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Hyers–Ulam Stability in Terms of Dichotomy of First Order Linear Dynamic Systems

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Abstract. In this paper, we establish connections between the Hyers–Ulam stability of the first order linear dynamic system and its dichotomy. The main tool for proving our results is the spectral decomposition theorem on time scales.

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1. Introduction

The study of stability problems for various functional equations was triggered by an intriguing and famous talk presented by Ulam in the fall of 1940, at Wisconsin University. In his talk, Ulam discussed a number of important unsolved mathematical problems. Among them, a question concerning the stability of homomorphisms seemed too abstract for anyone to reach any conclusion. The question was following(cf. [27, 28]):
Let G_1 be a group and G_2 be a metric group with metric $d(.,.)$. For a given $\epsilon > 0$, can

there be found $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality

$$d(f(xy), f(x)f(y)) \leq \epsilon, \forall\, x, y \in G_1,$$

then there exists a homomorphism $g : G_1 \to G_2$ such that

$$d(g(x), f(x)) \leq \delta, \forall\, x \in G_1.$$

If the answer is yes, then we say that the functional equation for homomorphism is stable on (G_1, G_2).

In the following year, Hyers was able to give a partial solution to Ulam's question and that was the first significant breakthrough and step toward more solutions in this area. For the case where G_1 and G_2 are assumed to be Banach spaces, Hyers [11] was the first mathematician who brilliantly answered to the question by direct approach and therefore this stability phenomena was named as "Hyers–Ulam Stability".

In 1978, Rassias extended the partial answer by Hyers in his paper [24] by using direct approach. In fact, he generalized Hyers answer, in more stronger way. This exciting result of Rassias attracted the attention of a large number of mathematicians across the glob and now this area has become a very active area of research and is known as Hyers–Ulam–Rassias stability. Since 1980's numerous number of papers dealing with the stability of different type of functional equations have been published, see [3, 4, 7, 12–16].

However, among the functional equations, Obloza seems to be the first mathematician who has investigated the Hyers–Ulam stability of linear differential equations (see [21,22]). Thereafter, Alsina and Ger published their paper which handles the Hyers–Ulam stability of the linear differential equation $y^{'}(t) = y(t)$. They proved that if a differentiable function $y(t)$ is a solution of the inequality $|y^{'}(t) - y(t)| \leq \varepsilon$ for some $\varepsilon \geq 0$ and for all $t \in (a, \infty)$, then there exists a constant c such that $|y(t) - ce^t| \leq 3\varepsilon$ for all $t \in (a, \infty)$, where $a \in \mathbf{R}$ (cf. [1]). Note that $y_c(t) = ce^t$ is one-parameter family of solutions of $y^{'}(t) = y(t)$. These results were generalized for second and higher order linear differential equations by different mathematicians, e.g see([17, 20, 26]). Recently in 2016, Li *et al.* generalized all these results to nth order linear homogeneous and non-homogeneous differential equations with non-constant coefficients using open-mapping approach (see [19]).

Serious work on the stability problem of differential equations has been initiated since 2000's and so far different classes of differential equations have been investigated for stability, with different approaches, we recommend [8, 10, 18, 23, 25, 29–32, 34].

Recently, Buşe *et al.* [5] established a relationship between Hyers–Ulam stability and dichotomy, i.e., they proved that $m \times m$ complex linear system is Hyers–Ulam stable if and only if it is dichotomic, i.e., its associated matrix has no eigenvalues on the imaginary axis. Thereafter, Barbu *et al.* in [2] extended this relationship to the discrete case.

The main purpose of this paper is to unify the results of [2] and [5] i.e. we give a relationship between the Hyers–Ulam stability and dichotomy of the first order linear dynamic system $x^{\Delta}(t) = Gx(t),\ t \in \mathbf{T}$, using the idea of time scale. Details about the time scale analysis is given in next section.

2. Preliminaries

The idea of time scale analysis was introduced by Hilger [9], in order to unify the discrete and continuous analysis. Here, we recall the main definitions of time scales.

