic fluid in a circular pipe. The unsteady unidirectional transient flow of rate fluid with non-integer order time derivatives, in an annulus region, produced by a fixed pressure gradient and a translation with constant velocity of the inner cylinder was studied by Mathur and Khandelwal [31]. Liu et al [28] studied some helical flows of rate type fluids with non-integer order time derivatives, in a space between two oscillating concentric cylinders and within an oscillating circular cylinder of infinite length. The most existing solutions in the literature correspond to the problems with boundary conditions on the velocity.

However, there are several particular problems with the specified force on the boundary [43], [44]. For example in [43], Renardy has studied the motion of Maxwell fluid across a strip bounded by parallel plates and proved that, to develop a well posed problem it is necessary to impose boundary conditions on the stresses at the inflow boundary. In [44], Renardy explained how well posed boundary value problems can be established using boundary conditions on stresses. Waters and King [52] were among the first specialists who used the shear stress at the boundary to find exact solutions for motions of Maxwell fluids. Other remarkable problems like unsteady unidirectional transient flows of non-Newtonian fluid in unbounded domains which geometrically are axisymmetric pipe-like [12], Jamil et al. [21] establish the results of shear stress for the motion of fluid between concentric cylinders using Hankel transform, the flow of an incompressible electrically conducting couple stress fluid generated by performing longitudinal and torsional oscillations of a porous circular cylinder subjected to constant suction/injection at the surface of the cylinder and in the presence of a radial magnetic field was discuss by Nagarajn et al. [36]. Moreover, axial flow of serval non-Newtonian fluids through circular cylinder are investigated by Vieru et al. [51]. More recently, Fetecau et al. [14], obtained exact solutions for flows of rate type fluids in a circular domain that applies constant couple to the fluid.

Nowadays, fractional calculus modeling in dynamical problems is gaining popularity. For the accurate modeling of physical and engineering processes the non-integer order derivative models and techniques are found to be the best and meticulous to the experimental results [17], [15]. Basically viscoelastic fluid models for example Maxwell, Oldroyd-B etc are the best expressed in terms of non-integer (fractional) order form. Within the context of viscoelasticity the use of non-integer order derivatives was firstly proposed by Germant [16] After that the theory of viscoelasticity in the setting of fractional calculus was further extended by Smith and de Vries [46], Sarwar [45], Yang [53] and Koeller [25] etc. As such, these models are consistent with basic theories and are not arbitrary constructions that happen to describe experimental data. Hence a number of researchers have used fractional calculus as an empirical method of describing the properties of viscoelastic materials. A detailed bibliography is contained in the book by Mainardi [30], including an historical perspective up to 1980's. For further studies see [27],[23], [42], [48], [5], [1], [3], [4].

Many experimental data highlighted that the state of a physical system depends not only upon its current state but, also depends of its history. Because the integer order differential operator is a local operator, the classical fluid models cannot give the best description of the fluids behavior. Since the fractional derivative operators have non-local properties, the fractional calculus has been successfully used in the description of several physical phenomena. Many authers have proposed fractional models obtained from the classical models by replacing the integer derivative operator by the fractional derivative operator [15], [32].

So, the results with ordinary derivative models have marginal scientific value and definitely insufficient to warrant suitable correlation with the experimental data. Moreover, non-integer order derivatives have the elegant property that, if the limit of the fractional parameter tends to integer they coincide with the classic derivative of that order.

Our goal here is to investigate the unsteady flow of Maxwell fluids with non-integer order derivatives through a circular cylinder of infinite length in a rotating frame. In the present paper, the governing equation of the flow is associated to the tension and we considered the boundary conditions on the shear stress as in [14] and [13]. The flow of the fluid is shear driven as the consequence of the circular motion of the cylinder about its axis, under the action of a time dependent shear stress given on the boundary. The obtained solutions which are new in the literature, for the motion of Maxwell fluids with non-integer order derivatives, allow us to recover the corresponding results for ordinary Maxwell fluids. Also the effect of the non-integer order parameters and that of Reynolds number on the profiles of shears tress and fluid's velocity is underlined by graphical illustrations.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

The Cauchy stress tensor $\mathbf{T}$ corresponding to Maxwell fluid [6] is given by the relations

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda\left(\dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T\right) = \mu\mathbf{A}, \tag{2. 1}$$

where p, **I**, **S**, λ, **L**, μ, and **A** are respectively the hydrostatic pressure (SI unit as pascal), identity tensor , the extra-stress tensor (having same unit as force, pressure, strain and density and SI unit as Newton per meter square) pressure, the relaxation time (SI unit as Second), the velocity gradient (per second), the dynamic viscosity (Newton second per meter square) and first Rivlin-Ericksen tensor. The superscript T denotes the transpose operator and the superpose dot denote the material time derivative. The model characteristic by the constitutive equation (1) contains as a special case the Newtonian fluids.

