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Generalization of Higher Order Homomorphism in Configuration Complexes

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Abstract. In this research, Grassmannian complex (Configuration Complex) is being introduced in a generalized form. Its 4^{th} and 5^{th} order complexes are being constructed at first, then N^{th} order generalization is being discussed. For this purpose, higher order mixed partial differential morphisms between free abelian group generated by projective configuration of points will used. Furthermore, in this work, the associated diagrams of these complexes are shown to be commutative.

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1. Introduction

Grassmannian chain complex was named after the mathematician Hermann Grassmannian, who presented his idea in general. For past few year, the Grassmannian complexes have been under consideration of many researchers to investigate the homology of general linear groups. Grassmannian chain complex of free abelian groups generated by the projective configuration of points was first introduced by Suslin [9], who connected free abelian

groups using two types of first order differential morphisms: differential map and projection map to form Grassmannian chain complex.
Goncharov [1, 2, 3] connected Grassmannian complex with Bloch-Suslin complex for weight 2 and also connected the former with his own complex, called Goncharov's complex, for weight 3.
Grassmannian complex and Cathelineau's complex for weight 2 and 3 are connected by Siddiqui [8]. Khalid [5, 6] introduced new morphisms to connect Grassmannian complex and Variant of Cathelineau complex upto weight N and proved that the corresponding diagrams are commutative. Recently, Hussain [4] connect Grassmannian complex and Tangent complex for second order.
In previous work, Khalid [7] introduced 2^{nd} and 3^{rd} order Grassmannian complexes. Both of which have been defined using 2^{nd} and 3^{rd} order mixed partial differential morphism between free abelian groups generated by projective configuration of points.
In the present work, 4^{th}, 5^{th} and N^{th} order Grassmannian complexes have been constructed and free abelian group are connected through 4^{th}, 5^{th} and N^{th} order mixed partial differential morphism. It is also shown that the associated diagrams of these complexes are bi-complex and commutative.

2. Preliminary and Background

Let us suppose $G_{n+1}(n)$ is a group of $n-$ dimensional vector space generated by $(n+1)$ vectors and it is free abelian. The first order Grassmannian chain complex [9] is given by

$$G_{n+1}(n) \xrightarrow{d} G_n(n) \xrightarrow{d} G_{n-1}(n)$$

$$G_{n+1}(n) \xrightarrow{p} G_n(n-1) \xrightarrow{p} G_{n-1}(n-2)$$

where d is a differential function called first order differential morphism, defined as

$$d : (\alpha_0, \ldots, \alpha_n) \mapsto \sum_{\hbar=0}^{n} (-1)^{\hbar} (\alpha_0, ..., \hat{\alpha}_{\hbar}, ..., \alpha_n) \tag{2. 1}$$

and p is called projective map, which is another type of differential morphism, defined by

$$p : (\alpha_0, \ldots, \alpha_n) \mapsto \sum_{\flat=0}^{n} (-1)^{\flat} (\alpha_{\flat} | \alpha_0, \ldots, \hat{\alpha}_{\flat}, \ldots, \alpha_n). \tag{2. 2}$$

2^{nd} and 3^{rd} order Grassmannian complex have been introduced with 2^{nd} and 3^{rd} order mixed partial differential morphisms (see [7]).

2.1. **Fourth Order Grassmannian Complex.** Let us construct fourth order Grassmannian complex by the following

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow p^{iv} & & \downarrow p^{iv} & & \downarrow p^{iv} \\
\cdots & \xrightarrow{d^{iv}} & G_{n+16}(n+11) & \xrightarrow{d^{iv}} & G_{n+12}(n+11) & \xrightarrow{d^{iv}} & G_{n+8}(n+11) \\
 & & \downarrow p^{iv} & & \downarrow p^{iv} & & \downarrow p^{iv} \\
\cdots & \xrightarrow{d^{iv}} & G_{n+12}(n+7) & \xrightarrow{d^{iv}} & G_{n+8}(n+7) & \xrightarrow{d^{iv}} & G_{n+4}(n+7) \\
 & & \downarrow p^{iv} & & \downarrow p^{iv} & & \downarrow p^{iv} \\
\cdots & \xrightarrow{d^{iv}} & G_{n+8}(n+3) & \xrightarrow{d^{iv}} & G_{n+4}(n+3) & \xrightarrow{d^{iv}} & G_{n}(n+3)
\end{array} \tag{A}$$

where d^{iv} is fourth order mixed partial differential map, defined as

$$d^{iv}(\alpha_0,\ldots,\alpha_n) \mapsto \sum_{\substack{\hbar_1\neq\hbar_2\neq\hbar_3\neq\hbar_4\\ \hbar_1=0\\ \hbar_2=\hbar_1+1\\ \hbar_3=\hbar_1+2\\ \hbar_4=\hbar_1+3}}^{n} (-1)^{\hbar_1+\hbar_2+\hbar_3+\hbar_4}(\alpha_0,...,\hat{\alpha}_{\hbar_1},\hat{\alpha}_{\hbar_2},\hat{\alpha}_{\hbar_3},\hat{\alpha}_{\hbar_4},...,\alpha_n) \tag{2. 3}$$