The arbitrary non–empty closed subset of real numbers is called a time scale denoted by $\mathbf{T}$. The forward jump operator $\sigma : \mathbf{T} \to \mathbf{T}$, backward jump operator $\rho : \mathbf{T} \to \mathbf{T}$ and the graininess function $\mu : \mathbf{T} \to [0, \infty)$ are respectively defined as:

$$\sigma(t) = \inf\{s \in \mathbf{T} : s > t\},\ \rho(t) = \sup\{s \in \mathbf{T} : s < t\},\ \mu(t) = \sigma(t) - t.$$

A point $s \in \mathbf{T}$ is called left scattered and left dense if $s > \rho(s)$ and $\rho(s) = s$, respectively. If $s < \sigma(s)$ and $\sigma(s) = s$, then such a point $s \in \mathbf{T}$ is termed right scattered and right dense, respectively. The set $\mathbf{T}^z$ is known as the derived form of time scale $\mathbf{T}$ and is defined as

$$\mathbf{T}^z = \begin{cases} \mathbf{T}\backslash(\rho(\sup \mathbf{T}), \sup \mathbf{T}], & \text{if } \sup \mathbf{T} < \infty, \\ \mathbf{T}, & \text{if } \sup \mathbf{T} = \infty. \end{cases}$$

A function $g : \mathbf{T} \to \mathbf{R}$ is said to be right–dense continuous if it is continuous at all right-dense points in $\mathbf{T}$ and its left–sided limit exists at all left–dense points in $\mathbf{T}$, where $\mathbf{R}$ is the set of real numbers. A function $g : \mathbf{T} \to \mathbf{R}$ is called regressive if $1 + \mu(t)g(t) \neq 0$ for all $t \in \mathbf{T}^z$ and if $1+\mu(t)g(t) > 0$, then the function g is termed positively regressive. The sets of all right–dense continuous, regressive and right-dense continuous, positively regressive functions are denoted by $\mathcal{R}_{\mathcal{F}}(\mathbf{T})$ and $\mathcal{R}_{\mathcal{F}}(\mathbf{T})^+$, respectively.

The delta derivative of the function $g : \mathbf{T} \to \mathbf{R}$ at $t \in \mathbf{T}^z$ is defined by

$$g^{\Delta}(t) = \lim_{s \to t,\ s \neq \sigma(t)} \frac{g(\sigma(t)) - g(s)}{\sigma(t) - s}.$$

The Δ–integral of the rd-continuous function $g : \mathbf{T} \to \mathbf{R}$ is defined by

$$\int_a^b g(t)\Delta t = G(b) - G(a),\ \forall\, a, b \in \mathbf{T},$$

where the rd–continuous function G is an anti–derivative of g, i.e. $G^{\Delta} = g$ on $\mathbf{T}^z$.

Definition 2.1. *If $g \in \mathcal{R}_{\mathcal{F}}(\boldsymbol{T})$ satisfies $\inf_{t\in \boldsymbol{T}} |1 + \mu(t)g(t)| > 0$, then g is called strongly regressive.*

Definition 2.2. *If $G \in \mathcal{R}_{\mathcal{F}}(\boldsymbol{T})$, then generalized exponential function $e_G(r, u)$ on $\boldsymbol{T}$ is defined as*

$$e_G(r, u) = \exp\left(\int_u^r \chi_{\mu(t)} G(t)\Delta t\right) \quad \forall\, r, u \in \boldsymbol{T},$$

with cylindrical transformation

$$\chi_{\mu(t)} G(t) = \begin{cases} \dfrac{\mathrm{Log}(1 + \mu(t)G(t))}{\mu(t)}, & \text{if } \mu(t) \neq 0, \\ G(t), & \text{if } \mu(t) = 0. \end{cases}$$

Definition 2.3. *Let $\boldsymbol{T}$ be an unbounded time scale such that for any $s_0 \in \boldsymbol{T}$, we have*

$$E_{\mathbb{C}}(\boldsymbol{T}) := \left\{\kappa \in \mathbb{C} : \lim_{S\to\infty} \sup \frac{1}{S - s_0} \int_{s_0}^{S} \lim_{u \to \mu(s)} \frac{\log|1 + u\kappa|}{u} \Delta s < 0 \right\},$$

and

$$E_{\boldsymbol{R}}(\boldsymbol{T}) := \{\kappa \in \boldsymbol{R} | \forall\, Q\ \in \boldsymbol{T} : \exists\, q \in \boldsymbol{T} \text{ with } q > Q \text{ such that } 1 + \mu(s)\kappa = 0\}.$$