Consider an infinite circular cylinder of radius R. At $t' = 0$, the cylinder and fluid are at rest. After time $t' = 0^+$, the cylinder instigates to turn about its axis as the consequence of a time dependent torque per unit length $2\pi R\tau'(R, t')$, where τ' is the non-trivial shear stress applied to the boundary of the cylinder. We infer that velocity and extra-stress tensor are of the form,

$$\mathbf{V} = \mathbf{V}(r', t') = w'(r', t')\hat{\mathbf{e}}_\theta, \quad \mathbf{S} = \mathbf{S}(r', t'), \tag{2. 2}$$

where $\hat{\mathbf{e}}_\theta$ is unit vector along θ-direction of cylindrical coordinate system. For such a flow the constrain of incompressibility is fulfilled. Since the model is at rest at time $t' = 0$, we have

$$w'(r', 0) = 0, \;\; \mathbf{S}(r', 0) = \mathbf{0}. \tag{2. 3}$$

Introducing (2. 2) in (2. 1) and using (2. 3), we get $S_{r'r'} = S_{r'z} = S_{z\theta} = S_{zz} = 0$ together with the following partial differential equation [14]

$$\left(1 + \lambda\frac{\partial}{\partial t'}\right)\tau'(r', t') = \mu\left(\frac{\partial}{\partial r'} - \frac{1}{r'}\right)w'(r', t'), \tag{2. 4}$$

where $\tau'(r', t') = S_{r'\theta}(r', t')$ is the non zero component of extra stress tensor.
With no body force, the balance of linear momentum, reduces to [14]

$$\rho\frac{\partial w'(r', t')}{\partial t'} = \left(\frac{\partial}{\partial r'} + \frac{2}{r'}\right)\tau'(r', t'), \tag{2. 5}$$

where ρ is the constant density of the fluid.

Driving out $w'(r', t')$ from Eqs. (2. 4) and (2. 5), we extract the subsequent governing equation for the shear stress [14], [13]

$$\left(1 + \lambda\frac{\partial}{\partial t'}\right)\frac{\partial\tau'(r', t')}{\partial t'} = \nu\left(\frac{\partial^2}{\partial r'^2} + \frac{1}{r'}\frac{\partial}{\partial r'} - \frac{4}{r'^2}\right)\tau'(r', t'), \tag{2. 6}$$

where ν is the kinematic viscosity (SI units are meter square per second) of the fluid.

The suitable initial and boundary conditions are

$$\tau'(r', 0) = \frac{\partial\tau'(r', t')}{\partial t'}\,|_{t'=0} = 0, \tag{2. 7}$$

$$\tau'(R, t') = fH(t')\frac{t'^{\delta-1}}{\lambda^{\delta-1}\Gamma(\delta)}; \quad \delta \geq 1, \tag{2. 8}$$

where $H(\cdot)$ is the Heaviside unit step function.
By introducing the following dimensionless quantities $t = \frac{t'}{\lambda}$, $r = \frac{r'}{R}$, $w = \frac{w'}{W_o}$, $W_o = \frac{\lambda f}{\rho R}$, $\tau = \frac{\tau'}{f}$,into the Eqs. (2. 5)–(2. 8) become

$$\frac{\partial w(r, t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau(r, t), \tag{2. 9}$$

$$Re\left(1+\frac{\partial}{\partial t}\right)\frac{\partial\tau(r,t)}{\partial t}=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{4}{r^2}\right)\tau(r,t), \tag{2. 10}$$

with $Re=\frac{R^2}{\lambda\nu}$, the Reynolds number and

$$\tau(r,0)=\frac{\partial\tau(r,t)}{\partial t}\mid_{t=0}=0,\ \ w(r,0)=0, \tag{2. 11}$$

$$\tau(1,t)=H(t)\frac{t^{\delta-1}}{\Gamma(\delta)};\quad t\geq 0,\ \ \delta\geq 1. \tag{2. 12}$$

The corresponding non-integer order model is characterized by

$$Re\,(1+D_t^{\alpha})\,\frac{\partial\tau(r,t)}{\partial t}=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{4}{r^2}\right)\tau(r,t), \tag{2. 13}$$

$$D_t^{\alpha}w(r,t)=(\frac{\partial}{\partial r}+\frac{2}{r})\tau(r,t), \tag{2. 14}$$

subject to the conditions

$$\tau(r,0)=\frac{\partial\tau(r,t)}{\partial t}\mid_{t=0}=0,\ \ w(r,0)=0, \tag{2. 15}$$

$$\tau(1,t)=H(t)\frac{t^{\delta-1}}{\Gamma(\delta)};\quad t\geq 0,\ \ \delta\geq 1, \tag{2. 16}$$

where $D_t^p f(t)=\begin{cases}\frac{1}{\Gamma(1-p)}\int_0^t\frac{f'(s)}{(t-s)^p}ds, & 0\leq p<1;\\ f'(t), & p=1,\end{cases}$ is the Caputo derivative operator with respect to t [7], [8], [24], [40].

