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Inequalities for Three Times Differentiable Functions

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Abstract. In this article, first we prove a new integral identity and present some general inequalities of Hadamard's type for the functions whose third derivative are concave (convex). Second applications for special means and some new error estimates of the Midpoint formula are given.

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1. Introduction

Let $\psi : J \to \mathbb{R}$, be a function defined on $J \subseteq \mathbb{R}$, then we say that ψ is convex on J if the inequality

$$\psi(rz + (1-r)w) \leq r\psi(z) + (1-r)\psi(w) \tag{1. 1}$$

holds for all $z, w \in J$ and $r \in [0,1]$. Also we say that ψ is concave, if the negative of ψ i.e. $-\psi$ is convex. A lot of celebrated inequalities have been obtained for the functions defined in (1. 1) and a huge part of literature has been devoted to this class of convex functions. But here we will present only one of them in following:

For the convex function $\psi : J \rightarrow \mathbb{R}$, defined in (1. 1) with $a_1, a_2 \in J \subseteq \mathbb{R}$ and $a_1 \neq a_2$, we have the following inequalities:

$$\psi\left(\frac{a_1+a_2}{2}\right) \leq \frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(t)dt \leq \frac{\psi(a_1)+\psi(a_2)}{2} \tag{1. 2}$$

The inequalities given in (1. 2) hold in the reversed direction at the same time if ψ is concave. This remarkable result was given in ([11], 1893) and is well known in the analysis of mathematical inequalities as Hermite-Hadamard inequality. This double inequality was become the center of interest for many prolific researchers and has received considerable attention, since it was appeared for the first time in print. Also a number of variants generalizations and extensions of (1. 2) have been appeared in the theory of convex analysis, for example see [1–6, 8–10, 13–20] and the references cited therein.

Now for the sake of brevity, here we introduce some notation to denote the following repeated hypotheses, which will be used in the rest of the paper:

$\mathbf{H_1}$: Let $\psi : J \subseteq \rightarrow \mathbb{R}$ be a three times differentiable function on J°, where $a_1, a_2 \in J^\circ$ with $a_1 < a_2$ and $\psi''' \in L[a_1, a_2]$.

$\mathbf{H_2}$: Suppose $\mathbf{H_1}$ holds and $|\psi'''|$ be a concave function on $[a_1, a_2]$ for all $z \in [a_1, a_2]$ and $r \in [0, 1]$.

$\mathbf{H_3}$: Suppose $\mathbf{H_1}$ holds and for $(q \geq 1)$ $|\psi'''|^q$ be a concave function on $[a_1, a_2]$ for all $z \in [a_1, a_2]$ and $r \in [0, 1]$.

$\mathbf{H_4}$: Suppose $\mathbf{H_1}$ holds and $|\psi'''|$ be a convex function on $[a_1, a_2]$ for all $z \in [a_1, a_2]$ and $r \in [0, 1]$.

$\mathbf{H_5}$: Suppose $\mathbf{H_1}$ holds and for $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ let $|\psi'''|^q$ be a convex function on $[a_1, a_2]$ for all $z \in [a_1, a_2]$ and $r \in [0, 1]$.

$\mathbf{H_6}$: Suppose $\mathbf{H_1}$ holds and for $p > 1$ $|\psi'''|^{\frac{p}{p-1}}$ be a convex function on $[a_1, a_2]$ for all $z \in [a_1, a_2]$ and $r \in [0, 1]$.

Dragomir and the co-author have proved the following important results in [7] associated with the right hand part of inequality (1. 2).

Lemma 1.1 ([7]). *Suppose* $\mathbf{H_1}$ *holds, then we have:*

$$\frac{\psi(a_1)+\psi(a_2)}{2} - \frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(z)dz = \frac{a_2-a_1}{2}\int_0^1(1-2r)\psi'(ra_1+(1-r)a_2)dr. \tag{1. 3}$$

Theorem 1.2. *Suppose* $\mathbf{H_2}$ *holds, then the following is valid:*

$$\left|\frac{\psi(a_1)+\psi(a_2)}{2} - \frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(z)dz\right| \leq \frac{(a_2-a_1)}{4}\left(\frac{|\psi'(a_1)|+|\psi'(a_2)|}{2}\right). \tag{1. 4}$$

In [12], U. S. Kirmaci gave the results given below:

Lemma 1.3 ([12]). *If* $\mathbf{H_1}$ *holds, then the equality given below is valid:*

$$\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(z)dz - \psi\left(\frac{a_1+a_2}{2}\right) = (a_2-a_1)\left[\int_0^{\frac{1}{2}} r\psi'(ra_1+(1-r)a_2)dr + \int_{\frac{1}{2}}^1 (r-1)\psi'(ra_1+(1-r)a_2)dr\right]. \tag{1. 5}$$

Theorem 1.4 ([12]). *The following inequality is valid under the hypothesis* $\mathbf{H_2}$*:*

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(z)dz-\psi\left(\frac{a_1+a_2}{2}\right)\right|\leq\frac{(a_2-a_1)(|\psi'(a_1)|+|\psi'(a_2)|)}{8}. \qquad (1.\ 6)$$

Here in this article, first we are going to prove an identity and by making use of it we present new inequalities for three times differentiable concave (convex) functions. Then applications of the main results are given to particular means of real numbers. At the last section of the paper new error estimates associated with the Midpoint formula are presented.