and p^{iv} is another fourth order mixed partial differential morphism called projection map, defined by

$$p^{iv}(\alpha_0,\ldots,\alpha_n) \mapsto \sum_{\substack{\flat_1\neq\flat_2\neq\flat_3\neq\flat_4\\ \flat_1=0\\ \flat_2=\flat_1+1\\ \flat_3=\flat_1+2\\ \flat_4=\flat_1+3}}^{n} (-1)^{\flat_1+\flat_2+\flat_3+\flat_4}(\alpha_{\flat_1},\alpha_{\flat_2},\alpha_{\flat_3},\alpha_{\flat_4}|\alpha_0,...,\hat{\alpha}_{\flat_1},\hat{\alpha}_{\flat_2},\hat{\alpha}_{\flat_3},\hat{\alpha}_{\flat_4},...,\alpha_n) \tag{2. 4}$$

Proposition 2.2. $(i)\quad d^{iv}\circ d^{iv}=0 \quad (ii)\quad p^{iv}\circ p^{iv}=0$

$$(i)\quad G_9(4)\xrightarrow{d^{iv}} G_5(4)\xrightarrow{d^{iv}} G_1(4)$$

$$(ii)\quad G_9(9)\xrightarrow{p^{iv}} G_5(5)\xrightarrow{p^{iv}} G_1(1)$$

Proof: (i) let $(\alpha_0,...,\alpha_8)\in G_9(4)$, apply map d^{iv} then

$$d^{iv}(\alpha_0,...,\alpha_8) = \sum_{\substack{\hbar_1=0\\ \hbar_2=\hbar_1+1\\ \hbar_3=\hbar_1+2\\ \hbar_4=\hbar_1+3}}^{8} (-1)^{\hbar_1+\hbar_2+\hbar_3+\hbar_4}(\alpha_0,...,\hat{\alpha}_{\hbar_1},\hat{\alpha}_{\hbar_2},\hat{\alpha}_{\hbar_3},\hat{\alpha}_{\hbar_4},...,\alpha_8) \tag{2. 5}$$

In Eq. (2. 5), there are 126 terms with each having 5 points, if expanded it becomes,

$$\begin{aligned} d^{iv}(\alpha_0, ..., \alpha_8) =& (\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8) - (\alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_8) + \\ & (\alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_8) - (\alpha_3, \alpha_4, \alpha_5, \alpha_7, \alpha_8) + \\ & (\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8) - (\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) + \\ & . \\ & . \\ & . \\ & (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) - (\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5) + \\ & (\alpha_0, \alpha_1, \alpha_3, \alpha_4, \alpha_5) - (\alpha_0, \alpha_1, \alpha_2, \alpha_4, \alpha_5) + \\ & (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_5) - (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \end{aligned} \quad (2.\ 6)$$

Now apply map d^{iv}, to get

$$d^{iv} \circ d^{iv} = \sum_{\substack{\hbar_5=\hbar_1+4 \\ \hbar_6=\hbar_1+5 \\ \hbar_7=\hbar_1+6 \\ \hbar_8=\hbar_1+7}}^{8} (-1)^{\hbar_5+\hbar_6+\hbar_7+\hbar_8} \sum_{\substack{\hbar_1=0 \\ \hbar_2=\hbar_1+1 \\ \hbar_3=\hbar_1+2 \\ \hbar_4=\hbar_1+3}}^{8} (-1)^{\hbar_1+\hbar_2+\hbar_3+\hbar_4} (\alpha_0, ..., \hat{\alpha}_{\hbar_1}, \hat{\alpha}_{\hbar_2}, \hat{\alpha}_{\hbar_3}, \hat{\alpha}_{\hbar_4}, \hat{\alpha}_{\hbar_5}, \hat{\alpha}_{\hbar_6}, \hat{\alpha}_{\hbar_7}, \hat{\alpha}_{\hbar_8}, ..., \alpha_8) \quad (2.\ 7)$$

In Eq.(2. 7), there are 630 terms, each of which has single points given by

$$\begin{aligned} d^{iv} \circ d^{iv} =& (\alpha_4) - (\alpha_5) + (\alpha_6) - (\alpha_7) + (\alpha_8) - (\alpha_3) + (\alpha_5) - (\alpha_6) + \\ & (\alpha_7) - (\alpha_8) + (\alpha_3) - (\alpha_4) + (\alpha_6) - (\alpha_7) + (\alpha_8) - (\alpha_3) + \\ & . \\ & . \\ & . \\ & + (\alpha_1) - (\alpha_2) + (\alpha_3) - (\alpha_4) + (\alpha_5) - (\alpha_0) + (\alpha_2) - (\alpha_3) + \\ & (\alpha_4) - (\alpha_5) + (\alpha_0) - (\alpha_1) + (\alpha_2) - (\alpha_3) + (\alpha_5) - (\alpha_0) + \\ & (\alpha_1) - (\alpha_2) + (\alpha_3) - (\alpha_4) \end{aligned} \quad (2.\ 8)$$

