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Application of the Srivastav-Owa Fractional Calculus Operator to Janowski Spiral-like Functions of Complex Order

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Abstract. We aim to introduce a new subfamily of Janowski spiral-like functions of complex order, based on Srivastava-Owa fractional calculus operator. For functions in this new subfamily, we establish a necessary and sufficient condition, Marx-Strohhäcker type inequalities as well as distortion and radius inequalities. A Fekete-Szegö problem for this new subfamily is also investigated. The results presented here, would extend, unify and improve some recent results in literature.

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1. Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ where $\mathbb{C}$ is set of all complex numbers. We let Ψ the family of all functions $f(z)$ analytic in Δ and normalized by $f(0) = f'(0) - 1 = 0$. By Π, we mean the family of all functions $p(z)$ analytic in the Δ with $p(0) = 1$. For fixed numbers Λ, Υ with $-1 \leq \Upsilon < \Lambda \leq 1$, we denote by $\Pi[\Lambda, \Upsilon]$ (see [15]) the class of all Janowski functions $p(z)$ analytic in Δ such that

$$p(z) = \frac{1 + \Lambda s(z)}{1 + \Upsilon s(z)}, \quad (z \in \Delta)$$

where $s(z)$ is the familiar Schwarz function satisfying $s(0) = 0$ and $|s(z)| < 1, \forall z \in \Delta$. Note that, $\Pi[1, -1] = \Pi$: the familiar class of Caratheodary functions with positive real part and $\Pi[1 - 2\delta, -1] = \Pi(\delta)$: the class of Caratheodary functions with $\Re\{p(z)\} > \delta, \ (0 \leq \delta < 1)$. Janowski [15] also defined the classes $\mathcal{C}[\Lambda, \Upsilon]$ and $\mathcal{S}^*[\Lambda, \Upsilon]$ of convex and starlike functions respectively. Also, $\mathcal{C}[1, -1] = \mathcal{C}$ and $\mathcal{S}^*[1, -1] = \mathcal{S}^*$, which are the familiar classes of convex and starlike functions respectively. We owed the following concepts of fractional calculus to Srivastava and Owa [23] (see also [1, 4, 11, 12, 13, 20] for applications).

Definition 1.1. *For $f(z) \in \Psi$, we define the fractional integral $\Omega_z^{-\alpha}$ of order α $(\alpha > 0)$ as*

$$\Omega_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(\gamma)}{(z-\gamma)^{1-\alpha}} d\gamma,$$

where the multiplicity of $(z-\gamma)^{\alpha-1}$ can be removed by demanding that $\log(z-\gamma)$ is real when $(z-\gamma) > 0$.

Definition 1.2. *For $f(z) \in \Psi$, we define the fractional derivative Ω_z^{α} of order α $(0 \leq \alpha < 1)$ as*

$$\Omega_z^{\alpha} f(z) = \frac{d}{dz}\left(\Omega_z^{\alpha-1} f(z)\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\gamma)}{(z-\gamma)^{\alpha}} d\gamma,$$

where the multiplicity of $(z-\gamma)^{\alpha-1}$ can be removed by demanding that $\log(z-\gamma)$ is real when $(z-\gamma) > 0$.

Following Definition 1.2, we have

Definition 1.3. *For $f(z) \in \Psi$, we define the fractional derivative $\Omega_z^{n+\alpha}$ of order $n+\alpha$ as*

$$\begin{aligned} \Omega_z^{n+\alpha} f(z) &= \frac{d^n}{dz^n}\left(\Omega_z^{\alpha} f(z)\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d^{n+1}}{dz^{n+1}} \int_0^z \frac{f(\gamma)}{(z-\gamma)^{\alpha}} d\gamma, \\ & \left(0 \leq \alpha < 1, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\right), \end{aligned}$$

where the multiplicity of $(z-\gamma)^{-\alpha}$ can be removed by demanding that $\log(z-\gamma)$ is real when $(z-\gamma) > 0$.

Now in view of the above definitions, we note that

$$\begin{aligned} \Omega_z^{-\alpha} z^m &= \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)} z^{m+\alpha}, \ (\alpha > 0, m > 0), \\ \Omega_z^{\alpha} z^m &= \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} z^{m-\alpha}, \ \ (0 \leq \alpha < 1, m > 0), \end{aligned}$$

and

$$\Omega_z^{n+\alpha} z^m = \frac{\Gamma(m+1)}{\Gamma(m+1-n-\alpha)} z^{m-n-\alpha}, \ (0 \leq \alpha < 1, m > 0, n \in \mathbb{N}_0, m-n \neq -1, -2, -3, ...).$$

Thus for any real α, we have

$$\Omega_z^{\alpha} z^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} z^{m-\alpha}, \ (\alpha > 1, m-\alpha \neq -1. -2, ...).$$

With the aid of above definitions, Owa and Srivastava [19, 23] (see also [8]) introduced the fractional calculus operator (called as Srivastava-Owa fractional calculus operator) Ω^{α} for $f(z) \in \Psi$, as follow

$$\Omega^{\alpha} f(z) = \Gamma(2-\alpha) z^{\alpha} \Omega_z^{\alpha} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\alpha)\Gamma(n+1)}{\Gamma(n+1-\alpha)} a_n z^n.$$

