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Refinements of Hardy-type Integral Inequalities with Kernels

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Abstract. The aim of this paper is to give new refinements of the Hardy-type inequality for arbitrary convex function with different kernels.

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1. Introduction

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a measurable and non-negative kernel, and

$$K(x) = \int_{\Omega_2} k(x,y)\, d\mu_2(y) < \infty,\ x \in \Omega_1. \tag{1. 1}$$

Throughout this paper we suppose $K(x) > 0$ a.e. on Ω_1 and by a weight function (shortly: a weight) we mean a non-negative measurable function on the actual set.

Let $U(k)$ denote the class of measurable functions $g : \Omega_1 \to \mathbb{R}$ with the representation

$$g(x) = \int_{\Omega_2} k(x,y) f(y) d\mu_2(y),$$

where $f : \Omega_2 \to \mathbb{R}$ is a measurable function.

In [16] K. Krulić et al. studied some new weighted Hardy-type inequalities on $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$, measure spaces with σ-finite measures with an integral operator A_k defined by

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y) d\mu_2(y), \tag{1. 2}$$

where $f : \Omega_2 \to \mathbb{R}$ is a measurable function, K be defined by (1.1) and they proved the following result:

THEOREM 1.1. *Let $u : \Omega_1 \to \mathbb{R}$ be a weight function. Assume that $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is locally integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v by*

$$v(y) := \int_{\Omega_1} u(x) \frac{k(x,y)}{K(x)} d\mu_1(x) < \infty. \tag{1. 3}$$

If Φ is a convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} u(x) \Phi(A_k f(x)) d\mu_1(x) \leq \int_{\Omega_2} v(y) \Phi(f(y)) d\mu_2(y) \tag{1. 4}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $Imf \subseteq I$, where A_k is defined by (1.2).

The following refinement of Theorem 1.1 is given in [5].

THEOREM 1.2. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by* (1.1)*. Suppose that the function $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$, and that v is defined on Ω_2 by* (1.3)*. If Φ is convex on the interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int\, I$, then the inequality*

$$\int_{\Omega_2} v(y) \Phi\left(f(y)\right) d\mu_2(y) - \int_{\Omega_1} u(x) \Phi\left(A_k f(x)\right) d\mu_1(x) \geq \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,y)$$
$$\times \left|\, |\Phi(f(y)) - \Phi(A_k f(x))| - |\varphi(A_k f(x))|.|f(y) - A_k f(x)| \,\right| d\mu_2(y)\, d\mu_1(x) \tag{1. 5}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $f(y) \in I$, for all fixed $y \in \Omega_2$ where A_k is defined by (1.2).

S. Iqbal et al. in their recent paper [8] proved an inequality for arbitrary convex function with some applications for fractional integrals and fractional derivatives. Also, recently Čižmešija et al. proved new Hardy-type inequalities and their refinements (see [5], [6], [7]). Now the area of fractional integrals and derivatives is investigated a lot (see [9]–[14]). This book [17] contains a lot of information concerning convex functions and related inequalities.

Here we want to give new improved results. For this purpose if we substitute $k(x,y)$ by $k(x,y)f_2(y)$ and f by f_1/f_2, where $f_i : \Omega_2 \to \mathbb{R}, (i = 1, 2)$ are measurable functions in Theorem 1.1 and Theorem 1.2 we obtain the following result.

THEOREM 1.3. *Let* $(\Omega_1, \Sigma_1, \mu_1)$ *and* $(\Omega_2, \Sigma_2, \mu_2)$ *be measure spaces with* σ*-finite measures,* u *be a weight function on* Ω_1 *and* k *be a non-negative measurable function on* $\Omega_1 \times \Omega_2$*. Suppose that the function* $x \mapsto u(x)\frac{k(x,y)}{g_2(x)}$ *is integrable on* Ω_1 *for each fixed* $y \in \Omega_2$*, and that* v *is defined on* Ω_2 *by*

$$v(y) := f_2(y) \int\limits_{\Omega_1} \frac{u(x)k(x,y)}{g_2(x)} d\mu_1(x) < \infty.$$