all the terms cancel each other, therefore

$$d^{iv} \circ d^{iv} = 0$$

Proof: (ii) let $(\alpha_0, ..., \alpha_8) \in G_9(9)$, apply map p^{iv} then

$$p^{iv}(\alpha_0, ..., \alpha_8) = \sum_{\substack{\flat_1=0 \\ \flat_2=\flat_1+1 \\ \flat_3=\flat_1+2 \\ \flat_4=\flat_1+3}}^{8} (-1)^{\flat_1+\flat_2+\flat_3+\flat_4} (\alpha_{\flat_1}, \alpha_{\flat_2}, \alpha_{\flat_3}, \alpha_{\flat_4} | \alpha_0, ..., \hat{\alpha}_{\flat_1}, \hat{\alpha}_{\flat_2}, \hat{\alpha}_{\flat_3}, \hat{\alpha}_{\flat_4}, ..., \alpha_8) \quad (2.\ 9)$$

in Eq.(2. 9), there are 126 terms with 5 points each with 4 projected points like such as

$$\begin{aligned} p^{iv}(\alpha_0,...,\alpha_8) =&(\alpha_0,\alpha_1,\alpha_2,\alpha_3|\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8)- \\ &(\alpha_0,\alpha_1,\alpha_2,\alpha_4|\alpha_3,\alpha_5,\alpha_6,\alpha_7,\alpha_8)+ \\ &. \\ &. \\ &. \\ &(\alpha_4,\alpha_6,\alpha_7,\alpha_8|\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_5)- \\ &(\alpha_5,\alpha_6,\alpha_7,\alpha_8|\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) \end{aligned} \quad (2.\ 10)$$

Now applying map p^{iv} again, to get

$$\begin{aligned} p^{iv}\circ p^{iv} = \sum_{\substack{\flat_5=\flat_1+4\\ \flat_6=\flat_1+5\\ \flat_7=\flat_1+6\\ \flat_8=\flat_1+7}}^{8} (-1)^{\flat_5+\flat_6+\flat_7+\flat_8} \sum_{\substack{\flat_1\neq\flat_2\neq\flat_3\neq\flat_4\\ \flat_1=0\\ \flat_2=\flat_1+1\\ \flat_3=\flat_1+2\\ \flat_4=\flat_1+3}}^{8} (-1)^{\flat_1+\flat_2+\flat_3+\flat_4}(\alpha_{\flat_1},\alpha_{\flat_2},\alpha_{\flat_3},\alpha_{\flat_4}, \\ \alpha_{\flat_5},\alpha_{\flat_6},\alpha_{\flat_7},\alpha_{\flat_8}|\alpha_0,...,\hat{\alpha}_{\flat_1},\hat{\alpha}_{\flat_2},\hat{\alpha}_{\flat_3},\hat{\alpha}_{\flat_4},\hat{\alpha}_{\flat_5},\hat{\alpha}_{\flat_6},\hat{\alpha}_{\flat_7},\hat{\alpha}_{\flat_8},...,\alpha_8) \end{aligned} \quad (2.\ 11)$$

In Eq.(2. 11), there are 630 terms with each having single points with 8 projected points, given by

$$\begin{aligned} p^{iv}\circ p^{iv} =&(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_6,\alpha_7,\alpha_8|\alpha_4)- \\ &(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_6,\alpha_7,\alpha_8|\alpha_5) \\ &. \\ &. \\ &. \\ &(\alpha_0,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8|\alpha_1)- \\ &(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8|\alpha_0) \end{aligned} \quad (2.\ 12)$$

all the terms cancel each other, therefore

$$p^{iv}\circ p^{iv}=0$$

Proposition 2.3. $p^{iv}\circ d^{iv}=d^{iv}\circ p^{iv}$.
Proof: Consider the subcomplex of the above diagram to be

$$\begin{array}{ccc} G_9(5) & \xrightarrow{d^{iv}} & G_5(5) \\ \downarrow{\scriptstyle p^{iv}} & & \downarrow{\scriptstyle p^{iv}} \\ G_5(1) & \xrightarrow{d^{iv}} & G_1(1) \end{array} \quad (B)$$

Let $(\alpha_0, ..., \alpha_8) \in G_9(5)$, apply map d^{iv}

$$d^{iv}(\alpha_0, \ldots, \alpha_8) = \sum_{\substack{\hbar_1=0 \\ \hbar_2=\hbar_1+1 \\ \hbar_3=\hbar_1+2 \\ \hbar_4=\hbar_1+3}}^{8} (-1)^{\hbar_1+\hbar_2+\hbar_3+\hbar_4}(\alpha_0, ..., \hat{\alpha}_{\hbar_1}, \hat{\alpha}_{\hbar_2}, \hat{\alpha}_{\hbar_3}, \hat{\alpha}_{\hbar_4}, ..., \alpha_8) \quad (2.\ 13)$$

then:

$$p^{iv} \circ d^{iv} = \sum_{\substack{\flat_1=\hbar_1+4 \\ \flat_2=\flat_1+1 \\ \flat_3=\flat_1+2 \\ \flat_4=\flat_1+3}}^{8} (-1)^{\flat_1+\flat_2+\flat_3+\flat_4} \sum_{\substack{\hbar_1=0 \\ \hbar_2=\hbar_1+1 \\ \hbar_3=\hbar_1+2 \\ \hbar_4=\hbar_1+3}}^{8} (-1)^{\hbar_1+\hbar_2+\hbar_3+\hbar_4}(\alpha_{\flat_1}, \alpha_{\flat_2}, \alpha_{\flat_3}, \alpha_{\flat_4}|\alpha_0, ..., \hat{\alpha}_{\hbar_1}, \hat{\alpha}_{\hbar_2}, \hat{\alpha}_{\hbar_3}, \hat{\alpha}_{\hbar_4}, \hat{\alpha}_{\flat_1}, \hat{\alpha}_{\flat_2}, \hat{\alpha}_{\flat_3}, \hat{\alpha}_{\flat_4}, ..., \alpha_8) \quad (2.\ 14)$$

now take $(\alpha_0, ..., \alpha_8) \in G_9(5)$ again and apply map p^{iv}

$$p^{iv}(\alpha_0, ..., \alpha_8) = \sum_{\substack{\flat_1=0 \\ \flat_2=\flat_1+1 \\ \flat_3=\flat_1+2 \\ \flat_4=\flat_1+3}}^{8} (-1)^{\flat_1+\flat_2+\flat_3+\flat_4}(\alpha_{\flat_1}, \alpha_{\flat_2}, \alpha_{\flat_3}, \alpha_{\flat_4}|\alpha_0, ..., \hat{\alpha}_{\flat_1}, \hat{\alpha}_{\flat_2}, \hat{\alpha}_{\flat_3}, \hat{\alpha}_{\flat_4}, ..., \alpha_8) \quad (2.\ 15)$$

now apply map d^{iv}

$$d^{iv} \circ p^{iv} = \sum_{\substack{\hbar_1=\flat_1+4 \\ \hbar_2=\hbar_1+1 \\ \hbar_3=\hbar_1+2 \\ \hbar_4=\hbar_1+3}}^{8} (-1)^{\hbar_1+\hbar_2+\hbar_3+\hbar_4} \sum_{\substack{\flat_1=0 \\ \flat_2=\flat_1+1 \\ \flat_3=\flat_1+2 \\ \flat_4=\flat_1+3}}^{8} (-1)^{\flat_1+\flat_2+\flat_3+\flat_4}(\alpha_{\flat_1}, \alpha_{\flat_2}, \alpha_{\flat_3}, \alpha_{\flat_4}|\alpha_0, ..., \hat{\alpha}_{\flat_1}, \hat{\alpha}_{\flat_2}, \hat{\alpha}_{\flat_3}, \hat{\alpha}_{\flat_4}, \hat{\alpha}_{\hbar_1}, \hat{\alpha}_{\hbar_2}, \hat{\alpha}_{\hbar_3}, \hat{\alpha}_{\hbar_4}, ..., \alpha_8) \quad (2.\ 16)$$

using dummy indices $\hbar_1, \hbar_2, \hbar_3, \hbar_4$ and $\flat_1, \flat_2, \flat_3, \flat_4$ in Eq.(2. 14) and Eq.(2. 16), it is observed that the diagram A is commutative.

2.4. **Fifth Order Grassmannian Complex.** Fifth order Grassmannian complex is constructed by the following

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow p^v & & \downarrow p^v & & \downarrow p^v \\
\cdots & \xrightarrow{d^v} & G_{n+25}(n+11) & \xrightarrow{d^v} & G_{n+20}(n+11) & \xrightarrow{d^v} & G_{n+15}(n+11) \\
 & & \downarrow p^v & & \downarrow p^v & & \downarrow p^v \\
\cdots & \xrightarrow{d^v} & G_{n+20}(n+6) & \xrightarrow{d^v} & G_{n+15}(n+6) & \xrightarrow{d^v} & G_{n+10}(n+6) \\
 & & \downarrow p^v & & \downarrow p^v & & \downarrow p^v \\
\cdots & \xrightarrow{d^v} & G_{n+15}(n+1) & \xrightarrow{d^v} & G_{n+10}(n+1) & \xrightarrow{d^v} & G_{n+5}(n+1)
\end{array} \tag{C}$$

where d^v is fifth order mixed partial differential map, defined as

$$d^v : (\alpha_0, \ldots, \alpha_n) \mapsto \sum_{\substack{\hbar_1 \neq \hbar_2 \neq \hbar_3 \neq \hbar_4 \neq \hbar_5 \\ \hbar_1 = 0 \\ \hbar_2 = \hbar_1 + 1 \\ \hbar_3 = \hbar_1 + 2 \\ \hbar_4 = \hbar_1 + 3 \\ \hbar_5 = \hbar_1 + 4}}^{n} (-1)^{\hbar_1 + \hbar_2 + \hbar_3 + \hbar_4 + \hbar_5} (\alpha_0, ..., \hat{\alpha}_{\hbar_1}, \hat{\alpha}_{\hbar_2}, \hat{\alpha}_{\hbar_3}, \hat{\alpha}_{\hbar_4}, \hat{\alpha}_{\hbar_5}, \ldots, \alpha_n) \tag{2. 17}$$