3. SOLUTION OF THE PROBLEM

3.1. **CALCULATION FOR SHEAR STRESS.** Implementing Laplace transform [11] to Eq. (2. 13) and utilizing (2. 15)$_1$ and (2. 16), we get

$$\overline{\tau}(r,q)=\frac{1}{Re}\frac{1}{q+q^{\alpha+1}}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{4}{r^2}\right)\overline{\tau}(r,q), \tag{3. 17}$$

$$\overline{\tau}(1,q)=\frac{1}{q^{\delta}}, \tag{3. 18}$$

where $\overline{\tau}(r,q)$ is the Laplace transform of $\tau(r,t)$ and q is the transform variable.

Applying Hankel transform [47] to Eqs. (3. 17) and (3. 18), we obtain

$$\overline{\tau}_H(r_n,q)=\frac{1}{Re}\frac{1}{q+q^{\alpha+1}}\left(-r_nJ_2'(r_n)\overline{\tau}(1,q)-r_n^2\overline{\tau}_H(r_n,q)\right), \tag{3. 19}$$

where $\overline{\tau}_H(r_n,q)=\displaystyle\int_0^1 r\overline{\tau}(r,q)J_2(rr_n)dr$ is the finite Hankel transform of the function $\overline{\tau}(r,q)$ and r_n, $n=1,2,....$ are the positive roots of the transcendental equation $J_2(x)=0$, $J_\nu(\cdot)$ being the Bessel function of first kind of order ν. Equation (3. 19) is equivalent to

$$\left[\frac{Re(q+q^{\alpha+1})+r_n^2}{Re(q+q^{\alpha+1})}\right]\overline{\tau}_H(r_n,q)=-\frac{1}{Re}\frac{1}{q+q^{\alpha+1}}\frac{r_nJ_2'(r_n)}{q^{\delta}}, \tag{3. 20}$$

or

$$\overline{\tau}_H(r_n,q) = -\frac{1}{Re(q+q^{\alpha+1})+r_n^2}\frac{r_nJ_2'(r_n)}{q^\delta}. \tag{3. 21}$$

Using the equivalent expressions

$$\frac{1}{Re(q+q^{\alpha+1})+r_n^2}-\frac{1}{r_n^2} = -\frac{q+q^{\alpha+1}}{r_n^2[(q^{\alpha+1}+A_n)+q]}, \tag{3. 22}$$

where $A_n = \frac{r_n^2}{Re}$ and

$$\frac{1}{Re(q+q^{\alpha+1})+r_n^2}-\frac{1}{r_n^2} = -\frac{q}{r_n^2}\sum_{k=0}^{\infty}(-1)^k\left[\frac{q^{\alpha+k}}{(q^{\alpha+1}+A_n)^{k+1}}+\frac{q^k}{(q^{\alpha+1}+A_n)^{k+1}}\right], \tag{3. 23}$$

into Eq. (3. 21), we get

$$\overline{\tau}_H(r_n,q) = -\frac{J_1(r_n)}{r_nq^\delta}+\frac{J_1(r_n)}{r_n}\sum_{k=0}^{\infty}(-1)^k\times$$
$$\times\left[\frac{q^{\alpha+k+1-\delta}}{(q^{\alpha+1}+A_n)^{k+1}}+\frac{q^{k+1-\delta}}{(q^{\alpha+1}+A_n)^{k+1}}\right]. \tag{3. 24}$$

Implementing the inverse Laplace transform [29] to the last equality, we get

$$\tau_H(r_n,t) = -H(t)\frac{t^{\delta-1}}{\Gamma(\delta)}\frac{J_1(r_n)}{r_n}+H(t)\frac{J_1(r_n)}{r_n}\sum_{k=0}^{\infty}(-1)^k\times$$
$$\times\,[G_{\alpha+1,\,\alpha+k+1-\delta,\,k+1}(-A_n,t)+G_{\alpha+1,\,k+1-\delta,\,k+1}(-A_n,t)], \tag{3. 25}$$

where $G_{a,b,c}(\cdot,t)$ is the generalized Lorenzo Hartley function [29], with $\mathcal{L}^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\} = G_{a,b,c}(d,t)$; Re $(ac-b)>0$, Re$(q)>0$, $\left|\frac{d}{q^a}\right|<1$, and $G_{a,b,c}(d,t) = \sum_{j=0}^{\infty}\frac{d^j\Gamma(c+j)}{\Gamma(c)\Gamma(j+1)}\frac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]}$.