If Φ *is convex on the interval* $I \subseteq \mathbb{R}$*, and* $\varphi : I \to \mathbb{R}$ *is any function, such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in Int\, I$*, then the inequality*

$$\int\limits_{\Omega_2} v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right) d\mu_2(y) - \int\limits_{\Omega_1} u(x)\Phi\left(\frac{g_1(x)}{g_2(x)}\right) d\mu_1(x) \geq \int\limits_{\Omega_1} \frac{u(x)}{g_2(x)} \int\limits_{\Omega_2} k(x,y) f_2(y)$$
$$\times \left| \left|\Phi\left(\frac{f_1(y)}{f_2(y)}\right) - \Phi\left(\frac{g_1(x)}{g_2(x)}\right)\right| - \left|\varphi\left(\frac{g_1(x)}{g_2(x)}\right)\right| \cdot \left|\frac{f_1(y)}{f_2(y)} - \frac{g_1(x)}{g_2(x)}\right| \right| d\mu_2(y)\, d\mu_1(x) \tag{1. 6}$$

holds for all measurable functions $f_i : \Omega_2 \to \mathbb{R}$*, such that* $\frac{f_1(y)}{f_2(y)} \in I$*.*

Let us emphasize on the next remark that connects our central result to [8, Theorem 2.1].

REMARK 1. If we take $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$ and $d\mu_2(y) = dy$ the inequality (1. 6) becomes the refinement of the inequality given in [8, Theorem 2.1].

This paper is organized in the following way: After introduction in Section 2, we give new refinements of the Hardy-type inequality for fractional integral of a function with respect to another increasing function, Riemann-Liouville, Hadamard-type and Erdélyi-Kober-type fractional integrals. In Section 3, we give refinements for fractional derivative of Riemann-Liouville, Canavati and Caputo-type involving an arbitrary convex function.

2. REFINEMENTS OF HARDY-TYPE INEQUALITIES FOR FRACTIONAL INTEGRALS

We recall the definition of the *fractional integrals of a function* f *with respect to given function* g. For details see e.g. [15, p. 99].

Let (a,b), $-\infty \leq a < b \leq \infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let g be an increasing function on $(a,b]$ and g' be a continuous function on (a,b). The left-sided fractional integral of a function f with respect to another function g in $[a,b]$ is given by

$$(I^{\alpha}_{a+;g} f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x (g(x) - g(y))^{\alpha-1} g'(y) f(y) dy, \qquad (x > a). \tag{2. 7}$$

Our first result about the fractional integrals is given in the following theorem.

THEOREM 2.1. *Let* u *be a weight function on* (a,b)*,* g *be an increasing function on* $(a,b]$ *such that* g' *be a continuous function on* (a,b) *and* $\alpha > 0$*.* $I^{\alpha}_{a+;g} f$ *denotes the left-sided fractional integral of a function* f *with respect to another function* g *in* $[a,b]$*. Define* v *on* (a,b) *by*

$$v(y) := \frac{f_2(y)}{\Gamma(\alpha)} \int\limits_y^b u(x) \frac{g'(y)(g(x) - g(y))^{\alpha-1}}{(I^{\alpha}_{a+;g} f_2)(x)} dx < \infty.$$

If Φ is convex on the interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int\, I$, then the following inequality holds:

$$\int_a^b v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)dy - \int_a^b u(x)\Phi\left(\frac{(I^\alpha_{a+;g}f_1)(x)}{(I^\alpha_{a+;g}f_2)(x)}\right)dx \geq \frac{1}{\Gamma(\alpha)}\int_a^b \frac{u(x)}{(I^\alpha_{a+;g}f_2)(x)} \times \int_a^x \frac{g'(y)f_2(y)}{(g(x)-g(y))^{1-\alpha}} \left| \left|\Phi\left(\frac{f_1(y)}{f_2(y)}\right) - \Phi\left(\frac{(I^\alpha_{a+;g}f_1)(x)}{(I^\alpha_{a+;g}f_2)(x)}\right)\right| - \left|\varphi\left(\frac{(I^\alpha_{a+;g}f_1)(x)}{(I^\alpha_{a+;g}f_2)(x)}\right)\right| \cdot \left|\frac{f_1(y)}{f_2(y)} - \frac{(I^\alpha_{a+;g}f_1)(x)}{(I^\alpha_{a+;g}f_2)(x)}\right|\right| dy\,dx. \quad (2.\ 8)$$