Note that

$$\Omega^0 f(z) = f(z), \quad \text{and} \quad \Omega^1 f(z) = z f'(z).$$

Moreover, for $\alpha \neq 2, 3, 4, ...$ and $\beta \neq 2, 3, 4, ...$ we note that

$$\Omega^{\beta}\left(\Omega^{\alpha} f\left(z\right)\right) = \Omega^{\alpha}\left(\Omega^{\beta} f\left(z\right)\right) = z + \sum_{n=2}^{\infty} \frac{\Gamma\left(2-\alpha\right)\Gamma\left(2-\beta\right)\left(\Gamma\left(n+1\right)\right)^{2}}{\Gamma\left(n+1-\alpha\right)\Gamma\left(n+1-\beta\right)} a_n z^n,$$

and

$$\Omega\left(\Omega^{\alpha} f\left(z\right)\right) = z\left(\Omega^{\alpha} f\left(z\right)\right)' = \Gamma\left(2-\alpha\right) z^{\alpha}\left[\alpha \Omega_z^{\alpha} f\left(z\right) + z\Omega_z^{\alpha+1} f\left(z\right)\right]. \qquad (1.\ 1)$$

We also recall the following concepts, which we shall require in our later investigation. Let $f, g \in \Psi$, then we say that f is subordinated to g, notationally $f \prec g$, if for some analytic function $s\left(z\right)$ there holds

$$f\left(z\right) = g\left(s\left(z\right)\right),\ \left(z \in \Delta\right),$$

where $s\left(z\right)$ is the Schwarz function with $w\left(0\right) = 0$ and $\left|s\left(z\right)\right| < 1, \forall z \in \Delta$. Note that, if $g(z)$ is univalent in Δ, then $f \prec g$ can be put equivalently in the form

$$f\left(0\right) = g\left(0\right) \text{ and } f\left(\Delta\right) \subset g\left(\Delta\right),\ z \in \Delta.$$

Using the Srivastava-Owa fractional calculus operator, Çağlar et al. [8] introduced the class $\mathcal{S}_{\alpha}^{*}\left[\Lambda, \Upsilon\right]\ (-1 \leq \Upsilon < \Lambda \leq 1)$ of Janowski starlike functions as follow

$$\mathcal{S}_{\alpha}^{*}\left[\Lambda, \Upsilon\right] = \left\{ f \in \Psi : \frac{z\left(\Omega^{\alpha} f\left(z\right)\right)'}{\Omega^{\alpha} f\left(z\right)} = p\left(z\right) \in \Pi\left[\Lambda, \Upsilon\right];\ \text{for } \alpha \neq 2, 3, 4, ... \right\}.$$

For functions in this class, they [8] obtained coefficient bounds, distortion inequalities and some other interesting inequalities, for details we refer to their cited paper.
On inspiring from the work of Çağlar et al. [8], we introduce the class $\mathcal{S}_{\alpha}^{*}\left(\tau, b, \Lambda, \Upsilon\right)$ of Janowski spiral-like functions of complex order $(b \neq 0)$ by means of Srivastava-Owa fractional calculus operator as follow.
Let $f \in \Psi$, then by $\mathcal{S}_{\alpha}^{*}\left(\tau, b, \Lambda, \Upsilon\right)$ we denote the family of all functions given as

$$\mathcal{S}_{\alpha}^{*}\left(\tau, b, \Lambda, \Upsilon\right) = \left\{ f \in \Psi : 1 + \frac{e^{i\tau}}{b\cos\tau}\left(\frac{z\left(\Omega^{\alpha} f\left(z\right)\right)'}{\Omega^{\alpha} f\left(z\right)} - 1\right) = p\left(z\right) \in \Pi\left[\Lambda, \Upsilon\right] \right\}$$

where τ is real with $|\tau| < \frac{\pi}{2}$, $b \in \mathbb{C}^{*} = \mathbb{C} - \{0\}$, $z \in \Delta$ and $\alpha \neq 2, 3, 4, \cdots$. Observe that one can also define the class $\mathcal{S}_{\alpha}^{*}\left(\tau, b, \Lambda, \Upsilon\right)$ with the help of identity (1.1) as follow

$$\mathcal{S}_{\alpha}^{*}\left(\tau, b, \Lambda, \Upsilon\right) = \left\{ f \in \Psi : 1 + \frac{e^{i\tau}}{b\cos\tau}\left(\alpha + \frac{z\Omega_z^{\alpha+1} f\left(z\right)}{\Omega^{\alpha} f\left(z\right)} - 1\right) = p\left(z\right) \in \Pi\left[\Lambda, \Upsilon\right] \right\}.$$