Applying the inverse Hankel transform to Eq. (3. 25) and using the inverse formula

$$\tau(r,t) = 2\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{[J_2'(r_n)]^2}\tau_H(r_n,t),$$

we get

$$\tau(r,t) = H(t)\frac{t^{\delta-1}}{\Gamma(\delta)}r^2+2H(t)\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{r_nJ_1(r_n)}\sum_{k=0}^{\infty}(-1)^k\times$$
$$\times\,[G_{\alpha+1,\,\alpha+k+1-\delta,\,k+1}(-A_n,t)+G_{\alpha+1,\,k+1-\delta,\,k+1}(-A_n,t)]. \tag{3. 26}$$

3.2. **CALCULATION FOR VELOCITY.** Using Eq. (3. 26) into Eq. (2. 14), we obtain the following non-integer order differential equation for velocity

$$D_t^\alpha w(r,t) = 4rH(t)\frac{t^{\delta-1}}{\Gamma(\delta)}+2H(t)\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{J_1(r_n)}\sum_{k=0}^{\infty}(-1)^k\times$$
$$\times\,[G_{\alpha+1,\,\alpha+k+1-\delta,\,k+1}(-A_n,t)+G_{\alpha+1,\,k+1-\delta,\,k+1}(-A_n,t)]. \tag{3. 27}$$

Implementing the Laplace transform to this equation, we have

$$\overline{w}(r,q) = 4r\frac{1}{q^{\delta+\alpha}} + \sum_{n=1}^{\infty}\frac{J_2(rr_n)}{J_1(r_n)}\sum_{k=0}^{\infty}(-1)^k \times \times\left[\frac{q^{k+1-\delta}}{(q^{\alpha+1}+A_n)^{k+1}} + \frac{q^{k-\alpha-\delta+1}}{(q^{\alpha+1}+A_n)^{k+1}}\right], \quad (3.\ 28)$$

with the inverse Laplace transform,

$$w(r,t) = 4r\frac{t^{\delta+\alpha-1}}{\Gamma(\delta+\alpha)} + 2H(t)\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{J_1(r_n)}\sum_{k=0}^{\infty}(-1)^k \times \times\left[G_{\alpha+1,\,k+1-\delta,\,k+1}(-A_n,t) + G_{\alpha+1,\,k-\alpha+1-\delta,\,k+1}(-A_n,t)\right]. \quad (3.\ 29)$$

4. LIMITING CASE

4.1. **ORDINARY MAXWELL FLUID MODEL.** Into Eqs. (3. 26) and (3. 29), letting $\alpha = 1$, we have

$$\tau_M(r,t) = H(t)\frac{t^{\delta-1}}{\Gamma(\delta)}r^2 + 2H(t)\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{J_1(r_n)}\frac{1}{r_n}\sum_{k=0}^{\infty}(-1)^k \times \times[G_{2,\,2+k-\delta,\,k+1}(-A_n,t) + G_{2,\,k+1-\delta,\,k+1}(-A_n,t)], \quad (4.\ 30)$$

$$w_M(r,t) = 4r\frac{t^{\delta+\alpha-1}}{\Gamma(\delta+1)} + 2H(t)\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{J_1(r_n)}\sum_{k=0}^{\infty}(-1)^k \times \times[G_{2,\,k+1-\delta,\,k+1}(-A_n,t) + G_{2,\,k-\delta,\,k+1}(-A_n,t)], \quad (4.\ 31)$$

the expressions of shear stress and velocity for ordinary Maxwell fluid.

5. NUMERICAL RESULTS AND DISCUSSION

In this article, unsteady rotational flows of Maxwell fluid with non-integer order derivatives which fills a straight circular cylinder of radius R and of infinite length are studied. Flows are produced by a time dependent torque applied to the boundary of the cylinder. As novelty, the governing equation related to the dynamic torsion is used. Closed form solutions of dimensionless shear stress and velocity fields are obtained by utilizing integral transforms. These solutions that satisfy all prescribed initial and boundary condition, allow us to contribute the exact solutions for the motion of a rate type fluid produced by the circular cylinder that exerts a constant or time dependent shear stress to the fluid. These solutions can easily be step down to the analogous solutions for ordinary Maxwell fluids as limiting cases. Further, the control of non-integer order parameter on the flow is investigated numerically and by graphical interpretation as follows.