Proof. Applying Theorem 1.3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_1(y) = dy$,

$$k(x,y) = \begin{cases} \frac{1}{\Gamma(\alpha)}\frac{g'(y)}{(g(x)-g(y))^{1-\alpha}}, & a \leq y \leq x; \\ 0, & x < y \leq b, \end{cases}$$

and replacing g_i by $I^\alpha_{a+;g}f_i$, $(i = 1,2)$, we obtain (2.8). ■

REMARK 2. Since right-hand side of inequality (2. 8) is non-negative, we obtain the following inequality:

$$\int_a^b u(x)\Phi\left(\frac{(I^\alpha_{a+;g}f_1)(x)}{(I^\alpha_{a+;g}f_2)(x)}\right)dx \leq \int_a^b v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)dy \quad (2.\ 9)$$

In particular for the weight function $u(x) = g'(x)I^\alpha_{a+;g}f_2(y)$, we obtain $v(y) = \frac{1}{\Gamma(\alpha+1)}f_2(y)g'(y)(g(b)-g(y))^\alpha$ and (2.9) reduces to

$$\int_a^b g'(x)(I^\alpha_{a+;g}f_2)(x)\Phi\left(\frac{(I^\alpha_{a+;g}f_1)(x)}{(I^\alpha_{a+;g}f_2)(x)}\right)dx \leq \frac{1}{\Gamma(\alpha+1)}\int_a^b g'(y)(g(b)-g(y))^\alpha f_2(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)dy. \quad (2.\ 10)$$

Since g is an increasing function, $(g(b)-g(y))^\alpha \leq (g(b)-g(a))^\alpha$, (2.10) becomes

$$\int_a^b g'(x)(I^\alpha_{a+;g}f_2)(x)\Phi\left(\frac{(I^\alpha_{a+;g}f_1)(x)}{(I^\alpha_{a+;g}f_2)(x)}\right)dx \leq \frac{(g(b)-g(a))^\alpha}{\Gamma(\alpha+1)}\int_a^b g'(y)f_2(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)dy. \quad (2.\ 11)$$

For $g(x) = x$ in (2. 7), we obtain the well known left-sided Riemann-Liouville fractional integral of order $\alpha > 0$ which is defined by

$$I^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}f(y)dy, \quad (x > a),$$

and we obtain the related result for Riemann-Liouville fractional integral in upcoming corollary.

COROLLARY 2.1. *Let u be a weight function on (a,b) and $\alpha > 0$. $I^{\alpha}_{a^+}f$ denotes the Riemann-Liouville fractional integral of f. Define v on (a,b) by*

$$v(y) := \frac{f_2(y)}{\Gamma(\alpha)} \int_y^b \frac{u(x)(x-y)^{\alpha-1}}{I^{\alpha}_{a^+}f_2(x)} dx < \infty.$$

If Φ is convex on the interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int\, I$, then the following inequality holds:

$$\int_a^b v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right) dy - \int_a^b u(x)\Phi\left(\frac{I^{\alpha}_{a^+}f_1(x)}{I^{\alpha}_{a^+}f_2(x)}\right) dx \geq \frac{1}{\Gamma(\alpha)} \int_a^b \frac{u(x)}{I^{\alpha}_{a^+}f_2(x)} \int_a^x (x-y)^{\alpha-1}$$
$$\times f_2(y) \left| \left| \Phi\left(\frac{f_1(y)}{f_2(y)}\right) - \Phi\left(\frac{I^{\alpha}_{a^+}f_1(x)}{I^{\alpha}_{a^+}f_2(x)}\right) \right| - \left| \varphi\left(\frac{I^{\alpha}_{a^+}f_1(x)}{I^{\alpha}_{a^+}f_2(x)}\right) \right| \cdot \left| \frac{f_1(y)}{f_2(y)} - \frac{I^{\alpha}_{a^+}f_1(x)}{I^{\alpha}_{a^+}f_2(x)} \right| \right| dy\, dx. \tag{2. 12}$$