and p^v is another fifth order mixed partial differential morphism, called projection map, given by

$$p^v(\alpha_0, \ldots, \alpha_n) = \sum_{\substack{\flat_1 \neq \flat_2 \neq \flat_3 \neq \flat_4 \neq \flat_5 \\ \flat_1 = 0 \\ \flat_2 = \flat_1 + 1 \\ \flat_3 = \flat_1 + 2 \\ \flat_4 = \flat_1 + 3 \\ \flat_5 = \flat_1 + 4}}^{n} (-1)^{\flat_1 + \flat_2 + \flat_3 + \flat_4 + \flat_5} (\alpha_{\flat_1}, \alpha_{\flat_2}, \alpha_{\flat_3}, \alpha_{\flat_4}, \alpha_{\flat_5} | \alpha_0, ..., \hat{\alpha}_{\flat_1}, \hat{\alpha}_{\flat_2}, \hat{\alpha}_{\flat_3}, \hat{\alpha}_{\flat_4}, \hat{\alpha}_{\flat_5}, \ldots, \alpha_n) \tag{2. 18}$$

Proposition 2.5. $(i)\ d^v \circ d^v = 0 \quad (ii)\ p^v \circ p^v = 0$

$$(i) \quad G_{11}(5) \xrightarrow{d^v} G_6(5) \xrightarrow{d^v} G_1(5)$$

$$(ii) \quad G_{11}(11) \xrightarrow{p^v} G_6(6) \xrightarrow{p^v} G_1(1)$$

Proof: (i) let $(\alpha_0, ..., \alpha_{10}) \in G_{11}(5)$ and apply map d^v to it then

$$d^v(\alpha_0, ..., \alpha_{10}) = \sum_{\substack{\hbar_1=0\\ \hbar_2=\hbar_1+1\\ \hbar_3=\hbar_1+2\\ \hbar_4=\hbar_1+3\\ \hbar_5=\hbar_1+4}}^{10} (-1)^{\hbar_1+\hbar_2+\hbar_3+\hbar_4+\hbar_5}(\alpha_0, ..., \hat{\alpha}_{\hbar_1}, \hat{\alpha}_{\hbar_2}, \hat{\alpha}_{\hbar_3}, \hat{\alpha}_{\hbar_4}, \hat{\alpha}_{\hbar_5}, ..., \alpha_{10}) \quad (2.\ 19)$$

using combinations in the Eq.(2. 19), there are 462 terms with each having 6 points as given below.

$$\begin{aligned} d^v(\alpha_0, ..., \alpha_{10}) = &+ (\alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}) \\ &- (\alpha_4, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}) \\ &+ (\alpha_4, \alpha_5, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}) \\ &. \\ &. \\ &. \\ &- (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6) \\ &+ (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6) \\ &- (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \end{aligned} \quad (2.\ 20)$$

Applying map d^v again, to get

$$d^v \circ d^v = \sum_{\substack{\hbar_6=\hbar_1+5\\ \hbar_7=\hbar_1+6\\ \hbar_8=\hbar_1+7\\ \hbar_9=\hbar_1+8\\ \hbar_{10}=\hbar_1+9}}^{10} (-1)^{\hbar_6+\hbar_7+\hbar_8+\hbar_9+\hbar_{10}} \sum_{\substack{\hbar_1=0\\ \hbar_2=\hbar_1+1\\ \hbar_3=\hbar_1+2\\ \hbar_4=\hbar_1+3\\ \hbar_5=\hbar_1+4}}^{10} (-1)^{\hbar_1+\hbar_2+\hbar_3+\hbar_4+\hbar_5}(\alpha_0, ..., \hat{\alpha}_{\hbar_1}, \hat{\alpha}_{\hbar_2}, \hat{\alpha}_{\hbar_3}, \hat{\alpha}_{\hbar_4}, \hat{\alpha}_{\hbar_5}, \hat{\alpha}_{\flat_6}, \hat{\alpha}_{\flat_7}, \hat{\alpha}_{\flat_8}, \hat{\alpha}_{\flat_9}, \hat{\alpha}_{\flat_{10}}, ..., \alpha_{10}) \quad (2.\ 21)$$

In the Eq.(2. 21), there are 2772 terms each having single points given by

$$\begin{aligned} d^v \circ d^v = &+ (\alpha_5) - (\alpha_6) + (\alpha_7) - (\alpha_8) + (\alpha_9) - (\alpha_{10}) \\ &- (\alpha_4) + (\alpha_6) - (\alpha_7) + (\alpha_8) - (\alpha_9) + (\alpha_{10}) \\ &. \\ &. \\ &. \\ &+ (\alpha_0) - (\alpha_1) + (\alpha_2) - (\alpha_3) + (\alpha_4) - (\alpha_6) \\ &- (\alpha_0) + (\alpha_1) - (\alpha_2) + (\alpha_3) - (\alpha_4) + (\alpha_5) \end{aligned} \quad (2.\ 22)$$

all the terms cancel each other, therefore:

$$d^v \circ d^v = 0$$

Proof: (ii) let $(\alpha_0, ..., \alpha_{10}) \in G_{11}(11)$, apply map p^v then