2. Main Results

We begin this section with the following lemma, which is needed for the establishment of our main results:

Lemma 2.1. *Under the hypothesis* $\mathbf{H_1}$*, the following identity is valid:*

$$\begin{aligned}
&\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(z)dz-\frac{(a_2-a_1)^2}{24}\psi''\left(\frac{a_1+a_2}{2}\right)-\psi\left(\frac{a_1+a_2}{2}\right)\\
=\;&\frac{(a_2-a_1)^3}{96}\left[\int_0^1(1-r)^3\psi'''\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)dr\right.\\
-\;&\left.\int_0^1 r^3\psi'''\left(\frac{ra_2}{2}+\frac{2-r}{2}a_1\right)dr\right].
\end{aligned}$$

Proof. Integrating by parts yields:

$$\begin{aligned}
I_1 \;=\;& \int_0^1(1-r)^3\psi'''\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)dt\\
=\;& \left.\frac{(1-r)^3\psi''\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)}{\frac{a_2-a_1}{2}}\right|_0^1\\
+\;& \frac{6}{a_2-a_1}\int_0^1(1-r)^2\psi''\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)dr\\
=\;& -\frac{2}{a_2-a_1}\psi''\left(\frac{a_1+a_2}{2}\right)-\frac{12}{(a_2-a_1)^2}\psi'\left(\frac{a_1+a_2}{2}\right)\\
-\;& \frac{48}{(a_2-a_1)^3}\psi\left(\frac{a_1+a_2}{2}\right)+\frac{48}{(a_2-a_1)^3}\int_0^1\psi\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)dr
\end{aligned}$$

and then changing of variables give us the following,

$$I_1 \;=\; -\frac{2}{a_2-a_1}\psi''\left(\frac{a_1+a_2}{2}\right)-\frac{12}{(a_2-a_1)^2}\psi'\left(\frac{a_1+a_2}{2}\right)$$

$$- \quad \frac{48}{(a_2-a_1)^3}\psi\left(\frac{a_1+a_2}{2}\right)+\frac{96}{(a_2-a_1)^4}\int\limits_{\frac{a_1+a_2}{2}}^{a_2}\psi(z)dz. \qquad (2.\ 7)$$

Equivalently we get,

$$I_2=\frac{2}{a_2-a_1}\psi''\left(\frac{a_1+a_2}{2}\right)-\frac{12}{(a_2-a_1)^2}\psi'\left(\frac{a_1+a_2}{2}\right)$$

$$-\frac{48}{(a_2-a_1)^3}\psi\left(\frac{a_1+a_2}{2}\right)+\frac{96}{(a_2-a_1)^4}\int\limits_{a_1}^{\frac{a_1+a_2}{2}}\psi(z)dz. \qquad (2.\ 8)$$

Finally the subtraction of (2. 7) from (2. 8) and then multiplying by $\frac{(a_2-a_1)^3}{96}$ leads to the required conclusion. □

Theorem 2.2. *Under the hypothesis* $\mathbf{H_2}$, *the following inequality holds:*

$$\left|\frac{1}{a_2-a_1}\int\limits_{a_1}^{a_2}\psi(z)dz-\frac{(a_2-a_1)^2}{24}\psi''\left(\frac{a_1+a_2}{2}\right)-\psi\left(\frac{a_1+a_2}{2}\right)\right|$$

$$\leq \quad \frac{(a_2-a_1)^3}{96}\left|\psi'''\left(\frac{5a_2+3a_1}{8}\right)\right|. \qquad (2.\ 9)$$

Proof. By applying triangular inequality on Lemma 2.1, we have

$$\left|\frac{1}{a_2-a_1}\int\limits_{a_1}^{a_2}\psi(z)dz \;-\; \frac{(a_2-a_1)^2}{24}\psi''\left(\frac{a_1+a_2}{2}\right)-\psi\left(\frac{a_1+a_2}{2}\right)\right|$$

$$\leq \quad \frac{(a_2-a_1)^3}{96}\left[\int\limits_0^1 r^3\left|\psi'''\left(\frac{ra_2}{2}+\frac{2-r}{2}a_1\right)\right|dr\right.$$

$$\left. + \quad \int\limits_0^1(1-r)^3\left|\psi'''\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)\right|dr\right].$$