REMARK 3. Inequality (2. 12) is a refinement of inequality

$$\int_a^b u(x)\Phi\left(\frac{I^{\alpha}_{a^+}f_1(x)}{I^{\alpha}_{a^+}f_2(x)}\right) dx \leq \int_a^b v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right) dy$$

which is given in [8]. Also, for $g(x) = x$ inequality (2.11) reduces to

$$\int_a^b I^{\alpha}_{a^+}f_2(x)\Phi\left(\frac{I^{\alpha}_{a^+}f_1(x)}{I^{\alpha}_{a^+}f_2(x)}\right) dx \leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \int_a^b f_2(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right) dy.$$

REMARK 4. If we set $g(x) = \log x$, then $I^{\alpha}_{a_+;x}f(x)$ reduces to $J^{\alpha}_{a_+}f(x)$ left-sided Hadamard-type fractional integral that is defined for $\alpha > 0$ by

$$(J^{\alpha}_{a_+}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log\frac{x}{y}\right)^{\alpha-1} \frac{f(y)dy}{y}, \quad x > a.$$

The inequality (2.8) becomes:

$$\int_a^b v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right) dy - \int_a^b u(x)\Phi\left(\frac{(J^{\alpha}_{a_+}f_1)(x)}{(J^{\alpha}_{a_+}f_2)(x)}\right) dx \geq \frac{1}{\Gamma(\alpha)} \int_a^b \frac{u(x)}{(J^{\alpha}_{a_+}f_2)(x)}$$
$$\times \int_a^x (\log x - \log y)^{\alpha-1} f_2(y) \left| \left| \Phi\left(\frac{f_1(y)}{f_2(y)}\right) - \Phi\left(\frac{(J^{\alpha}_{a_+}f_1)(x)}{(J^{\alpha}_{a_+}f_2)(x)}\right) \right| \right.$$
$$\left. - \left| \varphi\left(\frac{(J^{\alpha}_{a_+}f_1)(x)}{(J^{\alpha}_{a_+}f_2)(x)}\right) \right| \cdot \left| \frac{f_1(y)}{f_2(y)} - \frac{(J^{\alpha}_{a_+}f_1)(x)}{(J^{\alpha}_{a_+}f_2)(x)} \right| \right| \frac{dy}{y}\, dx$$

and the inequality (2.11) given in Remark 2 becomes

$$\int_a^b (J_{a_+}^\alpha f_2)(x)\Phi\left(\frac{(J_{a_+}^\alpha f_1)(x)}{(J_{a_+}^\alpha f_2)(x)}\right)\frac{dx}{x} \leq \frac{(\log b - \log a)^\alpha}{\Gamma(\alpha+1)}\int_a^b f_2(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)\frac{dy}{y}.$$

We also recall the definition of Erdélyi-Kober type fractional integrals. For details see [18].

Let $(a,b),(0 \leq a < b \leq \infty)$ be finite or infinite interval of $\mathbb{R}_+$. Let $\alpha > 0, \sigma > 0$, and $\eta \in \mathbb{R}$. The left-sided Erdélyi-Kober type fractional integral of order $\alpha > 0$ is defined by

$$(I_{a_+;\sigma;\eta}^\alpha f)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_a^x \frac{y^{\sigma\eta+\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha}} f(y)dy, \quad x > a.$$

.

COROLLARY 2.2. *Let u be a weight function on (a,b), ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function and $I_{a_+;\sigma;\eta}^\alpha f$ denotes the Erdélyi–Kober type fractional left-sided integral. Define v by*

$$v(y) := \frac{\sigma f_2(y)}{\Gamma(\alpha)}\int_y^b \frac{u(x)x^{-\sigma(\alpha+\eta)}y^{\sigma\eta+\sigma-1}}{(I_{a_+;\sigma;\eta}^\alpha f_2)(x)(x^\sigma - y^\sigma)^{1-\alpha}}dx < \infty.$$