Various well-known classes appear as a special case of this new class. Indeed, for $\tau = 0$, $b = 1$, we retrieve the class $\mathcal{S}_{\alpha}^{*}\left[\Lambda, \Upsilon\right]$ of Çağlar et al. [8]. Also, $\mathcal{S}_{1}^{*}\left(0, b, \Lambda, \Upsilon\right) = \mathcal{C}_b\left[\Lambda, \Upsilon\right]$, $\mathcal{S}_{0}^{*}\left(0, b, \Lambda, \Upsilon\right) = \mathcal{S}_{b}^{*}\left[\Lambda, \Upsilon\right]$ which are special classes (with $\lambda = 0$) of a class considered in [22]. Moreover, $\mathcal{S}_{0}^{*}\left(\tau, b, 1, -1\right) = \mathcal{S}^{\tau}\left(b\right)$ ([5]), $\mathcal{S}_{1}^{*}\left(\tau, b, 1, -1\right) = \mathcal{C}^{\tau}\left(b\right)$ ([5, 6]), $\mathcal{S}_{0}^{*}\left(0, b, 1, -1\right) = \mathcal{S}^{*}\left(b\right)$ ([17]) and $\mathcal{S}_{1}^{*}\left(0, b, 1, -1\right) = \mathcal{C}\left(b\right)$ ([25]): which are, respectively, the familiar subclasses of spiral-like, Robertson, starlike and convex functions of complex order $b \neq 0$. Furthermore, $\mathcal{S}_{0}^{*}\left(0, 1, 1-2\gamma, -1\right) = \mathcal{S}^{*}\left(\gamma\right)$ and $\mathcal{S}_{1}^{*}\left(0, 1, 1-2\gamma, -1\right) = \mathcal{C}\left(\gamma\right)$ are the classes of starlike and convex functions of order $\gamma\ (0 \leq \gamma < 1)$ respectively [2, 14] (see also [10]).

From now, we assume that $\alpha \neq 2, 3, 4, ..., n \in \mathbb{N}_2 = \{2, 3, 4, ..\}$, $b \in \mathbb{C}^{*} = \mathbb{C} - \{0\}$, $-1 \leq \Upsilon < \Lambda \leq 1$, and τ is real with $|\tau| < \frac{\pi}{2}$, unless otherwise stated.

2. Necessary and sufficient condition for $\mathcal{S}_{\alpha}^{*}(\tau, b, \Lambda, \Upsilon)$ and its consequence

Here, we prove a necessary and sufficient condition for the class $\mathcal{S}_{\alpha}^{*}(\tau, b, \Lambda, \Upsilon)$. We also obtain some Marx-Strohhäcker type inequalities as an interesting consequence of this condition. We begin with the following result.

Theorem 2.1. *Let $f \in \Psi$, then $f \in \mathcal{S}_{\alpha}^{*}(\tau, b, \Lambda, \Upsilon)$ if and only if*

$$\left(\frac{z\left(\Omega^{\alpha} f(z)\right)'}{\Omega^{\alpha} f(z)} - 1\right) \prec \begin{cases} \frac{be^{-i\tau}\cos\tau(\Lambda-\Upsilon)z}{1+\Upsilon z}, & \Upsilon \neq 0, \\ be^{-i\tau}\cos\tau\Lambda z, & \Upsilon = 0. \end{cases} \tag{2. 1}$$

Proof. First, we obtain the necessary condition. Let (2.1) holds, then the subordination principle yields

$$\left(\frac{z\left(\Omega^{\alpha} f(z)\right)'}{\Omega^{\alpha} f(z)} - 1\right) = \begin{cases} \frac{be^{-i\tau}\cos\tau(\Lambda-\Upsilon)s(z)}{1+\Upsilon w(z)}, & \Upsilon \neq 0, \\ be^{-i\tau}\cos\tau\Lambda s(z), & \Upsilon = 0. \end{cases}$$

where $s(z)$ is analytic in Δ. Consequently

$$1 + \frac{1}{be^{-i\tau}\cos\tau}\left(\frac{z\left(\Omega^{\alpha} f(z)\right)'}{\Omega^{\alpha} f(z)} - 1\right) = \begin{cases} \frac{1+\Lambda s(z)}{1+\Upsilon s(z)}, & \Upsilon \neq 0, \\ 1+\Lambda s(z) & \Upsilon = 0. \end{cases}$$

Hence $f \in \mathcal{S}_{\alpha}^{*}(\tau, b, \Lambda, \Upsilon)$. Conversely, assume $f \in \mathcal{S}_{\alpha}^{*}(\tau, b, \Lambda, \Upsilon)$. Then for $\Upsilon \neq 0$

$$1 + \frac{1}{be^{-i\tau}\cos\tau}\left(\frac{z\left(\Omega^{\alpha} f(z)\right)'}{\Omega^{\alpha} f(z)} - 1\right) = p(z) \in \Pi[\Lambda, \Upsilon].$$

Now in view of above equality, the boundary function $p_1(z) \in \Pi[\Lambda, \Upsilon]$ can be written as

$$p_1(z) = \frac{1+\Lambda s(z)}{1+\Upsilon s(z)}.$$

Thus we get

$$1 + \frac{1}{be^{-i\tau}\cos\tau}\left(\frac{z\left(\Omega^{\alpha} f(z)\right)'}{\Omega^{\alpha} f(z)} - 1\right) = \frac{1+\Lambda s(z)}{1+\Upsilon s(z)},$$

or

$$\left(\frac{z\left(\Omega^{\alpha} f(z)\right)'}{\Omega^{\alpha} f(z)} - 1\right) = \frac{be^{-i\tau}\cos\tau(\Lambda-\Upsilon)s(z)}{1+\Upsilon s(z)}.$$

Hence by virtue of subordination, we find that

$$\left(\frac{z\left(\Omega^{\alpha} f(z)\right)'}{\Omega^{\alpha} f(z)} - 1\right) \prec \frac{be^{-i\tau}\cos\tau(\Lambda-\Upsilon)z}{1+\Upsilon z}.$$

For $\Upsilon = 0$, there comes

$$\left(\frac{z\left(\Omega^{\alpha} f(z)\right)'}{\Omega^{\alpha} f(z)} - 1\right) \prec be^{-i\tau}\cos\tau\Lambda z.$$

This completes the proof. □

Next, we prove the Marx-Strohhäcker type inequalities for $\mathcal{S}_{\alpha}^{*}(\tau, b, \Lambda, \Upsilon)$. The following lemma we owe to Jack [14], which we need to establish the result.