$$p^v(\alpha_0, ..., \alpha_{10}) = \sum_{\substack{\flat_1=0\\ \flat_2=\flat_1+1\\ \flat_3=\flat_1+2\\ \flat_4=\flat_1+3\\ \flat_5=\flat_1+4}}^{10} (-1)^{\flat_1+\flat_2+\flat_3+\flat_4+\flat_5}(\alpha_{\flat_1}, \alpha_{\flat_2}, \alpha_{\flat_3}, \alpha_{i_4}, \alpha_{\flat_5}|\alpha_0, \dots, \hat{\alpha}_{\flat_1}, \hat{\alpha}_{\flat_2}, \hat{\alpha}_{\flat_3}, \hat{\alpha}_{\flat_4}, \hat{\alpha}_{\flat_5}, \dots, \alpha_{10}) \quad (2.\ 23)$$

In Eq.(2. 23), there are 462 terms and each have 6 points with 5 projected points like given below.

$$\begin{aligned} p^v(\alpha_0, ..., \alpha_{10}) =&(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4|\alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10})-\\ &(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_5|\alpha_4, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10})+\\ &.\\ &.\\ &.\\ &(\alpha_5, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}|\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6)-\\ &(\alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}|\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \end{aligned} \quad (2.\ 24)$$

Now applying map p^v again, to get

$$p^v \circ p^v = \sum_{\substack{\flat_6=\flat_1+5\\ \flat_7=\flat_1+6\\ \flat_8=\flat_1+7\\ \flat_9=\flat_1+8\\ \flat_{10}=\flat_1+9}}^{10} (-1)^{\flat_6+\flat_7+\flat_8+\flat_9+\flat_{10}} \sum_{\substack{\flat_1=0\\ \flat_2=\flat_1+1\\ \flat_3=\flat_1+2\\ \flat_4=\flat_1+3\\ \flat_5=\flat_1+4}}^{10} (-1)^{\flat_1+\flat_2+\flat_3+\flat_4+\flat_5}(\alpha_{\flat_1}, \alpha_{\flat_2}, \alpha_{\flat_3}, \alpha_{\flat_4}, \alpha_{\flat_5}, \alpha_{\flat_6}, \alpha_{\flat_7}, \alpha_{\flat_8}, \alpha_{\flat_9}, \alpha_{\flat_{10}}|\alpha_0, \dots, \hat{\alpha}_{\flat_1}, \hat{\alpha}_{\flat_2}, \hat{\alpha}_{\flat_3}, \hat{\alpha}_{\flat_4}, \hat{\alpha}_{\flat_5}, \hat{\alpha}_{\flat_6}, \hat{\alpha}_{\flat_7}, \hat{\alpha}_{\flat_8}, \hat{\alpha}_{\flat_9}, \hat{\alpha}_{\flat_{10}}, \dots, \alpha_{10}) \quad (2.\ 25)$$

In Eq.(2. 25), there are 2772 terms each have single point with 10 projected points given by

$$\begin{aligned} p^v \circ p^v =&(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9|\alpha_{10})-\\ &(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_{10}|\alpha_9)+\\ &.\\ &.\\ &.\\ &(\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}|\alpha_1)-\\ &(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}|\alpha_0) \end{aligned} \quad (2.\ 26)$$

all the terms cancel each other, therefore

$$p^v \circ p^v = 0$$

Proposition 2.6. $p^v \circ d^v = d^v \circ p^v$.
Proof: On taking the subcomplex of the above diagram given by

$$\begin{array}{ccc} G_{11}(6) & \xrightarrow{d^v} & G_6(6) \\ \downarrow{\scriptstyle p^v} & & \downarrow{\scriptstyle p^v} \\ G_6(1) & \xrightarrow{d^v} & G_1(1) \end{array} \tag{D}$$

Let $(\alpha_0, ..., \alpha_{10}) \in G_{11}(6)$, apply map d^v

$$d^v(\alpha_0, \ldots, \alpha_{10}) = \sum_{\substack{\hbar_1=0 \\ \hbar_2=\hbar_1+1 \\ \hbar_3=\hbar_1+2 \\ \hbar_4=\hbar_1+3 \\ \hbar_5=\hbar_1+4}}^{10} (-1)^{\hbar_1+\hbar_2+\hbar_3+\hbar_4+\hbar_5}(\alpha_0, ..., \hat{\alpha}_{\hbar_1}, \hat{\alpha}_{\hbar_2}, \hat{\alpha}_{\hbar_3}, \hat{\alpha}_{\hbar_4}, \hat{\alpha}_{\hbar_5}, ..., \alpha_{10}) \tag{2. 27}$$