We define the set of exponential stability on $\boldsymbol{T}$ as:

$$E(\boldsymbol{T}) = E_{\mathbb{C}}(\boldsymbol{T}) \cup E_{\boldsymbol{R}}(\boldsymbol{T}).$$

Lemma 2.4. *[33] Let $\boldsymbol{T}$ be a time scale and β be a positive number such that $\beta \in \mathcal{R}_{\mathcal{F}}(\boldsymbol{T})^+$. Then for the corresponding scalar system $z^{\Delta} = \beta z$ the following inequality holds*

$$e_\beta(u, v) \le e^{\beta(u-v)}\ for\ all\ u \ge v.$$

Let q_G be the characteristic polynomial of the regressive matrix G and let $\mathcal{S}(G) := \{\kappa_1, \kappa_2, \ldots, \kappa_k\}$, $k \leq m$ be its spectrum, where each $\kappa_1, \kappa_2, \ldots, \kappa_k$ are regressive. There exist integers $c_1, c_2, \ldots, c_k \geq 1$ such that

$$q_G(\kappa) = (\kappa - \kappa_1)^{c_1}(\kappa - \kappa_2)^{c_2} \ldots (\kappa - \kappa_k)^{c_k},\ c_1 + c_2 + \cdots + c_k = c.$$

Let $i = 1, 2, \ldots, k$ and $\mathcal{Z}_i := \ker(G - \kappa_i I)^{m_i}$. Clearly $\mathcal{Z}_i$ is an $e_G(t, 0)$–invariant subspace of $\mathbb{C}^m$ and $\dim(\mathcal{Z}_i) \geq 1$. So for Time Scale $\mathbf{T}$ we have the following Spectral Decomposition Theorem.

Theorem 2.5. *[33] For each $z \in \mathbb{C}^m$, there exist $z_i \in \mathcal{Z}_i$ $(i = 1, 2, \ldots, k)$ such that*

$$e_G(t, 0)z = e_G(t, 0)z_1 + e_G(t, 0)z_2 + \cdots + e_G(t, 0)z_k, \quad t \in \boldsymbol{T}.$$

Moreover, if $z_i(t) := e_G(t, 0)z_i$, then $z_i(t) \in \mathcal{Z}_i\ \forall\ t \in \boldsymbol{T}$ and there exists $\mathbb{C}^m$–valued polynomial $h_i(t)$ with degree less than or equal to $m_i - 1$ such that

$$z_i(t) = e_{\kappa i}(t, 0)h_i(t), \quad t \in \boldsymbol{T},\ i = 1, 2, \ldots, k.$$

Proof. From Cayley–Hamilton theorem and using the fact that

$$\ker[gh(G)] = \ker[g(G)] \oplus \ker[h(G)],$$

whenever complex valued polynomials g and h are relative prime and it follows that

$$\mathbb{C}^m = \mathcal{Z}_1 \oplus \mathcal{Z}_2 \oplus \cdots \oplus \mathcal{Z}_k. \tag{2. 1}$$

Let $z \in \mathbb{C}^m$, for each $i \in \{1, 2, \ldots, k\}$ there exists a unique $z_i \in \mathcal{Z}_i$ such that

$$z = z_1 + z_2 + \cdots + z_k,$$

and then

$$e_G(t, 0)z = e_G(t, 0)z_1 + e_G(t, 0)z_2 + \cdots + e_G(t, 0)z_k, \quad t \in \mathbf{T}.$$

Let $h_i(t) = e_{\ominus\kappa i}(t, 0)z_i(t)$. A simple calculation shows that

$$h_i^{\Delta^{mi}}(t) = \frac{e_{\ominus\kappa}(t, 0)(G - \kappa_i I)^{mi} z_i e_G(t, 0)}{(1 + \mu\kappa_i)^{mi}} = 0.$$

The last equality follows because $z_i(t)$ belongs to $\mathcal{Z}_i$ for each $t \in \mathbf{T}$. Then h_i is a $\mathbb{C}^m$-valued polynomial having degree less than m_i. □