In Fig. 1, we have prepared graphs in order to study the control/infulence of the non-integer order parameter α on the shear stress and fluid velocity for $\delta = 1$ (corresponding to uniform shear stress), $\delta = 2$ and 3 (for time dependent shear stress). The profiles corresponding to shear stress and velocity are plotted versus r for small time and miscellaneous values of the non-integer order parameter α, namely, $\alpha \in \{0.3, 0.5, 0.7\}$ and $Re = 3$. The curves show that the effect of the non-integer order parameter α is significant only near the boundary of the cylinder. It is observed that near the boundary, the shear stress and velocity decreases by increasing values of α. It is also observed that the magnitude of the shear stress and velocity decrease by increase the value of δ(the parameter of the external torsion).

In Fig. 2, we presented the effect of the Reynolds number on the shear stress and velocity of the fluid. The shear stress and velocity profiles are plotted versus r for separate values of Reynolds number Re, namely, $Re \in \{0.5, 1.5, 3.5\}$, and $\alpha = 0.7$. The curves show that the shear stress along with velocity decreases with increasing values of Re, but very near to the cylinder there is a critical value of r after which the velocity increases with the increasing values of Re. It is observed that the magnitude of the shear stress and velocity decreases by increasing the value of δ. In all the graphs, we have chosen the dimensionless material parameter $\lambda = 1.8$ and dimensionless time $t = 0.2$.
Furthermore to approximate the positive roots of the Bessel function $J_2(x) = 0$ we used the computer software Mathcad.

6. CONCLUSION

Some concluding remarks are:

• The effects of the non-integer order parameter α on the fluid motion is significant especially in the vicinity of the cylinder.
• The fluid velocity decreases for increasing values of α on the whole domain.
• The magnitude of the shear stress as well as that of velocity decreases with increasing value of δ.
• The shear stress and velocity admit a maximum value for certain values of δ at each moment of time t. These maximum values increase with the time t. Also, there are values of the δ for which the shear stress and velocity are zero.
• The variation of the shear stress and of velocity with the Reynolds number is small and approaches some constant value for each moment of the time t. It is noted that influence of the Reynolds number on velocity of the fluid and shear stress is significant only for small values of time t.
• Results for Ordinary Maxwell fluid are obtained by using limit $\alpha \to 0$

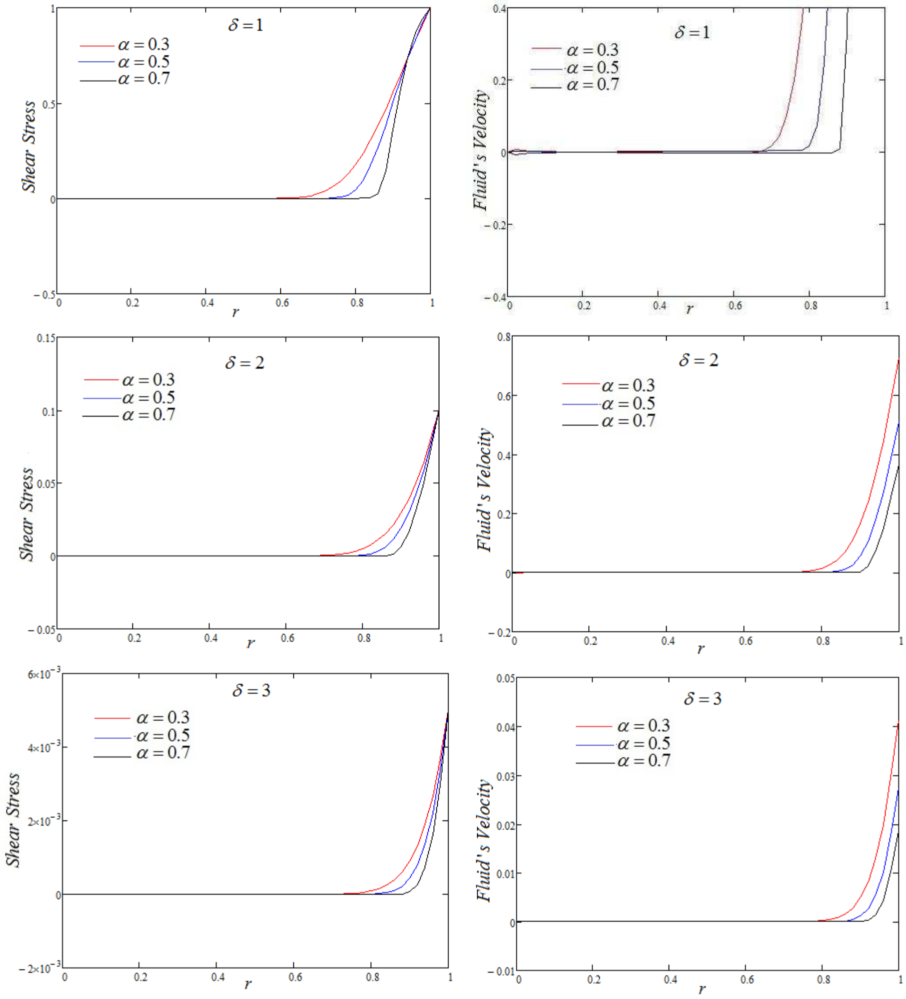

FIGURE 1. Profiles of shear stress and velocity versus r for α variation and different values of δ.