As $(1-r)^3\leq 1-r^3$ for all $r\in[0,1]$, therefore from above we can write

$$\left|\frac{1}{a_2-a_1}\int\limits_{a_1}^{a_2}\psi(z)dz-\frac{(a_2-a_1)^2}{24}\psi''\left(\frac{a_1+a_2}{2}\right)-\psi\left(\frac{a_1+a_2}{2}\right)\right|$$

$$\leq \quad \frac{(a_2-a_1)^3}{96}\left[\int\limits_0^1 r^3\left|\psi'''\left(\frac{ra_2}{2}+\frac{2-r}{2}a_1\right)\right|dr\right.$$

$$\left. + \quad \int\limits_0^1(1-r^3)\left|\psi'''\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)\right|dr\right]. \qquad (2.\ 10)$$

Since $|\psi'''|$ is concave, so the inequality (2. 10) becomes

$$\begin{aligned}
&\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(z)dz-\frac{(a_2-a_1)^2}{24}\psi''\left(\frac{a_1+a_2}{2}\right)-\psi\left(\frac{a_1+a_2}{2}\right)\right| \\
\leq\ & \frac{(a_2-a_1)^3}{96}\left[\int_0^1\left|\psi'''\Big(\frac{r^4a_2}{2}+\frac{2r^3-r^4}{2}a_1\right.\right. \\
+\ & \left.\left.\frac{(1-r^3)(1-r)}{2}a_1+\frac{(1-r^3)(1+r)}{2}a_2\Big)\right|dr\right].
\end{aligned}$$

Now by applying Jensen's inequality we have

$$\begin{aligned}
&\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(z)dz-\frac{(a_2-a_1)^2}{24}\psi''\left(\frac{a_1+a_2}{2}\right)-\psi\left(\frac{a_1+a_2}{2}\right)\right| \\
\leq\ & \frac{(a_2-a_1)^3}{96}\left[\left|\psi'''\int_0^1\Big(\frac{r^4a_2}{2}+\frac{2r^3-r^4}{2}a_1\right.\right. \\
+\ & \left.\left.\frac{(1-r^3)(1-r)}{2}a_1+\frac{(1-r^3)(1+r)}{2}a_2\Big)dr\right|\right] \\
=\ & \frac{(a_2-a_1)^3}{96}\left|\psi'''\left(\frac{21}{40}a_2+\frac{9}{40}a_1+\frac{1}{10}a_2+\frac{3}{20}a_1\right)\right| \\
=\ & \frac{(a_2-a_1)^3}{96}\left|\psi'''\left(\frac{5a_2+3a_1}{8}\right)\right|.
\end{aligned}$$

Hence the proof is completed. □

Corollary 2.3. *If we choose $\psi''\left(\frac{a_1+a_2}{2}\right)=0$ in Theorem 2.2, we obtain*

$$\begin{aligned}
&\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(z)dz-\psi\left(\frac{a_1+a_2}{2}\right)\right| \\
\leq\ & \frac{(a_2-a_1)^3}{96}\left|\psi'''\left(\frac{5a_2+3a_1}{8}\right)\right|. \qquad (2.\ 11)
\end{aligned}$$

Theorem 2.4. *Suppose $\mathbf{H_3}$ holds, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(z)dz-\frac{(a_2-a_1)^2}{24}\psi''\left(\frac{a_1+a_2}{2}\right)-\psi\left(\frac{a_1+a_2}{2}\right)\right| \\
\leq\ & \frac{(a_2-a_1)^3}{384}\left[\left|\psi'''\left(\frac{3a_2+2a_1}{5}\right)\right|+\left|\psi'''\left(\frac{2a_2+3a_1}{5}\right)\right|\right]. \qquad (2.\ 12)
\end{aligned}$$

Proof. First of all by the concavity of $|\psi'''|^q$ and then by power mean inequality, we have

$$\begin{aligned}
|\psi'''(tz+(1-t)w)|^q &\geq t|\psi'''(z)|^q+(1-t)|\psi'''(w)|^q \\
&\geq (t|\psi'''(z)|+(1-t)|\psi'''(w)|)^q
\end{aligned}$$

and hence

$$|\psi'''(tz+(1-t)w)| \geq t|\psi'''(z)| + (1-t)|\psi'''(w)|,$$

so $|\psi'''|$ is also concave. Now by applying triangular inequality on Lemma 2.1 we have