3. Exponential dichotomy

Let us decompose $\mathbb{C}$ into three sets:

$$E_{\mathbb{C}}(\mathbf{T}) := \left\{\kappa \in \mathbb{C} : \left(\limsup_{S\to\infty} \frac{1}{S - s_0} \int_{s_0}^{S} \lim_{u\to\mu(s)} \frac{\log|1 + u\kappa|}{u} \Delta s\right) < 0\right\},$$

$$E_{\mathbb{C}}^{+}(\mathbf{T}) := \left\{\kappa \in \mathbb{C} : \left(\limsup_{S\to\infty} \frac{1}{S - s_0} \int_{s_0}^{S} \lim_{u\to\mu(s)} \frac{\log|1 + u\kappa|}{u} \Delta s\right) > 0\right\}$$

and

$$E_{\mathbb{C}}^{0}(\mathbf{T}) := \left\{\kappa \in \mathbb{C} : \left(\limsup_{S\to\infty} \frac{1}{S - s_0} \int_{s_0}^{S} \lim_{u\to\mu(s)} \frac{\log|1 + u\kappa|}{u} \Delta s\right) = 0\right\}.$$

Clearly, $\mathbb{C} = E_{\mathbb{C}}(\mathbf{T}) \cup E_{\mathbb{C}}^{+}(\mathbf{T}) \cup E_{\mathbb{C}}^{0}(\mathbf{T})$.

Consider a linear system

$$x^{\Delta}(t) = Gx(t);\ x(t_0) = x_0,\ t,\ t_0 \in \mathbf{T},\ x_0 \in \mathbb{C}^m, \tag{G}$$

where G is a regressive matrix of order m.

Definition 3.1. *The system (G) is called*

- *Exponentially stable if all the eigenvalues of G are strongly regressive and $\mathcal{S}(G) \subset E_{\mathbb{C}}(\boldsymbol{T})$.*
- *Expansive if $\mathcal{S}(G) \subset E^+_{\mathbb{C}}(\boldsymbol{T})$.*
- *Dichotomic if $\mathcal{S}(G) \cap E^0_{\mathbb{C}}(\boldsymbol{T}) = \phi$.*

Remark 3.2. *Let us consider $\mathbb{C}^m = Y_s(G) \oplus Y_0(G) \oplus Y_u(G)$,*
where

$$Y_s(G) = \bigoplus_{i=1,\kappa_i \in E_{\mathbb{C}}(\boldsymbol{T})}^{k} \ker(G - \kappa_i I)^{m_i},$$

$$Y_0(G) = \bigoplus_{i=1,\kappa_i \in E^0_{\mathbb{C}}(\boldsymbol{T})}^{k} \ker(G - \kappa_i I)^{m_i},$$

$$Y_u(G) = \bigoplus_{i=1,\kappa_i \in E^+_{\mathbb{C}}(\boldsymbol{T})}^{k} \ker(G - \kappa_i I)^{m_i}.$$

The subspaces $Y_s(G)$ and $Y_u(G)$ are called stable and unstable subspaces of G, respectively. Now if G is a dichotomic matrix, then $Y_0(G) = \{0\}$ and so $\mathbb{C}^m = Y_s(G) \oplus Y_u(G)$.

Theorem 3.3. *The following three statements regarding system (G) are equivalent.*
*(**1**) System (G) is dichotomic.*
*(**2**) There exists a projection $\mathcal{V}$, positive constants $N_1,\ N_2$ and regressive functions (positive) $-v_1,\ v_2$ such that*
(i) $\|e_G(t,s)\mathcal{V}x\| \leq N_1 e_{-v_1}(t,s)\|\mathcal{V}x\|,\ \forall\, x \in \mathbb{C}^m,\ for\ every\ t \geq s,$ with $t,\ s \in \boldsymbol{T}$.
(ii) $||e_G(t,s)(I-\mathcal{V})x|| \leq N_2 e_{v_2}(t,s)||(I-\mathcal{V})x||,\ \forall\ x \in \mathbb{C}^m$, for every $t \leq s$ and $t,\ s \in \boldsymbol{T}$.
***3**) For each right–dense continuous and bounded function $\omega : \boldsymbol{T} \to \mathbb{C}^m$, the unique solution of the equation*

$$W^{\Delta}(t) = GW(t) + \omega(t),\ t \geq 0, \qquad (G,\omega)$$

is bounded with initial condition belonging to $Y_u(G)$.