If Φ is convex on the interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int\, I$, then the following inequality holds:

$$\int_a^b v(y)\Phi\left(\frac{f_1(y)}{f_2(y)}\right)dy - \int_a^b u(x)\Phi\left(\frac{(I_{a_+;\sigma;\eta}^\alpha f_1)(x)}{(I_{a_+;\sigma;\eta}^\alpha f_1)(x)}\right)dx \geq \frac{\sigma}{\Gamma(\alpha)}\int_a^b \frac{u(x)}{(I_{a_+;\sigma;\eta}^\alpha f_2)(x)}$$
$$\times \int_a^x \frac{x^{-\sigma(\alpha+\eta)}y^{\sigma\eta+\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha}} f_2(y)\left|\left|\Phi\left(\frac{f_1(y)}{f_2(y)}\right) - \Phi\left(\frac{(I_{a_+;\sigma;\eta}^\alpha f_1)(x)}{(I_{a_+;\sigma;\eta}^\alpha f_1)(x)}\right)\right|\right.$$
$$\left. - \left|\varphi\left(\frac{(I_{a_+;\sigma;\eta}^\alpha f_1)(x)}{(I_{a_+;\sigma;\eta}^\alpha f_1)(x)}\right)\right| \cdot \left|\frac{f_1(y)}{f_2(y)} - \frac{(I_{a_+;\sigma;\eta}^\alpha f_1)(x)}{(I_{a_+;\sigma;\eta}^\alpha f_1)(x)}\right|\right| dy\, dx.$$

Proof. Proof is similar to the proof of the Theorem 2.1. ■

REMARK 5. Here we give results only for left-sided fractional integrals. Likewise we can give results for right-sided fractional integrals, but here we omit the details.

3. REFINEMENTS OF HARDY-TYPE INEQUALITIES FOR FRACTIONAL DERIVATIVES

We continue with new refinements of Hardy-type inequalities for different fractional derivatives. First we give a result with respect to the generalized Riemann–Liouville fractional derivative. Let us recall the definition.

Let $\alpha > 0$ and $n = [\alpha] + 1$ where $[\cdot]$ is the integral part and we define the generalized Riemann-Liouville fractional derivative of f of order α by

$$(D_a^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-y)^{n-\alpha-1} f(y)\, dy\,.$$

In addition, we stipulate

$$D_a^0 f := f =: I_a^0 f, \quad I_a^{-\alpha} f := D_a^\alpha f \text{ if } \alpha > 0.$$

If $\alpha \in \mathbb{N}$ then $D_a^\alpha f = \frac{d^\alpha f}{dx^\alpha}$, the ordinary α-order derivative.

The following lemma summarizes conditions in identity for generalized Riemann-Liouville fractional derivative. For details see [2].

LEMMA 3.1. *Let $\beta > \alpha \geq 0$, $n = [\beta] + 1$, $m = [\alpha] + 1$. Identity*

$$D_a^\alpha f(x) = \frac{1}{\Gamma(\beta - \alpha)} \int_a^x (x-y)^{\beta-\alpha-1} D_a^\beta f(y)\, dy, \quad x \in [a,b].$$

is valid if one of the following conditions holds:

(i) $f \in I_a^\beta (L(a,b))$.
(ii) $I_a^{n-\beta} f \in AC^n[a,b]$ *and* $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots n$.
(iii) $D_a^{\beta-k} f \in C[a,b]$ *for* $k = 1, \ldots, n$, $D_a^{\beta-1} f \in AC[a,b]$ *and* $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots n$.
(iv) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta - \alpha \notin \mathbb{N}$, $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots, n$ *and* $D_a^{\alpha-k} f(a) = 0$ *for* $k = 1, \ldots, m$.
(v) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta - \alpha = l \in \mathbb{N}$, $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots, l$.
(vi) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$ *and* $f(a) = f'(a) = \cdots = f^{(n-2)}(a) = 0$.
(vii) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta \notin \mathbb{N}$ *and* $D_a^{\beta-1} f$ *is bounded in a neighborhood of* $t = a$.

COROLLARY 3.1. *Let u be a weight function on (a,b) and let assumptions in Lemma 3.1 be satisfied. Define v on (a,b) by*

$$v(y) := \frac{D_a^\beta f_2(y)}{\Gamma(\beta-\alpha)} \int_y^b \frac{u(x)(x-y)^{\beta-\alpha-1}}{D_a^\alpha f_2(x)} dx < \infty.$$