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(z)dz - \frac{(a_2-a_1)^2}{24}\psi''\left(\frac{a_1+a_2}{2}\right) - \psi\left(\frac{a_1+a_2}{2}\right)\right|$$
$$\leq \left[\frac{(a_2-a_1)^3}{96}\left[\left|\int_0^1 r^3\psi'''\left(\frac{ra_2}{2}+\frac{2-r}{2}a_1\right)dr\right|\right.\right.$$
$$\left.\left. + \left|\int_0^1 (1-r)^3\psi'''\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)dr\right|\right]. \quad (2.\ 13)$$

Now according to Jensen's integral inequality we have

$$\int_0^1 (1-r)^3\left|\psi'''\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)\right|dr$$

$$\leq \int_0^1 (1-r)^3dr\left|\psi'''\left(\frac{\int_0^1(1-r)^3\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)dr}{\int_0^1(1-r)^3dr}\right)\right| \quad (2.\ 14)$$

since $\int_0^1(1-r)^3dr = \frac{1}{4}$ and $\int_0^1\left(\frac{(1-r)^4}{2}a_1+\frac{(1-r)^3(1+r)}{2}a_2\right) = \frac{1}{10}a_1+\frac{3}{20}a_2$,
so (2. 14) becomes

$$\int_0^1 (1-r)^3|\psi'''\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)|dr \leq \frac{1}{4}\left|\psi'''\left(\frac{3a_2+2a_1}{5}\right)\right|, \quad (2.\ 15)$$

equivalently, we have

$$\int_0^1 r^3\psi'''\left(\frac{ra_2}{2}+\frac{2-r}{2}a_1\right)dr \leq \frac{1}{4}\left|\psi'''\left(\frac{2a_2+3a_1}{5}\right)\right|. \quad (2.\ 16)$$

By putting back(2. 15) and (2. 16) in (2. 19) we get the required result. □

Corollary 2.5. *By setting* $\psi''\left(\frac{a_1+a_2}{2}\right)=0$ *in Theorem 2.4, we get*

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(z)dz - \psi\left(\frac{a_1+a_2}{2}\right)\right|$$
$$\leq \frac{(a_2-a_1)^3}{384}\left[\left|\psi'''\left(\frac{3a_2+2a_1}{5}\right)\right| + \left|\psi'''\left(\frac{2a_2+3a_1}{5}\right)\right|\right]. \quad (2.\ 17)$$

Theorem 2.6. *Under the hypothesis* $\mathbf{H_4}$, *the following inequality is valid:*

$$\left|\frac{1}{a_2-a_1}\int\limits_{a_1}^{a_2}\psi(z)dz-\frac{(a_2-a_1)^2}{24}\psi''\left(\frac{a_1+a_2}{2}\right)-\psi\left(\frac{a_1+a_2}{2}\right)\right| \leq \frac{(a_2-a_1)^3(|\psi'''(a_1)|+|\psi'''(a_2)|)}{384}. \quad (2.\ 18)$$

Proof. Now by applying triangular inequality and definition of convex function on Lemma 2.1 we have,

$$\begin{aligned}
&\left|\frac{1}{a_2-a_1}\int\limits_{a_1}^{a_2}\psi(z)dz-\frac{(a_2-a_1)^2}{24}\psi''\left(\frac{a_1+a_2}{2}\right)-\psi\left(\frac{a_1+a_2}{2}\right)\right| \\
\leq\ & \frac{(a_2-a_1)^3}{96}\left[\int\limits_0^1 r^3\left|\psi'''\left(\frac{ra_2}{2}+\frac{2-r}{2}a_1\right)\right|dr\right. \\
+\ & \left.\int\limits_0^1(1-r)^3\left|\psi'''\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)\right|dr\right] \\
\leq\ & \frac{(a_2-a_1)^3}{96}\left[\int\limits_0^1 r^3\Big(\frac{r}{2}|\psi'''(a_2)|+\frac{2-r}{2}|\psi'''(a_1)|\Big)dr\right. \\
+\ & \left.\int\limits_0^1(1-r)^3\Big(\frac{1-r}{2}|\psi'''(a_1)|+\frac{1+r}{2}|\psi'''(a_2)|\Big)dr\right] \\
=\ & \frac{(a_2-a_1)^3}{96}\left[\frac{1}{10}|\psi'''(a_2)|+\frac{3}{20}|\psi'''(a_1)|+\frac{1}{10}|\psi'''(a_1)|+\frac{3}{20}|\psi'''(a_2)|\right] \\
=\ & \frac{(a_2-a_1)^3(|\psi'''(a_1)|+|\psi'''(a_2)|)}{384}.
\end{aligned}$$