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$ System (G) is dichotomic. By Remark 3.2, $\mathbb{C}^m = Y_s(G) \oplus Y_u(G)$ i.e. every $x \in \mathbb{C}^m$ can be written as $x = x_s + x_u$ with $x_s \in Y_s(G)$ and $x_u \in Y_u(G)$. Let $\mathcal{V} : \mathbb{C}^m \to \mathbb{C}^m$ defined by $\mathcal{V}x = x_s$. Obviously $\mathcal{V}$ is a projection and by using Theorem 2.5, we can easily verify that (i) and (ii) are satisfied for $N_1 > 0,\ N_2 > 0$ and positive regressive functions $-v_1$ and v_2.

$(\mathbf{2}) \Rightarrow (\mathbf{1})$ Suppose on contrary that (G) is not dichotomic. So there exists $l \in \{1,2,\ldots,k\}$ such that $\kappa_l \in E^0_{\mathbb{C}}(\mathbf{T})$. Let $x_0 \in \mathbb{C}^m$ such that $x_0 = 0+0+\cdots+0+x_l+0+\cdots+0$, where $x_l \neq 0$. Here two cases arises (a) $x_l \in Y_s(G)$ or (b) $x_l \in Y_u(G)$.
Case (a). If $x_l \in Y_s(G)$ then $e_G(t,s)\mathcal{V}x_0 = e_G(t,s)x_l$ and thus by Theorem 2.5, $e_G(t,s)\mathcal{V}x_0 = e_{\kappa_l}(t,s)p_l(t), \forall\ t \in \mathbf{T}$, where $p_l(t)$ is a finite degree polynomial with $\deg(p_l) \leq m_l - 1$. Hence,

$$||e_G(t,s)\mathcal{V}x_0|| = ||e_{\kappa_l}(t,s)p_l(t)|| = ||p_l(t)||,$$

i.e. we can not find constants N_1, N_2 and positive regressive functions $-v_1,\ v_2$ which satisfies (i), thus we arrived at a contradiction.
Case (b). If $x_l \in Y_u(G)$ then $e_G(t,s)(I-\mathcal{V})x_0 = e_G(t,s)x_l$ and thus by Theorem 2.5,

$e_G(t,s)(I-\mathcal{V})x_0 = e_{\kappa_l}(t,s)q_l(t), \forall\, t \in \mathbf{T}$, where $q_l(t)$ is a finite degree polynomial with degree less than or equal to $m_l - 1$. Thus we have

$$||e_G(t,s)(I-\mathcal{V})x_0|| = ||e_{\kappa_l}(t,s)q_l(t)|| = ||q_l(t)||,$$

i.e. in this case again we can not find constants N_1, N_2 and positive regressive functions $-v_1,\ v_2$ which satisfies (ii).

Thus in both cases we arrived at contradiction so we accept that (G) is dichotomic.

$(\mathbf{1}) \Rightarrow (\mathbf{3})$ Since system (G) is dichotomic, thus the map

$$t \mapsto W(t) := \int_0^t e_G(t,\sigma(s))\mathcal{V}\omega(s)\Delta s - \int_t^\infty e_G(t,\sigma(s))(I-\mathcal{V})\omega(s)\Delta s,$$

is a solution of (G,ω)(see [6]). Consider the second integral, from (ii), we have

$$\begin{aligned}
\int_t^\infty ||e_G(t,\sigma(s))(I-\mathcal{V})\omega(s)||\Delta s &\leq \int_t^\infty N_2 e_{v_2}(t,\sigma(s))||I-\mathcal{V}||||\omega||_\infty \Delta s \\
&= \frac{N_2}{v_2}||I-\mathcal{V}||||\omega||_\infty \int_t^\infty v_2 e_{v_2}(t,\sigma(s))\Delta s \\
&= \frac{N_2}{v_2}||I-\mathcal{V}||||\omega||_\infty (e_{v_2}(t,t) - \lim_{T\to\infty} e_{v_2}(t,T)) \\
&= \frac{N_2}{v_2}||I-\mathcal{V}||||\omega||_\infty (1-0) \\
&= \frac{N_2}{v_2}||I-\mathcal{V}||||\omega||_\infty .
\end{aligned}$$