then

$$p^v \circ d^v = \sum_{\substack{\flat_1=\hbar_1+5 \\ \flat_2=\flat_1+1 \\ \flat_3=\flat_1+2 \\ \flat_4=\flat_1+3 \\ \hbar_5=\flat_1+4}}^{10} (-1)^{\flat_1+\flat_2+\flat_3+\flat_4+\flat_5} \sum_{\substack{\hbar_1=0 \\ \hbar_2=\hbar_1+1 \\ \hbar_3=\hbar_1+2 \\ \hbar_4=\hbar_1+3 \\ \hbar_5=\hbar_1+4}}^{10} (-1)^{\hbar_1+\hbar_2+\hbar_3+\hbar_4+\hbar_5}(\alpha_{\flat_1}, \alpha_{\flat_2}, \alpha_{\flat_3}, \alpha_{\flat_4}, \alpha_{\flat_5}|\alpha_0, ..., \hat{\alpha}_{\hbar_1}, \hat{\alpha}_{\hbar_2}, \hat{\alpha}_{\hbar_3}, \hat{\alpha}_{\hbar_4}, \hat{\alpha}_{\hbar_5}, \hat{\alpha}_{\flat_1}, \hat{\alpha}_{\flat_2}, \hat{\alpha}_{\flat_3}, \hat{\alpha}_{\flat_4}, \hat{\alpha}_{\flat_5}, ..., \alpha_{10}) \tag{2. 28}$$

now take again $(\alpha_0, ..., \alpha_{10}) \in G_{11}(6)$, apply map p^v

$$p^v(\alpha_0, ..., \alpha_{10}) = \sum_{\substack{\flat_1=0 \\ \flat_2=\flat_1+1 \\ \flat_3=\flat_1+2 \\ \flat_4=\flat_1+3 \\ \flat_5=\flat_1+4}}^{10} (-1)^{\flat_1,+\flat_2+\flat_3+\flat_4+\flat_5}(\alpha_{\flat_1}, \alpha_{\flat_2}, \alpha_{\flat_3}, \alpha_{\flat_4}, \alpha_{\flat_5}|\alpha_0, ..., \hat{\alpha}_{\flat_1}, \hat{\alpha}_{\flat_2}, \hat{\alpha}_{\flat_3}, \hat{\alpha}_{\flat_4}, \hat{\alpha}_{\flat_5}, ..., \alpha_{10}) \tag{2. 29}$$

now apply map d^v

$$d^v \circ p^v = \sum_{\substack{\hbar_1=\flat_1+5 \\ \hbar_2=\hbar_1+1 \\ \hbar_3=\hbar_1+2 \\ \hbar_4=\hbar_1+3 \\ \hbar_5=\hbar_1+4}}^{10} (-1)^{\hbar_1+\hbar_2+\hbar_3+\hbar_4+\hbar_5} \sum_{\substack{\flat_1=0 \\ \flat_2=\flat_1+1 \\ \flat_3=\flat_1+2 \\ \flat_4=\flat_1+3 \\ \flat_5=\flat_1+4}}^{10} (-1)^{\flat_1+\flat_2+\flat_3+\flat_4+\flat_5}(\alpha_{\flat_1}, \alpha_{\flat_2}, \alpha_{\flat_3}, \alpha_{\flat_4}, \alpha_{\flat_5}|\alpha_0, ..., \hat{\alpha}_{\flat_1}, \hat{\alpha}_{\flat_2}, \hat{\alpha}_{\flat_3}, \hat{\alpha}_{\flat_4}, \hat{\alpha}_{\flat_5}, \hat{\alpha}_{\hbar_1}, \hat{\alpha}_{\hbar_2}, \hat{\alpha}_{\hbar_3}, \hat{\alpha}_{\hbar_4}, \hat{\alpha}_{\hbar_5}, ..., \alpha_{10}) \tag{2. 30}$$

use dummy indices $\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5$ and $\flat_1, \flat_2, \flat_3, \flat_4, \flat_5$ in equation Eq.(2. 28) and Eq.(2. 30), it is proved that the diagram C is commutative.

2.7. N^{th} **Order Grassmannian Complex.** Let us generalize Grassmannian as N^{th} order Grassmannian complex, shown below:

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow p^n & & \downarrow p^n & & \downarrow p^n \\ \cdots & \xrightarrow{d^n} & G_{4n+1}(4n+1) & \xrightarrow{d^n} & G_{3n+1}(4n+1) & \xrightarrow{d^n} & G_{2n+1}(4n+1) \\ & & \downarrow p^n & & \downarrow p^n & & \downarrow p^n \\ \cdots & \xrightarrow{d^n} & G_{3n+1}(3n+1) & \xrightarrow{d^n} & G_{2n+1}(3n+1) & \xrightarrow{d^n} & G_{n+1}(3n+1) \\ & & \downarrow p^n & & \downarrow p^n & & \downarrow p^n \\ \cdots & \xrightarrow{d^n} & G_{2n+1}(2n+1) & \xrightarrow{d^n} & G_{n+1}(2n+1) & \xrightarrow{d^n} & G_1(2n+1) \end{array} \tag{E}$$

where

$$d^n : (\alpha_0, \ldots, \alpha_n) \mapsto \sum_{\substack{\hbar_1 \neq \hbar_2 \ldots \neq \hbar_n \\ \hbar_1 = 0 \\ \hbar_2 = \hbar_1 + 1 \\ \vdots \\ \hbar_n = \hbar_1 + (n-1)}}^{n} (-1)^{\hbar_1 + \hbar_2 + \ldots + \hbar_n} (\alpha_0, ..., \hat{\alpha}_{\hbar_1}, ..., \hat{\alpha}_{\hbar_n}, ..., \alpha_n) \tag{2. 31}$$