7. ACKNOWLEDGMENTS

The authors are highly thankful and grateful to the department of Mathematics, University of Management and Technology Lahore, Pakistan for supporting and faciliting the research work.

REFERENCES

[1] K. A. Abro and M. A. Solangi, *Heat Transfer in Magnetohydrodynamic Second Grade Fluid with Porous Impacts using Caputo-Fabrizoi Fractional Derivatives*, Punjab Univ. J. Math. Vol. **49**, No. 2 (2017) 113-125

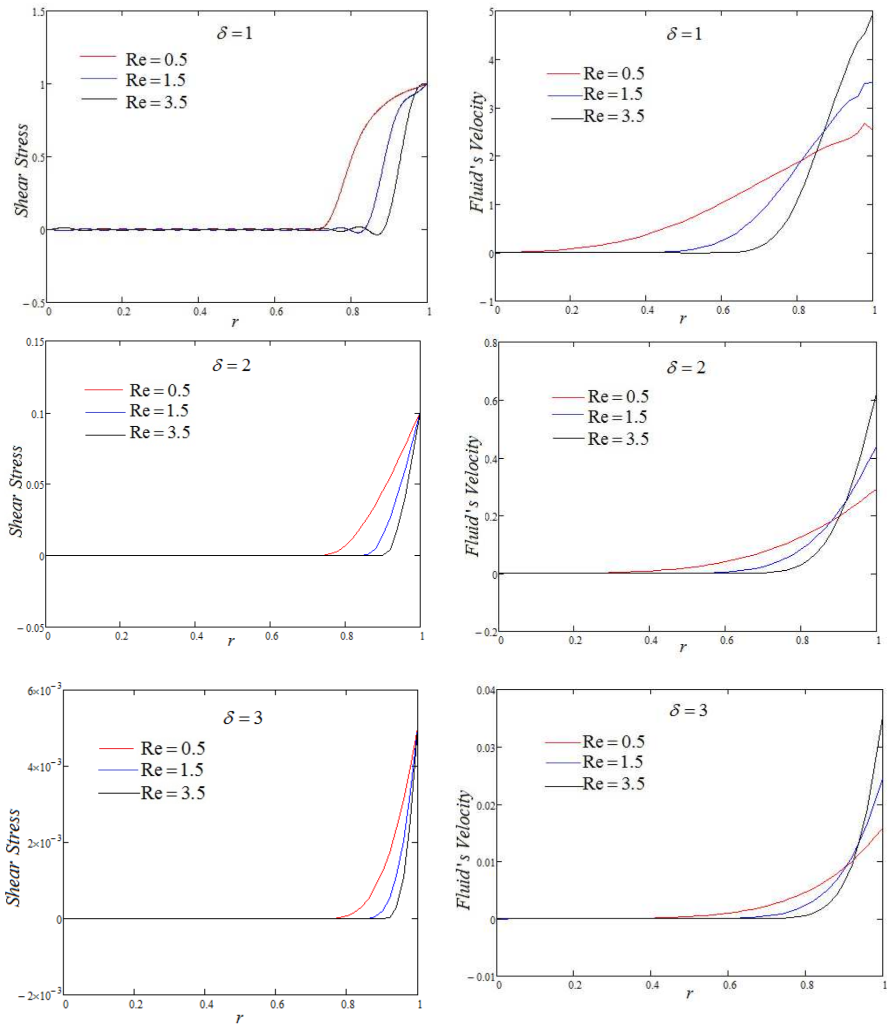

FIGURE 2. Profiles of shear stress and velocity versus r for Re variation and different values of δ.

[2] K. S. Adegbie, A. J. Omowaye, A. B. Disu and I. L. Animasaun, *Heat and Mass Transfer of Upper Convected Maxwell Fluid Flow with Variable Thermo-Physical Properties overa Horizontal Melting Surface*, Applied Mathematics. **6**, (2015) 1362-1379.

[3] A. U. Awan, M. Imran, M. Athar and M. Kamran, *Exact analytical solutions for a longitudinal flow of a fractional Maxwell fluid between two coaxial cylinders*, Punjab Univ. J. Math. Vol. **45**, (2014) 9-23.

[4] A. U. Awan, Rabia Safdar, M. Imran and Aneela Shaukat, *Effects of Chemical Reaction on the Unsteady Flow of an Incompressible Fluid over a Vertical Oscillating Plate*, Punjab Univ. J. Math. Vol. **48**, No. 2 (2016) 167-182.