Lemma 2.2. [14, Jack's Lemma] *Let $s(z)$ is a non-constant analytic function in Δ with $s(0) = 0$. If $|s(z)|$ attains its maximum on the circle $|z| = r$ at z_1, then*

$$z_1 s'(z_1) = ks(z_1), \text{ for } k \geq 1 \ (k \in \mathbb{R}).$$

Theorem 2.3. *Let $f \in \mathcal{S}^*_\alpha(\tau, b, \Lambda, \Upsilon)$, then*

$$\left|\left(\Gamma(2-\alpha) z^{\alpha-1} \Omega^\alpha_z f(z)\right)^{\frac{\Upsilon}{be^{-i\tau}\cos\tau(\Lambda-\Upsilon)}} - 1\right| < 1, \ \Upsilon \neq 0,$$

$$\left|\log\left(\Gamma(2-\alpha) z^{\alpha-1} \Omega^\alpha_z f(z)\right)^{\frac{1}{be^{-i\tau}\cos\tau\Lambda}}\right| < 1, \ \Upsilon = 0.$$

Proof. Let $s(z)$ is define by

$$\frac{\Omega^\alpha f(z)}{z} = \begin{cases} (1 + \Upsilon s(z))^{\frac{be^{-i\tau}\cos\tau(\Lambda-\Upsilon)}{\Upsilon}}, & \Upsilon \neq 0, \\ e^{be^{-i\tau}\cos\tau\Lambda s(z)}, & \Upsilon = 0, \end{cases}$$

where $(1 + \Upsilon s(z))^{\frac{be^{-i\tau}\cos\tau(\Lambda-\Upsilon)}{\Upsilon}}$ and $e^{be^{-i\tau}\cos\tau\Lambda s(z)}$ has value 1 at $z = 0$. Thus $s(z)$ is analytic in Δ with $s(0) = 0$ and consequently

$$\left(\frac{z(\Omega^\alpha f(z))'}{\Omega^\alpha f(z)} - 1\right) = \begin{cases} \frac{be^{-i\tau}\cos\tau(\Lambda-\Upsilon) zs'(z)}{1+\Upsilon s(z)}, & \Upsilon \neq 0, \\ be^{-i\tau}\cos\tau\Lambda zs'(z), & \Upsilon = 0. \end{cases}$$

Now by virtue of subordination, we find that $|s(z)| < 1, \forall z \in \Delta$. In particular, assume on contrary to this, and let $z_1 \in \Delta$ such that $|s(z_1)| = 1$. Then from Jack's lemma, we easily conclude that $z_1 s'(z_1) = ks(z_1) \quad (z_1 \in \Delta)$, for some real $k \geq 1$. Thus

$$\left(\frac{z(\Omega^\alpha f(z_1))'}{\Omega^\alpha f(z_1)} - 1\right) = \begin{cases} \frac{be^{-i\tau}\cos\tau(\Lambda-\Upsilon) ks(z_1)}{1+\Upsilon s(z_1)} = \mathcal{F}(s(z_1)) \notin \mathcal{F}(\Delta), & \Upsilon \neq 0, \\ be^{-i\tau}\cos\tau\Lambda ks(z_1) = \mathcal{G}(s(z_1)) \notin \mathcal{G}(\Delta), & \Upsilon = 0. \end{cases}$$

But, this contradicts assertion (2) and hence $|s(z)| < 1, \forall z \in \Delta$. Now it follows that

$$\left|\left(\frac{\Omega^\alpha f(z)}{z}\right)^{\frac{\Upsilon}{be^{-i\tau}\cos\tau(\Lambda-\Upsilon)}} - 1\right| = |\Upsilon s(z)| = |\Upsilon| < 1, \ \Upsilon \neq 0,$$

$$\left|\log\left(\frac{\Omega^\alpha f(z)}{z}\right)^{\frac{1}{be^{-i\tau}\cos\tau\Lambda}}\right| = |s(z)| < 1, \ \Upsilon = 0.$$

Or, equivalently

$$\left|\left(\Gamma(2-\alpha) z^{\alpha-1} \Omega^\alpha_z f(z)\right)^{\frac{\Upsilon}{be^{-i\tau}\cos\tau(\Lambda-\Upsilon)}} - 1\right| < 1, \ \Upsilon \neq 0,$$

$$\left|\log\left(\Gamma(2-\alpha) z^{\alpha-1} \Omega^\alpha_z f(z)\right)^{\frac{1}{be^{-i\tau}\cos\tau\Lambda}}\right| < 1, \ \Upsilon = 0.$$

This completes the proof. □

For some recent work related to Marx-Strohhäcker type results, see [18].