Also

$$\begin{aligned}
\int_0^t ||e_G(t,\sigma(s))\mathcal{V}\omega(s)||\Delta s &\leq \int_0^t N_1 e_{-v_1}(t,\sigma(s))||\mathcal{V}||||\omega||_\infty \Delta s \\
&= \frac{N_1}{-v_1}||\mathcal{V}||||\omega||_\infty \int_0^t -v_1 e_{-v_1}(t,\sigma(s))\Delta s \\
&= \frac{N_1}{-v_1}||\mathcal{V}||||\omega||_\infty (e_{-v_1}(t,0) - e_{-v_1}(t,t)) \\
&= \frac{N_1}{-v_1}||\mathcal{V}||||\omega||_\infty (0-1) \\
&= \frac{N_1}{v_1}||\mathcal{V}||||\omega||_\infty .
\end{aligned}$$

So,

$$\sup_{t\geq 0}|W(t)| \leq \left(\frac{N_1}{v_1}||\mathcal{V}|| + \frac{N_2}{v_2}||I-\mathcal{V}||\right)\sup_{t\geq 0}|\omega(t)|.$$

Hence, the equation (G,ω) has a bounded solution. Also,

$$\begin{aligned}
W(0) &= -\int_0^\infty e_G(0,\sigma(s))(I-\mathcal{V})\omega(s)\Delta s \\
&= -\int_0^\infty e_{\ominus G}(\sigma(s),0)(I-\mathcal{V})\omega(s)\Delta s,
\end{aligned}$$

and thus $W(0) \in Y_u$ because Y_u is a closed subspace.

Now we need to prove uniqueness. Suppose $W_1(.)$ and $W_2(.)$ be the solutions of (G,ω) on $\mathbf{T}$. Then

$$W_1(t) = e_G(t,0)z_1 + \int_0^t e_G(t,\sigma(s))\omega(s)\Delta s,\ t \geq 0,$$

and

$$W_2(t) = e_G(t,0)z_2 + \int_0^t e_G(t,\sigma(s))\omega(s)\Delta s,\ t \geq 0,$$

with $z_1,\ z_2 \in Y_u$. Since $W_1(t) - W_2(t) = e_G(t,0)(z_1 - z_2),\ W_1(.) - W_2(.)$ is bounded on $\mathbf{T}$ and since (G) is dichotomic, so $z_1 - z_2 \in Y_s$. On the other hand, by the assumption, we have $z_1, z_2 \in Y_u$. This yields $z_1 - z_2 \in Y_u$ But $Y_u \cap Y_s = \{0\}$ and therefore $z_1 = z_2$.
$(\mathbf{3}) \Rightarrow (\mathbf{1})$ Suppose on contrary that the system (G) is not dichotomic. Then there exists $l \in \{1, 2, \ldots, k\}$ such that $\kappa_l \in E^0_{\mathbb{C}}(\mathbf{T})$. Let $x_0 \in \mathbb{C}^m$ such that $x_0 = 0 + 0 + \cdots + 0 + x_l + 0 + \cdots + 0$, where $x_l \neq 0$, then by using Theorem 2.5, we have $e_G(t,0)x_0 = e_{\kappa_l}(t,0)x_l, \forall\, t \in \mathbf{T}$. Let $\omega(t) := (1+\mu(t)\kappa_l)e_{\kappa_l}(t,0)x_l$ for $t \geq 0$, $t \in \mathbf{T}$ and take $z_0 \in Y_u$ such that the map

$$t \mapsto e_G(t,0)z_0 + \int_0^t e_G(t,\sigma(s))\omega(s)\Delta s,$$

is bounded on $\mathbf{T}$. But for $\omega(t) := (1+\mu(t)\kappa_l)e_{\kappa_l}(t,0)x_l$, we have

$$\begin{aligned}
e_G(t,0)z_0 + \int_0^t e_G(s,\sigma(s))\omega(s)\Delta s &= e_G(t,0)z_0 + \\
&\quad \int_0^t e_G(t,\sigma(s))(1+\mu(s)\kappa_l)e_{\kappa_l}(s,0)x_l\Delta s \\
&= e_G(t,0)z_0 + \\
&\quad \int_0^t e_G(t,\sigma(s))x_l(1+\mu(s)\kappa_l)e_{\kappa_l}(s,0)x_l\Delta s \\
&= e_G(t,0)z_0 + \\
&\quad \int_0^t e_{\kappa_l}(t,\sigma(s))(1+\mu(s)\kappa_l)e_{\kappa_l}(s,0)x_l\Delta s \\
&= e_G(t,0)z_0 + \\
&\quad \int_0^t \frac{e_{\kappa_l}(t,s)}{1+\mu(s)\kappa_l}(1+\mu(s)\kappa_l)e_{\kappa_l}(s,0)x_l\Delta s \\
&= e_G(s,0)z_0 + \int_0^t e_{\kappa_l}(t,s)e_{\kappa_l}(s,0)x_l\Delta s \\
&= e_G(t,0)z_0 + e_{\kappa_l}(t,0)tx_l.
\end{aligned}$$