[5] S. Bashir, M. U. G. Khan, A. A. Shah and A. Nasir, *Towards Computational Model of Human Brain Memory*, Punjab Univ. J. Math. Vol. **46**, No. 2 (2014) 35-45.

[6] R. B. Bird, et al., *Dynamics of Polymeric Liquids*: Vol. **1**, Fluid Mechanics, Wiley, New York, 1987.
[7] M. Caputo, *Elasticita e Dissipazione*, Zanichelli, Bologna 1969.
[8] M. Caputo, *Linear model of dissipation whose Q is almost frequency independent-II*, Geophys. J. R. Astr. Soc. **13**, (1967) 529-539.
[9] P. R. N. Childs, *Rotating Flow*, Butterworth-Heinemann, Oxford 2011.
[10] L. Debnath, *On the hydromagnetic spin-up flows in a rotating fluid*, Lett. Appl. Eng. Sci. **1**, (1973) 451-463.
[11] L. Debnath and D. Bhatta, *Integral transforms and their applications*, 2nd Ed. Capman and Hall/CRC (2007).
[12] C. Fetecau, *Analytical solutions for non-Newtonian fluid flows in pipe like domains*, Int. J. Non-Lin. Mech. **39**, (2004) 225–231.
[13] C. Fetecau, Q. Rubbab, S. Akhter and C. Fetecau, *New methods to provide exact solutions for some unidirectional motions of rate type fluids*, Thermal Sci. **20**, No. 1 (2016) 7-20. doi:10.2298/TSCI130225130F.
[14] Corina Fetecaua, M. Rana, N. Nigar and C. Fetecau, *First exact solutions for flows of rate type fluids in a circular duct that applies a Constant couple to the fluid*, Z. Naturforsch **69**a, (2014) 232–238. $/DOI:$ $10.5560/ZNA.2014-0022$
[15] C. H. R. Friedrich, *Relaxation and retardation functions of the Maxwell model with fractional derivatives*, Reol. Acta. **30**, (1991) 151–158.
[16] A. Germant, *On fractional differentials*, Philosophical Magazine **25**, (1938) 540–549.
[17] A. Heibig and L. I. Palade, *On the rest state stability of an objective fractional derivative viscoelastic fluid model*, Journal of Mathematical Physics, **49**, (2008) 043101-22.
[18] H. Ito and T. Motai, *Secondary flow in a rotating curved pipe*, Rep. Inst. High Speed Mech. **29**, (1974) 33–57.
[19] M. Jamil, *Effects of slip on oscillating fractionalized Maxwell fluid*, Nonlinear Engineering. aop, doi: 10.1515/nleng-2015-0030 (2016).
[20] M. Jamil, C. Fetecau and C. Fetecau, *Unsteady flow of viscoelastic fluid between two cylinders using fractional Maxwell model*. Acta Mech. Sin. **28**, (2012) 274-280.
[21] M. Jamil, C. Fetecau and M. Rana, *Some exact solutions for Oldroyd-B fluid due to time dependent prescribed shear stress*, J. Theor.Appl. Mech. **50**, No. 2 (2012) 549–562.
[22] M. Jamil, K. A. Abro and N. A. Khan, *Helices of fractionalized Maxwell fluid*, Nonlinear Engineering. **4**, (2015) 191-201.
[23] H. Khalil, R. A. Khan, D. Baleanu and M. M. Rashidi, *Some New Operational Matrices and Its Application to Fractional Order Poisson Equations with Integral Type Boundary Constrains*. Computers and Mathematics with Applications, in press.
[24] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Amsterdam, Netherlands: Elsevier 2006.
[25] R. C. Koeller, *Applications of fractional calculus to the theory of viscoelasticity*, J. Appl. Mech. **51**, No. 2 (1984) 299-307.
[26] O. K. Korko, K. S. Adegbie, A. J. Omowaye and I. L. Animasaun, *Boundary layer analysis of upper connected Maxwell fluid flow with variable thermo-physical properties over amelting thermally stratified surface*, FUTA Journal of Research in Sciences. **12**, No. 2 (2016) 287-298.
[27] S. Kumar, D. Kumar, S. Abbasbandy and M. M. Rashidi, *Analytical solution of fractional Navier-Stokes equation by using modified Laplace decomposition method*, Ain Shams Engineering Journal **5**, (2014) 569-574.
[28] Y. Liu, F. Zong and J. Dai, *Unsteady helical flow of a generalized Oldroyd-B fluid with fractional derivative*, Int. J. Math. Trends Technology **5**, (2014) 66–76.
[29] C. F. Lorenzo and T. T. Hartley, *Generalized functions for fractional calculus*, NASA/TP-1999-209424 1999.
[30] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press-World Scientific, London, 2010.
[31] V. Mathur and K. Khandelwal, *Exact solution for the flow of Oldroyd-B fluid between coaxial cylinders*, Int. J. Eng. Res. Technol. (IJERT), **3**, No. 1 (2014) 949–954.
[32] F. C. Meral, T. J. Royston and R. Magin, *Fractional calculus in viscoelasticity: An experimental study*, Commun Nonlinear Sci Numer Simulat. **15**, (2010) 939–945.