3. Distortion and radius inequalities for $\mathcal{S}_{\alpha}^{*}(\tau, b, \Lambda, \Upsilon)$

In this section, we aim to obtain some distortion and radius inequalities for $\mathcal{S}_{\alpha}^{*}(\tau, b, \Lambda, \Upsilon)$.

Theorem 3.1. *Let $f \in \mathcal{S}_{\alpha}^{*}(\tau, b, \Lambda, \Upsilon)$, then for $|z| = r\ (0 < r < 1)$, $\Upsilon \neq 0$,*

$$M(r, b, \tau, \alpha, \Lambda, \Upsilon) \leq |\Omega_z^{\alpha} f(z)| \leq N(r, b, \tau, \alpha, \Lambda, \Upsilon),$$

where

$$M(r, b, \tau, \alpha, \Lambda, \Upsilon) = \frac{r^{1-\alpha}}{\Gamma(2-\alpha)} \frac{(1-\Upsilon r)^{[|b|+\Re(b)\cos\tau]\left(\frac{\Lambda-\Upsilon}{2\Upsilon}\right)\cos\tau}}{(1+\Upsilon r)^{[|b|-\Re(b)\cos\tau]\left(\frac{\Lambda-\Upsilon}{2\Upsilon}\right)\cos\tau}},$$

$$N(r, b, \tau, \alpha, \Lambda, \Upsilon) = \frac{r^{1-\alpha}}{\Gamma(2-\alpha)} \frac{(1+\Upsilon r)^{[|b|+\Re(b)\cos\tau]\left(\frac{\Lambda-\Upsilon}{2\Upsilon}\right)\cos\tau}}{(1-\Upsilon r)^{[|b|-\Re(b)\cos\tau]\left(\frac{\Lambda-\Upsilon}{2\Upsilon}\right)\cos\tau}},$$

and for $\Upsilon = 0$,

$$\frac{r^{1-\alpha}}{\Gamma(2-\alpha)} e^{-r|b|\Lambda\cos\tau} \leq |\Omega_z^{\alpha} f(z)| \leq \frac{r^{1-\alpha}}{\Gamma(2-\alpha)} e^{r|b|\Lambda\cos\tau}.$$

This result is sharp.

Proof. For $p(z) \in \Pi[\Lambda, \Upsilon]$, Janowski [15] proved that

$$\begin{aligned} \left| p(z) - \frac{1-\Lambda\Upsilon r^2}{1-\Upsilon^2 r^2} \right| &\leq \frac{(\Lambda-\Upsilon) r}{1-\Upsilon^2 r^2}, \ \Upsilon \neq 0, \\ |p(z) - 1| &\leq \Lambda r, \qquad \Upsilon = 0. \end{aligned}$$

Thus by definition of $\mathcal{S}_{\alpha}^{*}(\tau, b, \Lambda, \Upsilon)$, for $\Upsilon \neq 0$ there comes

$$\left| 1 + \frac{1}{be^{-i\tau}\cos\tau} \left(\frac{z(\Omega^{\alpha} f(z))'}{\Omega^{\alpha} f(z)} - 1 \right) - \frac{1-\Lambda\Upsilon r^2}{1-\Upsilon^2 r^2} \right| \leq \frac{(\Lambda-\Upsilon) r}{1-\Upsilon^2 r^2},$$

which implies

$$\left| \frac{z(\Omega^{\alpha} f(z))'}{\Omega^{\alpha} f(z)} - \frac{1-\Upsilon\left[\Upsilon + be^{-i\tau}\cos\tau(\Lambda-\Upsilon)\right] r^2}{1-\Upsilon^2 r^2} \right| \leq \frac{|b|\cos\tau(\Lambda-\Upsilon) r}{1-\Upsilon^2 r^2}.$$

Now upon simple manipulation, the preceding inequality gives

$$m_1(r) \leq \Re\left(\frac{z(\Omega^{\alpha} f(z))'}{\Omega^{\alpha} f(z)} \right) \leq m_2(r), \tag{3. 1}$$

where

$$m_1(r) = \frac{1-\Upsilon\left[\Upsilon + \Re(b)(\Lambda-\Upsilon)\cos^2\tau\right] r^2 - |b|\cos\tau(\Lambda-\Upsilon) r}{(1+\Upsilon r)(1-\Upsilon r)},$$

and

$$m_2(r) = \frac{1-\Upsilon\left[\Upsilon + \Re(b)(\Lambda-\Upsilon)\cos^2\tau\right] r^2 + |b|\cos\tau(\Lambda-\Upsilon) r}{(1+\Upsilon r)(1-\Upsilon r)}.$$

Since

$$\Re\left(\frac{z(\Omega^{\alpha} f(z))'}{\Omega^{\alpha} f(z)} \right) = r\frac{\partial}{\partial r}\log|\Omega^{\alpha} f(z)|.$$

Thus, we get

$$\frac{m_1(r)}{r} \leq \frac{\partial}{\partial r} \log |\Omega^\alpha f(z)| \leq \frac{m_2(r)}{r}.$$