and

$$p^n : (\alpha_0, \ldots, \alpha_n) \mapsto \sum_{\substack{\flat_1 \neq \flat_2 \ldots \neq \flat_n \\ \flat_1 = 0 \\ \flat_2 = \flat_1 + 1 \\ \vdots \\ \flat_n = \flat_1 + (n-1)}}^{n} (-1)^{\flat_1 + \flat_2 + \ldots + \flat_n} (\alpha_{\flat_1}, ..., \alpha_{\flat_n} | \alpha_0, ..., \hat{\alpha}_{\flat_1}, ..., \hat{\alpha}_{\flat_n}, ..., \alpha_n) \tag{2. 32}$$

Proposition 2.8. $p^n \circ d^n = d^n \circ p^n$.

$$\begin{array}{ccc} G_{2n+1}(2n+1) & \xrightarrow{d^n} & G_{n+1}(2n+1) \\ \downarrow p^n & & \downarrow p^n \\ G_{n+1}(n+1) & \xrightarrow{d^n} & G_1(n+1) \end{array} \tag{F}$$

Proof: Let $(\alpha_0, ..., \alpha_{2n}) \in G_{2n+1}(2n+1)$, apply map d^n

$$d^n(\alpha_0, \ldots, \alpha_{2n}) = \sum_{\substack{\hbar_1 = 0 \\ \hbar_2 = \hbar_1 + 1 \\ \vdots \\ \hbar_n = \hbar_1 + (n-1)}}^{2n} (-1)^{\hbar_1 + \hbar_2 + \ldots + \hbar_n} (\alpha_0, ..., \hat{\alpha}_{\hbar_1}, ..., \hat{\alpha}_{\hbar_n}, ..., \alpha_{2n}) \tag{2. 33}$$

then

$$p^n \circ d^n = \sum_{\substack{\flat_1=\hbar_1+n \\ \flat_2=\flat_1+1 \\ \vdots \\ \flat_n=\flat_1+(2n-1)}}^{2n} (-1)^{\flat_1+\flat_2+\ldots+\flat_n} \sum_{\substack{\hbar_1=0 \\ \hbar_2=\hbar_1+1 \\ \vdots \\ \hbar_n=\hbar_1+(n-1)}}^{2n} (-1)^{\hbar_1+\hbar_2+\ldots+\hbar_n} (\alpha_{\flat_1}, ..., \alpha_{\flat_n} | \alpha_0, ..., \hat{\alpha}_{\hbar_1}, ..., \hat{\alpha}_{\hbar_n}, \hat{\alpha}_{\flat_1}, ..., \hat{\alpha}_{\flat_n}, ..., \alpha_{2n}) \tag{2. 34}$$

now take again $(\alpha_0, ..., \alpha_{2n}) \in G_{2n+1}(2n+1)$, apply map p^n

$$p^n(\alpha_0, ..., \alpha_{2n}) = \sum_{\substack{\flat_1=0 \\ \flat_2=\flat_1+1 \\ \vdots \\ \flat_n=\flat_1+(n-1)}}^{2n} (-1)^{\flat_1+\flat_2+\ldots+\flat_n} (\alpha_{\flat_1}, ..., \alpha_{\flat_n} | \alpha_0, ..., \hat{\alpha}_{\flat_1}, ..., \hat{\alpha}_{\flat_n}, ..., \alpha_{2n}) \tag{2. 35}$$

now apply map d^n

$$d^n \circ p^n = \sum_{\substack{\hbar_1=\flat_1+n \\ \hbar_2=\hbar_1+1 \\ \vdots \\ \hbar_n=\hbar_1+(2n-1)}}^{2n} (-1)^{\hbar_1+\hbar_2+\ldots+\hbar_n} \sum_{\substack{\flat_1=0 \\ \flat_2=\flat_1+1 \\ \vdots \\ \flat_n=\flat_1+(n-1)}}^{2n} (-1)^{\flat_1+\flat_2+\ldots+\flat_n} (\alpha_{\flat_1}, ..., \alpha_{\flat_n} | \alpha_0, ..., \hat{\alpha}_{\flat_1}, ..., \hat{\alpha}_{\flat_n}, \hat{\alpha}_{\hbar_1}, ..., \hat{\alpha}_{\hbar_n}, ..., \alpha_{2n}) \tag{2. 36}$$

use dummy indices $\hbar_1, \hbar_2, ...\hbar_n$ and $\flat_1, \flat_2, ..., \flat_n$ in Eq.(2. 34) and Eq.(2. 36), it is observe that the diagram E is commutative.

3. Conclusion

This study generalized the Grassmannian complex construction upto N^{th} order. Free abelian group have been connected through higher order mixed partial differential morphisms, through which the associated diagrams of these complexes shown commutative and bi-complex. This paper is a comprehensive contribution to the field of algebraic complexes. Also this work will serve as pioneer in the study of generalized Grassmannian chain complexes.

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