□

Corollary 2.7. *For the selection of* $\psi''\left(\frac{a_1+a_2}{2}\right)=0$ *in Theorem 2.6, we have*

$$\left|\frac{1}{a_2-a_1}\int\limits_{a_1}^{a_2}\psi(z)dz-\psi\left(\frac{a_1+a_2}{2}\right)\right| \leq \frac{(a_2-a_1)^3(|\psi'''(a_1)|+|\psi'''(a_2)|)}{384}. \quad (2.\ 19)$$

Theorem 2.8. *For the hypothesis* $\mathbf{H_5}$, *we have the inequality:*

$$\left|\frac{1}{a_2-a_1}\int\limits_{a_1}^{a_2}\psi(z)dz-\frac{(a_2-a_1)^2}{24}\psi''\left(\frac{a_1+a_2}{2}\right)-\psi\left(\frac{a_1+a_2}{2}\right)\right|$$

$$\leq \frac{(a_2-a_1)^3}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\psi'''(a_1)|^q+3|\psi'''(a_2)|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{|\psi'''(a_2)|^q+3|\psi'''(a_1)|^q}{4}\right)^{\frac{1}{q}}\right]. \tag{2. 20}$$

Proof. Now by applying triangular inequality and Holder inequality on Lemma 2.1, we have

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(z)dz-\frac{(a_2-a_1)^2}{24}\psi''\left(\frac{a_1+a_2}{2}\right)-\psi\left(\frac{a_1+a_2}{2}\right)\right|$$
$$\leq \frac{(a_2-a_1)^3}{96}\left[\int_0^1(1-r)^3\left|\psi'''\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)\right|dr\right.$$
$$\left.+\int_0^1 r^3\left|\psi'''\left(\frac{ra_2}{2}+\frac{2-r}{2}a_1\right)\right|dr\right]$$
$$\leq \frac{(a_2-a_1)^3}{96}\left[\left(\int_0^1(1-r)^{3p}dr\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi'''\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)\right|^q dr\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 r^{3p}dr\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi'''\left(\frac{ra_2}{2}+\frac{2-r}{2}a_1\right)\right|^q dr\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$. Now using the convexity of $|\psi'''|^q$, we have

$$\int_0^1\left|\psi'''\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)\right|^q dr$$
$$\leq \int_0^1\left[\frac{1-r}{2}|\psi'''(a_1)|^q+\frac{1+r}{2}|\psi'''(a_2)|^q\right]dr$$
$$= \frac{|\psi'''(a_1)|^q+3|\psi'''(a_2)|^q}{4}, \tag{2. 21}$$

similarly we have

$$\int_0^1\left|\psi'''\left(\frac{ra_2}{2}+\frac{2-r}{2}a_1\right)\right|^q dr \leq \frac{3|\psi'''(a_1)|^q+|\psi'''(a_2)|^q}{4}, \tag{2. 22}$$

and

$$\int_0^1 t^{3p}dr=\int_0^1(1-r)^{3p}=\frac{1}{3p+1}. \tag{2. 23}$$

Combining (2. 21), (2. 22) and (2. 23), we obtain the required result. □

Corollary 2.9. *By taking* $\psi''\left(\frac{a_1+a_2}{2}\right)=0$ *in Theorem 2.8, we obtain*

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(z)dz-\psi\left(\frac{a_1+a_2}{2}\right)\right|$$
$$\leq \frac{(a_2-a_1)^3}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\psi'''(a_1)|^q+3|\psi'''(a_2)|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{|\psi'''(a_2)|^q+3|\psi'''(a_1)|^q}{4}\right)^{\frac{1}{q}}\right]. \quad (2.\ 24)$$

Theorem 2.10. *Suppose* $\mathbf{H_6}$ *holds, then the inequality given below is valid:*

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(z)dz-\frac{(a_2-a_1)^2}{24}\psi''\left(\frac{a_1+a_2}{2}\right)-\psi\left(\frac{a_1+a_2}{2}\right)\right|$$
$$\leq \frac{(a_2-a_1)^3}{384}\left[\left(\frac{2|\psi'''(a_1)|^q+3|\psi'''(a_2)|^q}{5}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{2|\psi'''(a_2)|^q+3|\psi'''(a_1)|^q}{5}\right)^{\frac{1}{q}}\right]. \quad (2.\ 25)$$