Now upon integration from 0 to r, the above inequality yields

$$\frac{r(1-\Upsilon r)^{[|b|+\Re(b)\cos\tau]\left(\frac{\Lambda-\Upsilon}{2\Upsilon}\right)\cos\tau}}{(1+\Upsilon r)^{[|b|-\Re(b)\cos\tau]\left(\frac{\Lambda-\Upsilon}{2\Upsilon}\right)\cos\tau}} \leq |\Omega^\alpha f(z)| \leq \frac{r(1+\Upsilon r)^{[|b|+\Re(b)\cos\tau]\left(\frac{\Lambda-\Upsilon}{2\Upsilon}\right)\cos\tau}}{(1-\Upsilon r)^{[|b|-\Re(b)\cos\tau]\left(\frac{\Lambda-\Upsilon}{2\Upsilon}\right)\cos\tau}},$$

or

$$M(r,b,\tau,\alpha,\Lambda,\Upsilon) \leq |\Omega_z^\alpha f(z)| \leq N(r,b,\tau,\alpha,\Lambda,\Upsilon).$$

For $\Upsilon = 0$, the result is obvious. This completes the proof. □

Remark 3.2. *For sharpness, take the extremal function given as*

$$\Omega^\alpha f(z) = \begin{cases} z(1+\Upsilon z)^{\frac{(\Lambda-\Upsilon)be^{-i\tau}\cos\tau}{\Upsilon}}; & \Upsilon \neq 0 \\ ze^{be^{-i\tau}\cos\tau \Lambda z}; & \Upsilon = 0. \end{cases}$$

$$\Omega_z^\alpha f(z) = \begin{cases} \frac{1}{\Gamma(2-\alpha)} z^{1-\alpha}(1+\Upsilon z)^{\frac{(\Lambda-\Upsilon)be^{-i\tau}\cos\tau}{\Upsilon}}; & \Upsilon \neq 0 \\ \frac{1}{\Gamma(2-\alpha)} z^{1-\alpha} e^{be^{-i\tau}\cos\tau \Lambda z}; & \Upsilon = 0. \end{cases}$$

Remark 3.3. $(\mathbf{i})$. *For $b=1$, $\tau=0$, we receive immediately the distortion inequalities of Çağlar et al.* [8].
$(\mathbf{ii})$. *On letting $b = 1-\beta$ $(0 \leq \beta < 1)$, $\tau = 0$, we obtain*

$$\begin{aligned} \frac{r^{1-\alpha}}{\Gamma(2-\alpha)}(1-\Upsilon r)^{(1-\beta)\left(\frac{\Lambda-\Upsilon}{\Upsilon}\right)} &\leq |\Omega_z^\alpha f(z)| \leq \frac{r^{1-\alpha}}{\Gamma(2-\alpha)}(1+\Upsilon r)^{(1-\beta)\left(\frac{\Lambda-\Upsilon}{\Upsilon}\right)}; \ \Upsilon \neq 0, \\ \frac{r^{1-\alpha}}{\Gamma(2-\alpha)} e^{-(1-\beta)\Lambda} &\leq |\Omega_z^\alpha f(z)| \leq \frac{r^{1-\alpha}}{\Gamma(2-\alpha)} e^{(1-\beta)\Lambda}; \quad \Upsilon = 0. \end{aligned}$$

$(\mathbf{iii})$. *Also, for $b=1$, $\tau=0$, $\Lambda=1$, $\Upsilon=-1$, we receive immediately the distortion inequalities of classes $\mathcal{S}^*$ (with $\alpha=0$) and $\mathcal{C}$ (with $\alpha=1$), see* [10].

The next theorem presents the radius of largest disk in which $f(z)$ is starlike.

Theorem 3.4. *Let $f \in \mathcal{S}_\alpha^*(\tau,b,\Lambda,\Upsilon)$, then the radius of starlikeness $r_{\mathcal{S}^*}$ for $|z| = r < r_{\mathcal{S}^*}$ $(0<r<1)$ is given by*

$$r_{\mathcal{S}^*} = \frac{2}{|b|(\Lambda-\Upsilon)\cos\tau + \sqrt{|b|^2(\Lambda-\Upsilon)^2\cos^2\tau + 4\Upsilon\left[\Upsilon + \Re(b)(\Lambda-\Upsilon)\cos^2\tau\right]}}.$$

This result is sharp.

Proof. From (3.1), we have

$$\Re\left(\frac{z(\Omega^\alpha f(z))'}{\Omega^\alpha f(z)}\right) \geq \frac{1-\Upsilon\left[\Upsilon + \Re(b)(\Lambda-\Upsilon)\cos^2\tau\right]r^2 - |b|\cos\tau(\Lambda-\Upsilon)r}{1-\Upsilon^2 r^2}.$$

Now the right hand side of the preceding inequality is positive for $r < r_{\mathcal{S}^*}$, if

$$r_{\mathcal{S}^*} = \frac{|b|(\Lambda-\Upsilon)\cos\tau - \sqrt{|b|^2(\Lambda-\Upsilon)^2\cos^2\tau + 4\Upsilon\left[\Upsilon + \Re(b)(\Lambda-\Upsilon)\cos^2\tau\right]}}{-2\Upsilon\left[\Upsilon + \Re(b)(\Lambda-\Upsilon)\cos^2\tau\right]}.$$

The desired result follows now at once. □

Remark 3.5. *The sharpness can be seen for the function given as*

$$\Omega^\alpha f(z) = z(1+\Upsilon z)^{\frac{be^{-i\tau}\cos\tau(\Lambda-\Upsilon)}{\Upsilon}}.$$

4. Fekete-Szegö type problem for the class $\mathcal{S}^*_\alpha(\tau, b, \Lambda, \Upsilon)$

Now we investigate the Fakete-Szegö type problem for the class $\mathcal{S}^*_\alpha(\tau, b, \Lambda, \Upsilon)$. First we recall the following.