If Φ is convex on the interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int\, I$, then the following inequality holds:

$$\begin{aligned} \int_a^b v(y)\Phi\left(\frac{D_a^\beta f_1(y)}{D_a^\beta f_2(y)}\right) dy - \int_a^b u(x)\Phi\left(\frac{D_a^\alpha f_1(x)}{D_a^\alpha f_2(x)}\right) dx \geq \frac{1}{\Gamma(\beta-\alpha)} \int_a^b \frac{u(x)}{D_a^\alpha f_2(x)} \\ \times \int_a^x (x-y)^{\beta-\alpha-1} D_a^\beta f_2(y) \left| \left| \Phi\left(\frac{D_a^\beta f_1(y)}{D_a^\beta f_2(y)}\right) - \Phi\left(\frac{D_a^\alpha f_1(x)}{D_a^\alpha f_2(x)}\right) \right| \right. \\ \left. - \left| \varphi\left(\frac{D_a^\alpha f_1(x)}{D_a^\alpha f_2(x)}\right) \right| \cdot \left| \frac{D_a^\beta f_1(y)}{D_a^\beta f_2(y)} - \frac{D_a^\alpha f_1(x)}{D_a^\alpha f_2(x)} \right| \right| dy\, dx. \end{aligned} \tag{3. 13}$$

Proof. Applying Theorem 1.3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}, & a \leq y \leq x; \\ 0, & x < y \leq b, \end{cases}$$

and replacing f_i by $D_a^\beta f_i$ then, by Lemma 3.1, $g_i = D_a^\alpha f_i, (i = 1, 2)$ and we get (3. 13). ■

REMARK 6. Since right-hand side of inequality (3. 13) is non-negative, we obtain the following inequality:

$$\int_a^b u(x)\Phi\left(\frac{D_a^\alpha f_1(x)}{D_a^\alpha f_2(x)}\right)dx \le \int_a^b v(y)\Phi\left(\frac{D_a^\beta f_1(y)}{D_a^\beta f_2(y)}\right)dy \tag{3. 14}$$

that is given in [8].

Now we recall the Canavati-type fractional derivative ($\nu-$fractional derivative of f). We consider

$$C^\nu([0,1]) = \{f \in C^n([0,1]) : I_{1-\bar{\nu}}f^{(n)} \in C^1([0,1])\},$$

$\nu > 0,\ n = [\nu], [.]$ is the integral part, and $\bar{\nu} = \nu - n, 0 \le \bar{\nu} < 1$.
For $f \in C^\nu([0,1])$, the Canavati-ν fractional derivative of f is defined by

$$D^\nu f = DI_{1-\bar{\nu}}f^{(n)},$$

where $D = d/dx$. For further details and proof see [3, Theorem 2.1].

LEMMA 3.2. *Let* $\nu > \gamma > 0,\ n = [\nu],\ m = [\gamma]$. *Let* $f \in C^\nu([0,1])$, *be such that* $f^{(i)}(0) = 0,\ i = m, m+1, ..., n-1$. *Then*

(i) $f \in C^\gamma([0,1])$,

(ii) $(D^\gamma f)(x) = \frac{1}{\Gamma(\nu-\gamma)}\int_0^x (x-t)^{\nu-\gamma-1}(D^\nu f)(t)dt$,

for every $x \in [0,1]$.

COROLLARY 3.2. *Let* u *be a weight function on* (a,b) *and let assumptions in Lemma* 3.2 *be satisfied. Define* v *on* (a,b) *by*

$$v(y) := \frac{D^\nu f_2(x)}{\Gamma(\nu-\gamma)}\int_y^b \frac{u(x)(x-y)^{\nu-\gamma-1}}{D^\gamma f_2(x)}dx < \infty.$$

If Φ *is convex on the interval* $I \subseteq \mathbb{R}$, *and* $\varphi : I \to \mathbb{R}$ *is any function, such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in Int\, I$, *then the following inequality holds:*

$$\begin{aligned}\int_a^b v(y)\Phi\left(\frac{D^\nu f_1(y)}{D^\nu f_2(y)}\right)dy - \int_a^b u(x)\Phi\left(\frac{D^\gamma f_1(x)}{D^\gamma f_2(x)}\right)dx \ge \frac{1}{\Gamma(\nu-\gamma)}\int_a^b \frac{u(x)}{D^\gamma f_2(x)} \\ \times \int_a^x (x-y)^{\nu-\gamma-1}D^\nu f_2(x)\left|\left|\Phi\left(\frac{D^\nu f_1(y)}{D^\nu f_2(y)}\right) - \Phi\left(\frac{D^\gamma f_1(x)}{D^\gamma f_2(x)}\right)\right|\right. \\ \left. - \left|\varphi\left(\frac{D^\gamma f_1(x)}{D^\gamma f_2(x)}\right)\right| \cdot \left|\frac{D^\nu f_1(y)}{D^\nu f_2(y)} - \frac{D^\gamma f_1(x)}{D^\gamma f_2(x)}\right|\right| dy\,dx.\end{aligned}$$