If $z_0 = 0$, then we have a contradiction because the map

$$t \mapsto e_{\kappa_l}(t,0)tx_l$$

is unbounded. If $z_0 \neq 0$, we know that $z_0 \in Y_u$ and using the definition of Y_u there exist $N > 0$ and regressive function v such that

$$\|e_G(t,0)z_0\| \geq Ne_v(t,0), \forall\, t \geq 0,$$

i.e. in this case again the solution will be unbounded and thus we arrived at a contradiction.
□

4. Exponential dichotomy and Hyers–Ulam stability

We can see a δ–approximate solution of $x^{\Delta}(t) = Gx(t)$ as an exact solution of (G,ω) corresponding to $\omega(\cdot)$ bounded by δ. Thus with the help of Theorem 3.3, we give the definition of Hyers–Ulam stability as:

Definition 4.1. *Let δ be any positive real number. The system (G) is Hyers–Ulam stable if and only if there exists a non–negative constant K such that for every $\mathbb{C}^m$–valued right–dense continuous map $\omega = \omega(t)$ bounded by δ on **T**, and every $x \in \mathbb{C}^m$ there exists $x_0 \in \mathbb{C}^m$ such that*

$$\sup_{t\geq 0} ||e_G(t,0)(x-x_0) + \int_0^t e_G(t,\sigma(s))\omega(s)\Delta s|| \leq K\delta.$$

Theorem 4.2. *The system (G) is Hyers–Ulam stable if and only if it is exponentially dichotomic.*

Proof. **Necessity:** Suppose that the system (G) is not dichotomic i.e. $Y_0(G) \neq \{0\}$. Then, there exists κ_i in $\mathcal{S}(G)$, with $\kappa_i \in E^0_{\mathbb{C}}(\mathbf{T})$. Let $\delta > 0$ be fixed and set $\omega(t) = (1 + \mu(s)\kappa_i)e_{\kappa_i}(t,0)u_0$, with $||u_0|| \leq \delta$. Obviously, the function ω is right–dense continuous and bounded by δ. By assumption, the regressive matrix G or the system (G) is Hyers–Ulam stable. Hence, the solution

$$W(t) = e_G(t,0)(x-x_0) + \int_0^t e_G(t,\sigma(s))\omega(s)\Delta s,\ x,\ x_0 \in \mathbb{C}^m,$$

of the Cauchy problem

$$\begin{cases} W^{\Delta}(t) = GW(t) + \omega(t), t \geq 0 \\ \ W(0) = x - x_0, \end{cases} \qquad (G,\omega,x_0)$$

is bounded by $K\delta$.

By using the spectral decomposition theorem, there exists an $m \times m$ matrix–valued polynomial $P_i(t)$ having the degree at most $m_i - 1$, such that

$$\mathcal{V}e_G(t,0) = e_{\kappa_i}(t,0)P_i(t),\ \forall\ t \geq 0. \qquad (4.\ 2)$$

Then the map

$$t \mapsto \mathcal{V}\left[e_G(t,0)(x-x_0) + \int_0^t e_G(t,\sigma(s))\omega(s)\Delta s\right], \quad x,\ x_0 \in \mathbb{C}^m,$$

should also be bounded by $K\delta$.

On the other hand,

$$\mathcal{V}\left[e_G(t,0)(x-x_0) + \int_0^t e_G(t,\sigma(s))\omega(s)\Delta s\right]$$

$$= e_{\kappa_i}(t,0)P_i(t)(x-x_0) + \int_0^t \mathcal{V}e_G(t,\sigma(s))\omega(s)\Delta s,$$