Proof. Now by applying triangular inequality and Power mean inequality on Lemma 2.1, we have

$$\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(z)dz-\frac{(a_2-a_1)^2}{24}\psi''\left(\frac{a_1+a_2}{2}\right)-\psi\left(\frac{a_1+a_2}{2}\right)\right|$$
$$\leq \frac{(a_2-a_1)^3}{96}\left[\int_0^1(1-r)^3\left|\psi'''\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)\right|dr\right.$$
$$\left.+\int_0^1 r^3\left|\psi'''\left(\frac{ra_2}{2}+\frac{2-r}{2}a_1\right)\right|dr\right]$$
$$\leq \frac{(a_2-a_1)^3}{96}\left[\left(\int_0^1(1-r)^3dr\right)^{1-\frac{1}{q}}\left(\int_0^1(1-r)^3\left|\psi'''\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)\right|^q dr\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 r^3dr\right)^{1-\frac{1}{q}}\left(\int_0^1 r^3\left|\psi'''\left(\frac{ra_2}{2}+\frac{2-r}{2}a_1\right)\right|^q dr\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.
Now using convexity of $|\psi'''|^q$, we have

$$\int_0^1(1-r)^3\left|\psi'''\left(\frac{1-r}{2}a_1+\frac{1+r}{2}a_2\right)\right|^q dr$$

$$\begin{aligned} &\leq \int_0^1 \Big[(1-r)^3\Big(\frac{1-r}{2}|\psi'''(a_1)|^q + \frac{1+r}{2}|\psi'''(a_2)|^q\Big)\Big]dr \\ &= \frac{2|\psi'''(a_1)|^q + 3|\psi'''(a_2)|^q}{20}, \end{aligned} \tag{2. 26}$$

similarly we have

$$\int_0^1 r^3\left|\psi'''\left(\frac{ra_2}{2} + \frac{2-r}{2}a_1\right)\right|^q dr \leq \frac{3|\psi'''(a_1)|^q + 2|\psi'''(a_2)|^q}{20}, \tag{2. 27}$$

and

$$\int_0^1 r^3 dr = \int_0^1 (1-r)^3 = \frac{1}{4}. \tag{2. 28}$$

Combining (2. 26), (2. 27) and (2. 28), we obtain the required result.

Corollary 2.11. *If we set $\psi''\left(\frac{a_1+a_2}{2}\right) = 0$ in Theorem 2.10, then we have*

$$\begin{aligned} &\left|\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\psi(z)dz - \psi\left(\frac{a_1+a_2}{2}\right)\right| \\ &\leq \frac{(a_2-a_1)^3}{384}\left[\left(\frac{2|\psi'''(a_1)|^q + 3|\psi'''(a_2)|^q}{5}\right)^{\frac{1}{q}}\right. \\ &+ \left.\left(\frac{2|\psi'''(a_2)|^q + 3|\psi'''(a_1)|^q}{5}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

□

3. Applications to Means

The following definitions of means of real numbers given in [7] will be used in this section of the paper:

For any $a_1, a_2 \in \mathbb{R}$ with $a_1 \neq a_2$, we have:

$$\begin{aligned} A(a_1,a_2) &= \frac{a_1+a_2}{2} \quad a_1, a_2 > 0, \\ \bar{L}(a_1,a_2) &= \frac{a_2-a_1}{\ln a_2 - \ln a_1} \quad a_1 \neq a_2, \quad a_1, a_2 > 0, \\ L_n(a_1,a_2) &= \left[\frac{a_2^{n+1} - a_1^{n+1}}{(n+1)(a_2-a_1)}\right]^{\frac{1}{n}} \quad a_1, a_2 \in \mathbb{R}, \quad a_1 < a_2, \quad n \in \mathbb{N}. \end{aligned}$$

Proposition 3.1. *Let $0 < a_1 < a_2$, $n \in \mathbb{N}$, and $n \geq 4$. Then we have:*

$$\begin{aligned} &\left|L_n(a_1,a_2)^n - n(n-1)\frac{(a_2-a_1)^2}{24}A^{n-2}(a_1,a_2) - A^n(a_1,a_2)\right| \\ &\leq \frac{n(n-1)(n-2)(a_2-a_1)^3}{192}A\left(|a_1|^{n-3}, |a_2|^{n-3}\right). \end{aligned} \tag{3. 29}$$

Proof. Using the convex function $\psi(z) = z^n, z > 0$ in Theorem 2.6, the result is obvious. □

Proposition 3.2. *Let* $0 < a_1 < a_2$*,* $n \in \mathbb{N}$*,and* $n \geq 4$*. Then the inequality given below is valid:*

$$\begin{aligned}
& \left|L^{-1}(a_1,a_2)^n - 2A^{-3}(a_1,a_2) - A^{-1}(a_1,a_2)\right| \\
\leq \quad & \frac{(a_2-a_1)^3}{32} A\left(|a_1|^{-4}, |a_2|^{-4}\right). \qquad (3.\ 30)
\end{aligned}$$

Proof. The result can be obtained from Theorem 2.6 under the utility of convex function $\psi(z) = \frac{1}{z}, z > 0$. ☐