Proof. Proof is similar to the proof of Corollary 3.1. ■

Now we recall Caputo fractional derivative, for details see [1, p. 449].

Let $\nu \ge 0, n = \lceil \nu \rceil$, $f \in AC^n[a,b]$. The Caputo fractional derivative is given by

$$D_{*a}^\nu f(t) = \frac{1}{\Gamma(n-\nu)}\int_a^x \frac{f^{(n)}(y)}{(x-y)^{\nu-n+1}}dy,$$

for all $x \in [a,b]$. The above function exists almost everywhere for $x \in [a,b]$.

We continue with the following lemma that is given in [4].

LEMMA 3.3. *Let $\nu > \gamma \geq 0$, $n = [\nu]+1$, $m = [\gamma]+1$ and $f \in AC^n[a,b]$. Suppose that one of the following conditions hold:*

(a) $\nu, \gamma \notin \mathbb{N}_0$ *and* $f^i(a) = 0$ *for* $i = m, ..., n-1$
(b) $\nu \in \mathbb{N}_0, \gamma \notin \mathbb{N}_0$ *and* $f^i(a) = 0$ *for* $i = m, ..., n-2$
(c) $\nu \notin \mathbb{N}_0, \gamma \in \mathbb{N}_0$ *and* $f^i(a) = 0$ *for* $i = m-1, ..., n-1$
(d) $\nu \in \mathbb{N}_0, \gamma \in \mathbb{N}_0$ *and* $f^i(a) = 0$ *for* $i = m-1, ..., n-2$.

Then

$$D^{\gamma}_{*a} f(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-y)^{\nu-\gamma-1} D^{\nu}_{*a} f(y) dy$$

for all $a \leq x \leq b$.

COROLLARY 3.3. *Let u be a weight function on (a,b) and let assumptions in Lemma 3.3 be satisfied. Define v on (a,b) by*

$$v(y) := \frac{D^{\nu}_{*a} f_2(x)}{\Gamma(\nu-\gamma)} \int_y^b \frac{u(x)(x-y)^{\nu-\gamma-1}}{D^{\gamma}_{*a} f_2(x)} dx < \infty.$$

If Φ is convex on the interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int\, I$, then the following inequality holds:

$$\int_a^b v(y) \Phi\left(\frac{D^{\nu}_{*a} f_1(y)}{D^{\nu}_{*a} f_2(y)}\right) dy - \int_a^b u(x) \Phi\left(\frac{D^{\gamma}_{*a} f_1(x)}{D^{\gamma}_{*a} f_2(x)}\right) dx \geq \frac{1}{\Gamma(\nu-\gamma)} \int_a^b \frac{u(x)}{D^{\gamma}_{*a} f_2(x)}$$
$$\times \int_a^x (x-y)^{\nu-\gamma-1} D^{\nu}_{*a} f_2(y) \left| \left| \Phi\left(\frac{D^{\nu}_{*a} f_1(y)}{D^{\nu}_{*a} f_2(y)}\right) - \Phi\left(\frac{D^{\gamma}_{*a} f_1(x)}{D^{\gamma}_{*a} f_2(x)}\right) \right| \right.$$
$$\left. - \left| \varphi\left(\frac{D^{\gamma}_{*a} f_1(x)}{D^{\gamma}_{*a} f_2(x)}\right) \right| \cdot \left| \frac{D^{\nu}_{*a} f_1(y)}{D^{\nu}_{*a} f_2(y)} - \frac{D^{\gamma}_{*a} f_1(x)}{D^{\gamma}_{*a} f_2(x)} \right| \right| dy\, dx.$$

Proof. Similar to the proof of Corollary 3.1. ■

4. ACKNOWLEDGMENTS

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