and

$$\begin{aligned}\int_0^t \mathcal{V} e_G(t,\sigma(s))\omega(s)\Delta s &= \int_0^t \mathcal{V} e_G(t,\sigma(s))(1+\mu(s)\kappa_i)e_{\kappa_i}(s,0)u_0\Delta s \\ &= \int_0^t e_{\kappa_i}(s,0)e_{\kappa_i}(t,\sigma(s))(1+\mu(s)\kappa_i)P_i(t-\sigma(s))u_0\Delta s \\ &= \int_0^t \frac{e_{\kappa_i}(t,s)e_{\kappa_i}(s,0)(1+\mu(s)\kappa_i)}{(1+\mu(s)\kappa_i)}P_i(t-\sigma(s))u_0\Delta s \\ &= e_{\kappa_i}(t,0)\int_0^t P_i(t-\sigma(s))u_0\Delta s \\ &= e_{\kappa_i}(t,0)q_i(t),\end{aligned}$$

where $q_i(t)=\int_0^t P_i(t-\sigma(s))u_0\Delta s$ is a polynomial as well. Now choosing an appropriate vector $u_0\neq 0$,

$$\deg[P_i(t)(x-x_0)] \leq \deg[P_i(t)] = \deg[P_i(t)u_0] < 1+\deg[P_i(t)] = \deg[q_i(t)].$$

Therefore the solution $W(t)=e_{\kappa_i}(t,0)P_i(t)(x-x_0)+e_{\kappa_i}(t,0)q_i(t)$ is unbounded and we have a contradiction.

Sufficiency: Let $\omega:\mathbf{T}\to\mathbb{C}^m$ be a right–dense continuous function, with $\|\omega\|_\infty\leq\delta$. By Theorem 3.3, the solution $W(\cdot)$ starting from the subspace $Y_u(G)$ of (G,ω,x_0) is unique and bounded. Let $u_0=W(0)\in Y_u(G)$ and since (G) is dichotomic, the map

$$t\mapsto \int_0^t e_G(t,\sigma(s))\mathcal{V}\omega(s)\Delta s - \int_t^\infty e_G(t,\sigma(s))(I-\mathcal{V})\omega(s)\Delta s,$$

is a bounded solution on **T** of (G,ω,x_0). Then,

$$\begin{aligned}\|W(t)\| &= \|e_G(t,0)u_0+\int_0^t e_G(t,\sigma(s))\omega(s)\Delta s\| \\ &= \|\int_0^t e_G(t,\sigma(s))\mathcal{V}\omega(s)\Delta s - \int_t^\infty e_G(t,\sigma(s))(I-\mathcal{V})\omega(s)\Delta s\| \\ &\leq \left(\frac{N_1}{v_1}\|\mathcal{V}\|+\frac{N_2}{v_2}\|I-\mathcal{V}\|\right)\delta.\end{aligned}$$

The desired assertion follows by choosing $K=\left(\frac{N_1}{v_1}\|\mathcal{V}\|+\frac{N_2}{v_2}\|I-\mathcal{V}\|\right)$ and $x_0=x-u_0$. □

Example 4.3. *Show that the system $x^\Delta(t)=Gx(t),\ t\in$ **T** has the Hyer–Ulam stability on time scale **T**, where G is the 2×2 matrix defined by:*

$$G=\begin{pmatrix} -3 & 0 \\ 0 & 2\end{pmatrix}.$$

Solution: Since the eigenvalues of the coefficient matrix are $\kappa_1=2$ and $\kappa_2=-3$, we can see that the matrix G is regressive when $\mu(t)\neq -1/2, 1/3$. The regressive matrix G is dichotomic due to $\mathcal{S}(G)\cap E^0_{\mathbb{C}}(\mathbf{T})=\phi$. In this case, the matrix exponential function $e_G(t,\sigma(s))$ is given as:

$$e_G(t,\sigma(s))=\begin{pmatrix} e_{-3}(t,\sigma(s)) & 0 \\ 0 & e_2(t,\sigma(s))\end{pmatrix}.$$

So by using Theorems 2.5 and 3.3, it can be easily shown that

$$\sup_{t\geq 0} ||e_G(t,0)x_0 + \int_0^t e_G(t,\sigma(s))\omega(s)\Delta s|| \leq K\delta.$$

So the regressive matrix G is Hyers–Ulam stable.

5. Conclusion

In this paper, we unified the results of Hyers–Ulam stability and exponential dichotomy of first order linear differential and difference equations by using time scale i.e. we show that the first order linear dynamic system (G) is Hyers–Ulam stable if and only if it is dichotomic. This relationship is proved in terms of boundedness of solution of the Cauchy problem (G, ω, x_0).

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