Proposition 3.3. *Let* $0 < a_1 < a_2$*,* $n \in \mathbb{N}$*, and* $n \geq 4$*. Then we have:*

$$\begin{aligned}
& \left|L_n(a_1,a_2)^n - n(n-1)\frac{(a_2-a_1)^2}{24} A^{n-2}(a_1,a_2) - A^n(a_1,a_2)\right| \\
\leq \quad & \frac{n(n-1)(n-2)(a_2-a_1)^3}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}} \left[\left(|a_1|^{(n-3)q} + 3|a_2|^{(n-3)q}\right)^{\frac{1}{q}} \right. \\
& \left. + \left(|a_2|^{(n-3)q} + 3|a_1|^{(n-3)q}\right)^{\frac{1}{q}} \right]. \qquad (3.\ 31)
\end{aligned}$$

Proof. Using the convex function $\psi(z) = z^n, z > 0$ in Theorem 2.8, one has the result. ☐

Proposition 3.4. *Let* $0 < a_1 < a_2$*,* $n \in \mathbb{N}$*, and* $n \geq 4$*. Then the inequality given below is valid:*

$$\begin{aligned}
& \left|L^{-1}(a_1,a_2)^n - 2A^{-3}(a_1,a_2) - A^{-1}(a_1,a_2)\right| \\
\leq \quad & \frac{n(n-1)(n-2)(a_2-a_1)^3}{16}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}} \left[\left(|a_1|^{-4q} + 3|a_2|^{-4q}\right)^{\frac{1}{q}} \right. \\
+ \quad & \left. \left(|a_2|^{-4q} + 3|a_1|^{-4q}\right)^{\frac{1}{q}} \right]. \qquad (3.\ 32)
\end{aligned}$$

Proof. One can get the conclusion from Theorem 2.8 under the utility of convex function $\psi(z) = \frac{1}{z}, z > 0$. ☐

Proposition 3.5. *Let* $0 < a_1 < a_2$*,* $n \in \mathbb{N}$*,and* $n \geq 4$*. Then the following is true:*

$$\begin{aligned}
& \left|L_n(a_1,a_2)^n - n(n-1)\frac{(a_2-a_1)^2}{24} A^{n-2}(a_1,a_2) - A^n(a_1,a_2)\right| \\
\leq \quad & \frac{n(n-1)(n-2)(a_2-a_1)^3}{96} \left[\left(2|a_1|^{(n-3)q} + 3|a_2|^{(n-3)q}\right)^{\frac{1}{q}} \right. \\
+ \quad & \left. \left(3|a_1|^{(n-3)q} + 2|a_2|^{(n-3)q}\right)^{\frac{1}{q}} \right]. \qquad (3.\ 33)
\end{aligned}$$

Proof. Using the convex function $\psi(z) = z^n, z > 0$ in Theorem 2.10, one can easily obtain the result. ☐

Proposition 3.6. *Let* $0 < a_1 < a_2$*,* $n \in \mathbb{N}$*,and* $n \geq 4$*. Then we have the inequality:*

$$\begin{aligned}
& \left|L^{-1}(a_1,a_2)^n - 2A^{-3}(a_1,a_2) - A^{-1}(a_1,a_2)\right| \\
\leq \quad & \frac{n(n-1)(n-2)(a_2-a_1)^3}{16} \left[\left(2|a_1|^{-4q} + 3|a_2|^{-4q}\right)^{\frac{1}{q}} \right. \\
+ \quad & \left. \left(3|a_1|^{-4q} + 2|a_2|^{-4q}\right)^{\frac{1}{q}} \right]. \qquad (3.\ 34)
\end{aligned}$$

Proof. The statement of the proposition can be validated easily from Theorem 2.10 under the utility of convex function $\psi(z) = \frac{1}{z}, z > 0$. □

4. Applications to Midpoint Formula

Let $\mathcal{P} = \{z_1, z_2, .., z_n\}$ be a partition of points $z_i \in [a_1, a_2]$, $i = \overline{1, n}$ with $a_1 = z_0$, $z_n = a_2$ and $z_i < z_{i+1}$ for $i = \overline{1, n}$.
Then the well known Midpoint formula for the partition $\mathcal{P}$ is given by:

$$T(\psi, \mathcal{P}) = \sum_{i=0}^{n-1} \psi\left(\frac{z_i + z_{i+1}}{2}\right)(z_{i+1} - z_i)$$