Lemma 4.1. [3] *Let $s(z) = s_1z + s_2z^2 + s_3z^3 + ...\ (z \in \Delta)$ be Schwarz function, then for any real ϕ*

$$\left|s_2 - \phi s_1^2\right| \le \begin{cases} -\phi & \phi < -1 \\ 1 & -1 \le \phi \le 1 \\ \phi & \phi > 1 \end{cases}$$

These estimates are sharp and attains for $\phi > 1$ or $\phi < -1$ iff $s(z) = z$ or one of its rotation. If $-1 < \phi < 1$ then equality occurs iff $s(z) = z^2$ or one of its rotation. Equality also occurs for $\phi = -1$ iff $s(z) = \frac{z(z+\lambda)}{1+\lambda z}$ $(0 \le \lambda \le 1)$ or one of its rotation while, for $\phi = 1$ iff $s(z) = -\frac{z(z+\lambda)}{1+\lambda z}$ $(0 \le \lambda \le 1)$ or one of its rotation.

Lemma 4.2. [3] *Let $s(z) = s_1z + s_2z^2 + s_3z^3 + ...\ (z \in \Delta)$ be Schwarz function, then for any complex number ϕ*

$$\left|s_2 - \phi s_1^2\right| \le \max\{1, |\phi|\}$$

This estimate is sharp and attains for $s(z) = z$ or $s(z) = z^2$.

Theorem 4.3. *Let $f \in \mathcal{S}^*_\alpha(\tau, b, \Lambda, \Upsilon)$ and $\Upsilon \neq 0$. Then*

$$|a_2| \le \frac{|b|}{2}(2-\alpha)(\Lambda-\Upsilon)\cos\tau, \tag{4. 1}$$

$$|a_3| \le \frac{|b|}{12}\left(6-\alpha^2-5\alpha\right)(\Lambda-\Upsilon)\cos\tau\left(1+|b|(\Lambda-\Upsilon)\cos\tau\right). \tag{4. 2}$$

Also, for any real ϕ

$$\left|a_3 - \phi a_2^2\right| \le \begin{cases} -|b|(\Lambda-\Upsilon)\cos\tau\left(\frac{(3-\alpha)(2-\alpha)}{12}\right)\sigma & \phi \le \sigma_1, \\ |b|(\Lambda-\Upsilon)\cos\tau\left(\frac{(3-\alpha)(2-\alpha)}{12}\right) & \sigma_1 \le \phi \le \sigma_2, \\ |b|(\Lambda-\Upsilon)\cos\tau\left(\frac{(3-\alpha)(2-\alpha)}{12}\right)\sigma & \phi \ge \sigma_2, \end{cases} \tag{4. 3}$$

where

$$\sigma = b(\Lambda-\Upsilon)e^{-i\tau}\cos\tau\left[\frac{\phi}{3}\left(\frac{3-\alpha}{2-\alpha}\right)+1\right] - \Upsilon \tag{4. 4}$$

and

$$\sigma_1 = 3\left(\frac{3-\alpha}{2-\alpha}\right)\left[\frac{(\Upsilon-1)\,e^{i\tau}}{b\,(\Lambda-\Upsilon)\cos\tau}-1\right], \quad \sigma_2 = 3\left(\frac{3-\alpha}{2-\alpha}\right)\left[\frac{(\Upsilon+1)\,e^{i\tau}}{b\,(\Lambda-\Upsilon)\cos\tau}-1\right].$$

Furthermore, if ϕ is a complex number, then

$$\left|a_3-\phi a_2^2\right| \le |b|\,(\Lambda-\Upsilon)\cos\tau\left(\frac{(3-\alpha)\,(2-\alpha)}{12}\right)\max\{1,|\sigma|\} \qquad (4.\ 5)$$

where σ is defined by (4.4).

Proof. For $f \in \mathcal{S}_\alpha^*(\tau,b,\Lambda,\Upsilon)$ and $\Upsilon \neq 0$, then (2.1) implies

$$\frac{z\left(\Omega^\alpha f(z)\right)'}{\Omega^\alpha f(z)}-1 = \frac{be^{-i\tau}\cos\tau\,(\Lambda-\Upsilon)\,s(z)}{1+\Upsilon s(z)},$$

where $s(z)$ is the Schwarz function satisfying $s(0)=0$ and $|s(z)|<1$ for $z\in\Delta$. On substituting $s(z)=s_1z+s_2z^2+s_3z^3+...$, with simple calculations we get

$$\begin{aligned} &2\left[\frac{\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right]a_2z+\left[-4\left(\frac{\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)^2a_2^2+12\left(\frac{\Gamma(2-\alpha)}{\Gamma(4-\alpha)}\right)a_3\right]z^2+...\\ =\ & be^{-i\tau}\cos\tau\left[(\Lambda-\Upsilon)\,s_1z+(\Lambda-\Upsilon)\left(s_2-\Upsilon s_1^2\right)z^2+...\right].\end{aligned}$$