[33] H. Miyazaki, *Combined free and force convective heat transfer and fluid flow in rotating curved circular tube*, Int. J. Heat Mass Transfer. **14**, (1971) 1295–1309.

[34] H. Miyazaki, *Combined free and forced convective heat transfer and fluid flow in a rotating curved rectangular tube*, Trans. ASME J. Heat Transfer. Vol. 95, (1973) 64–71.

[35] S. Nadeem, R. U. Haq and Z. Khan, *Numerical study of MHD boundary layer flow of a Maxwell fluid past a stretching sheet in the presence of nanoparticles*, Journal of the Taiwan Institute of Chemical Engineers, **45**, No. 1 (2014) 121-126.

[36] G. Nagaraju and J. V. R. Murthy, MHD flow of longitudinal and torsional oscillations of a circular cylinder with suction in a couple stress fluid, Int. J. Appl. Mech. Eng. Vol. 18(4), (2013) 1099–1114.

[37] F. Olsson and J. Ystrm, *Some properties of the upper convected Maxwell model for viscoelastic fluid flow*, J. Non-Newtonian Fluid Mech. **48**, (1993) 125-145.

[38] A. J. Omowaye and I. L. Animasaun, *Upper-Convected Maxwell Fluid Flow with VariableThermo-Physical Properties over a Melting SurfaceSituated in Hot Environment Subject to ThermalStratification*, Journal of Applied Fluid Mechanics **9**, No. 4 (2016) 1777-1790.

[39] A. G. Petrov and L. V. Cherepanov, *Exact solutions of the problem of unsteady flow of a viscoplastic medium in a circular pipe*, Fluid Dynamics, **38**, (2003) 175–185.

[40] I. Podlubny, *Fractional differential equations*, Academic Press. San Diego1999.

[41] H. T. Qi and J. G. Liu, *Some duct flows of a fractional Maxwell fluid*, Eur. Phys. J. Special Topics. **193**, (2011) 71-79.

[42] N. Raza, *Unsteady Rotational Flow of a Second Grade Fluid with Non-Integer Caputo Time Fractional Derivative*, Punjab Univ. J. Math. Vol. **49**, No. 3 (2017) 15-25.

[43] M. Renardy, *Inflow boundary condition for steady flow of viscoelastic fluids with differential constitutive laws*, Rocky Mount. J. Math. **18**, (1998) 445–453.

[44] M. Renardy *An alternative approach to inflow boundary conditions for Maxwell fluids in three space dimensions*, J. Non-Newtonian Fluid Mech. **36**, (1990) 419–425.

[45] S. Sarwar and M. M. Rashidi, *Approximate solution of two-term fractional-order diffusion, wave-diffusion and telegraph models arising in mathematical physics using optimal homotopy asymptotic method*, Waves in Random and Complex Media. **26**, No. 3 (2016) 365-382.

[46] W. Smith and H. de Vries, *Rheological models containing fractional derivatives*, Rheological Acta. **9**, (1970) 525-534.

[47] I. N. Sneddon, Fourier transforms, McGraw-Hill Book Company, Inc., New York, Toronto, London (1951).

[48] M. Suleman and S. Riaz, *Unconditionally Stable Numerical Scheme to Study the Dynamics of Hepatitis B Disease*, Punjab Univ. J. Math. Vol. **49**, No. 3 (2017) 99-118.

[49] M. Takashima, *The effect of a magnetic field on thermal instability in a layer of Maxwell fluid*. Phys. Lett. A. **33**, (1970) 371-372 .

[50] D. Tripathi, *Peristaltic transport of fractional Maxwell fluids in uniform tubes: Applications in endoscopy*, Comput. Math. Appl. **62**, (2011) 1116-1126.

[51] D. Vieru and I. Siddique, *Axial flow of several non-Newtonian fluids through a circular cylinder*, Int. J. Appl. Mech. **2**, No. 3 (2010) 543–556.

[52] N. D. Waters and M. J. King, *Unsteady flow of an elastico-viscous liquid*, Rheol. Acta. **93**, (1970) 345–355.

[53] A. M. Yang, Y. Z. Zhang, C. Cattani, G. N. Xie, M. M. Rashidi, Y. J. Zhou and X. J. Yang, *Application of Local Fractional Series Expansion Method to Solve Klein-Gordon Equations on Cantor Sets*, Abstract and Applied Analysis, Article ID 372741,(2014) 6 pages.