It is known to us that if the second derivative of the function $\psi : [a_1, a_2] \to \mathbb{R}$ is defined on the open interval (a_1, a_2) and

$$M = \max_{t \in (a_1, a_2)} |\psi''(z)| < \infty, \quad \text{then}$$

$$\int_{a_1}^{a_2} \psi(z)dz = T(\psi, \mathcal{P}) + E(\psi, \mathcal{P})$$

where $E(\psi, \mathcal{P})$ represent the approximate error of the integral $\int_{a_1}^{a_2} \psi(z)dz$ by the Midpoint formula $T(\psi, \mathcal{P})$ and satisfies

$$|E(\psi, \mathcal{P})| \leq \frac{M}{12} \sum_{i=0}^{n-1} (z_{i+1} - z_i)^3$$

Proposition 4.1. *Suppose* $\mathbf{H_2}$ *holds, then we have following inequality:*

$$|E(\psi, \mathcal{P})| \quad \leq \quad \frac{1}{96} \sum_{i=0}^{n-1} (z_{i+1} - z_i)^4 \left|\psi'''\left(\frac{5z_{i+1} + 3z_i}{8}\right)\right|.$$

Proof. Utilizing Corollary 2.3 over the subinterval $[z_i, z_{i+1}]$ $(i = \overline{0, n-1})$ of the partition $\mathcal{P}$, we get

$$\left|\psi\left(\frac{z_i + z_{i+1}}{2}\right)(z_{i+1} - z_i) - \int_{z_i}^{z_{i+1}} \psi(z)dz\right| \leq \frac{1}{96}(z_{i+1} - z_i)^4 \left|\psi'''\left(\frac{5z_{i+1} + 3z_i}{8}\right)\right|.$$

Taking the summation on both sides of the inequality over i from 0 to $n-1$ and using the triangle inequality we deduce, that

$$\left|T(\psi, \mathcal{P}) - \int_{a_1}^{a_2} \psi(z)dz\right| \quad \leq \quad \frac{1}{96} \sum_{i=0}^{n-1} (z_{i+1} - z_i)^4 \left|\psi'''\left(\frac{5z_{i+1} + 3z_i}{8}\right)\right|.$$

□

Proposition 4.2. *Under the hypothesis* $\mathbf{H_3}$ *, the following inequality holds:*

$$|E(\psi, \mathcal{P})| \quad \leq \quad \frac{1}{384} \sum_{i=0}^{n-1} (z_{i+1} - z_i)^4 \left[\left|\psi'''\left(\frac{2z_i + 3z_{i+1}}{5}\right)\right| + \left|\psi'''\left(\frac{3z_i + 2z_{i+1}}{5}\right)\right|\right].$$

Proof. We can validate the proof by utilizing Corollary 2.5 and is analogous to the previous Proposition. □

Proposition 4.3. *Suppose* $\mathbf{H_4}$ *holds, then the inequality given below is valid:*

$$\begin{aligned}|E(\psi,\mathcal{P})| &\leq \frac{1}{384}\sum_{i=0}^{n-1}(z_{i+1}-z_i)^4\Big[|\psi'''(z_i)|+|\psi'''(z_{i+1})|\Big]\\ &\leq \frac{\max\{\psi'''(a_1),\psi'''(a_2)\}}{384}\sum_{i=0}^{n-1}(z_{i+1}-z_i)^4. \qquad (4.\ 35)\end{aligned}$$

Proof. Likewise Proposition 3.1 we can prove the result only by using Corollary 2.7 instead of Corollary 2.3. □

Proposition 4.4. *For the hypothesis* $\mathbf{H_5}$*, we have the inequality:*

$$\begin{aligned}|E(\psi,\mathcal{P})| &\leq \frac{1}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}\sum_{i=0}^{n-1}(z_{i+1}-z_i)^4\left[\left(\psi'''\left(\frac{|z_i|^q+3|z_{i+1}|^q}{4}\right)\right)^{\frac{1}{q}}\right.\\ &+ \left.\left(\psi'''\left(\frac{3|z_i|^q+|z_{i+1}|^q}{4}\right)\right)^{\frac{1}{q}}\right]. \qquad (4.\ 36)\end{aligned}$$

Proof. The statement can be proved easily by applying Corollary 2.9 and the procedure is similar to that of Proposition 3.1. □

Proposition 4.5. *Using* $\mathbf{H_6}$*, we have the following:*

$$\begin{aligned}|E(\psi,\mathcal{P})| &\leq \frac{1}{384}\sum_{i=0}^{n-1}(z_{i+1}-z_i)^4\left[\left(\psi'''\left(\frac{2|z_i|^q+3|z_{i+1}|^q}{5}\right)\right)^{\frac{1}{q}}\right.\\ &+ \left.\left(\psi'''\left(\frac{3|z_i|^q+2|z_{i+1}|^q}{5}\right)\right)^{\frac{1}{q}}\right]. \qquad (4.\ 37)\end{aligned}$$

Proof. The result can be obtained easily by utilizing Corollary 2.11 and the procedure is parallel to that of Proposition 3.1. □

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