Now equating coefficients of like powers of z gives us

$$\begin{aligned} a_2 &= \frac{1}{2}\left[be^{-i\tau}\cos\tau\,(\Lambda-\Upsilon)\left(\frac{\Gamma(3-\alpha)}{\Gamma(2-\alpha)}\right)s_1\right]\\ a_3 &= \frac{\Gamma(4-\alpha)}{12\Gamma(2-\alpha)}\left[be^{-i\tau}\cos\tau\,(\Lambda-\Upsilon)\left(s_2-\Upsilon s_1^2\right)+b^2e^{-2i\tau}\cos^2\tau\,(\Lambda-\Upsilon)^2\,s_1^2\right].\end{aligned}$$

Thus using $|s_1|\le 1$ and Lemma 4.1

$$\begin{aligned} |a_2| &\le \frac{|b|}{2}\,(2-\alpha)\,(\Lambda-\Upsilon)\cos\tau,\\ |a_3| &\le \frac{|b|}{12}\left(6-\alpha^2-5\alpha\right)(\Lambda-\Upsilon)\cos\tau\,(1+|b|\,(\Lambda-\Upsilon)\cos\tau).\end{aligned}$$

Simplification also leads us to

$$\left|a_3-\phi a_2^2\right| \le |b|\,(\Lambda-\Upsilon)\cos\tau\left(\frac{(3-\alpha)\,(2-\alpha)}{12}\right)\left|s_2-\sigma s_1^2\right|, \qquad (4.\ 6)$$

where σ is given by (4.4). Hence, from Lemma 4.1 the first inequality in (4.3) is established, when

$$b\,(\Lambda-\Upsilon)\,e^{-i\tau}\cos\tau\left[\frac{\phi}{3}\left(\frac{3-\alpha}{2-\alpha}\right)+1\right]-\Upsilon \le -1,$$

or

$$\phi \le \sigma_1 = 3\left(\frac{3-\alpha}{2-\alpha}\right)\left[\frac{(\Upsilon-1)\,e^{i\tau}}{b\,(\Lambda-\Upsilon)\cos\tau}-1\right].$$

Similarly by application of Lemma 4.1, the third inequality in (4.3) is established, when

$$b\,(\Lambda-\Upsilon)\,e^{-i\tau}\cos\tau\left[\frac{\phi}{3}\left(\frac{3-\alpha}{2-\alpha}\right)+1\right]-\Upsilon \ge 1,$$

or

$$\phi \geq \sigma_2 = 3\left(\frac{3-\alpha}{2-\alpha}\right)\left[\frac{(\Upsilon+1)\,e^{i\tau}}{b\,(\Lambda-\Upsilon)\cos\tau}-1\right].$$

Now the second inequality in (4.3) follows at once by Lemma 4.1, when

$$\sigma_1 \leq \phi \leq \sigma_2.$$

Moreover, applying Lemma 4.2 to (4.6) for the complex number ϕ, the inequality in (4.5) is straightforward. This completes the proof. □

Remark 4.4. *The functional* $a_3 - \phi a_2^2$ *is also known as Hankel determinant with Fekete-Szegö parameter and read as* $H_2^{\phi}(1)$, *(see* [18] *and the citation therein). For some recent results see also* [9, 16, 21, 24].

Remark 4.5. *Note that, our result (Theorem* 4.3*) with* $(\alpha = 0 = \tau)$ *brings improvement over the corresponding results of Srivastava et al.* [22, Theorem 1 with $\lambda = 0$]. *Since*

$$j + |b|\,(\Lambda-\Upsilon) \leq j + \frac{2\,|b|\,(\Lambda-\Upsilon)}{1-\Upsilon}, \quad (j = 0, 1; \quad -1 \leq \Upsilon < \Lambda \leq 1).$$

Remark 4.6. *On assigning specific values to the involved parameters in Theorem* 4.3*, one can deduce the Fekete-Szegö inequalities for the classes* $\mathcal{C}_b\,[\Lambda,\Upsilon]$, $\mathcal{S}_b^*\,[\Lambda,\Upsilon]$, $\mathcal{C}\,[\Lambda,\Upsilon]$, $\mathcal{S}^*\,[\Lambda,\Upsilon]$, $\mathcal{S}^\tau\,(b)$, $\mathcal{C}^\tau\,(b)$, $\mathcal{S}^*\,(b)$, $\mathcal{C}\,(b)$, $\mathcal{C}\,(\gamma)$ *and* $\mathcal{S}^*\,(\gamma)$.

5. Conclusion

In this paper, we have introduced a certain new family of starlike functions of complex order by using the well known Srivastava-Owa fractional calculus operator. For functions in this family, we have thoroughly investigated various properties like, necessary and sufficient condition, Marx-Strohhäcker type inequalities, distortion and radius inequalities, and Fekete-Szegö problem. Various earlier works, appeared as special cases to our reported results. We hope that, the present work may motivate various researchers working in this field.

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