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(λ, v) -Statistical Convergence on a Product Time Scale

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Abstract. We give definition of (λ,v)- statistical convergence on a product time scale. Furthermore, we generalize de la Vallée Poussin mean and define strongly (V,λ,v) and $[V,\lambda,v]^{p}_{\Psi^2}$- summable functions, statistical limit superior and inferior on a product time scale. Then, a few inclusion relations are expressed between the sets of (V,λ,v)-summable, p-strongly $[V,\lambda,v]^{p}_{\Psi^2}$-summable and (λ,v)-statistical convergent functions. Furthermore, some theorems are proved related to statistical limit superior and inferior on a product time scale.

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1. Introduction

The basic idea of statistical convergence was introduced by Zygmund [41] in 1935. Its notion was given by Steinhaus [35] and Fast [13] and later on reintroduced by Schoenberg [32] independently. They used natural density of the set $A \subset \mathbb{N}$ defined by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : k \in A\}\right|, \tag{1.1}$$

provided that above limit exists, where $|.|$ indicates the cardinality of the set of $k \in A$ which satisfy $k \le n$. The idea of statistical convergence depends on asymptotic density of subset of natural numbers. For many years, some concepts in mathematical analysis are generalized by using density such as statistical convergence. Because of this reason, they have an important relation. Statistical convergence is defined by using density in classical case as below:

Definition 1.1. [14] A complex sequence $x = (x_k)_{k\in\mathbb{N}}$ is statistically convergent to a number L if $\delta(\{k \in \mathbb{N}\colon |x_k - L| \geq \varepsilon\})$ has natural density zero for $\forall\varepsilon > 0$. L is necessarily unique and it is called statistical limit of $x = (x_k)_{k\in\mathbb{N}}$, and written as $st_{-}\lim x_k = L$. S denotes the space of all statistically convergent sequences.

Density and statistical convergence led to applications in summability and sequence spaces. For instance, statistical limits of measurable functions were introduced by Moricz [24]. He applied these concepts to strong Cesàro summability. For more details and related notions we refer to [11] and [30]. In the following years, de la Vallée-Poussin mean was introduced by Leindler [20]. Other than that, some authors studied (V,λ) and $(C,1)$ _summabilities, strongly (V,λ) and $(C,1)$ _summabilities and their properties in classical case (see [9], [21]). Mursaleen [25] took the initiative to introduce λ_density and λ_statistical convergence. After almost λ_statistical convergence was studied by Savaş [31]. Furthermore, Nuray [28] defined λ_strong summable and λ_statistical convergent functions. And it is generalized by Et *et al.* [12]. Subsequently, statistical convergence is defined in different forms as follows: The statistical convergence for double sequences was first studied by Mohiuddine *et al.* [22], Moricz [23], Mursaleen and Edely [27] and Tripathy [36].

Definition 1.2. [29] A double sequence $x = (x_{j,k})_{j,k=0}^{\infty}$ is convergent in Pringsheim sense, if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_{j,k} - L| < \varepsilon$ whenever $j,k > N$. In this case, one can write $P_{-}\lim\limits_{k} x_k = L$.

Definition 1.3. [27] The sequence $x = (x_{j,k})_{j,k=0}^{\infty}$ is bounded if there exists a number $M > 0$ such that $|x_{j,k}| < M$ for each j and k, that is, $\|x\| = \sup\limits_{j,k\geq 0} |x_{j,k}| < \infty$. The set ℓ_∞^2 denotes the set of all double bounded sequences.

Definition 1.4. [26] Let $K(m,n) = \{(j,k) : j \leq m, k \leq n\}$. The double density of $K \subseteq \mathbb{N}\times\mathbb{N}$ is defined by

$$\delta_2(K) = P_{-}\lim_{m,n} \frac{1}{mn} |K(m,n)|, \text{ if the limit exists.} \tag{1.2}$$

$x = (x_{j,k})$ is statistically convergent to L if for $\forall\varepsilon > 0$,

$$\{(j,k) : j \leq m, k \leq n : |x_{j,k} - L| \geq \varepsilon\}, \tag{1.3}$$

has double density as zero. In this case, one can write $st_{2-}\lim x_{j,k} = L$ and st_2 denotes the set of all statistically convergent double sequences. Now, we will remind (λ, v) _density and (λ, v) _statistical convergence in classical case.

Definition 1.5. [26] Let $\lambda = (\lambda_m)$ and $v = (v_n)$ be non-decreasing sequences of positive real numbers approaching to ∞ such that $\lambda_{m+1} \leq \lambda_m + 1$, $\lambda_1 = 0$ and $v_{n+1} \leq v_n + 1$, $v_1 = 0$. Throughout this study, Λ is supposed as the set of all such sequences. $(\lambda, v)-$density of $K \subset \mathbb{N}\times\mathbb{N}$ is defined by

$$\delta_{\lambda,v}(K) = P_{-}\lim_{m,n} \frac{1}{\lambda_m v_n} |\{m - \lambda_m + 1 \leq j \leq m, n - v_n + 1 \leq k \leq n : (j,k) \in K\}|, \tag{1.4}$$

provided that above limit exists. In case of $\lambda_m = m$, $v_n = n$, (λ, v)_density reduces to the double density. At the same time, since $\left(\frac{\lambda_m}{m}\right) \leq 1$ and $\left(\frac{v_n}{n}\right) \leq 1$, it yields $\delta_2(K) \leq \delta_{\lambda,v}(K)$ for $\forall K$.

Definition 1.6. The double de la Vallée Poussin mean was introduced by Mursaleen *et al.* [26] as follows

$$t_{m,n} = \frac{1}{\lambda_m v_n} \sum_{j \in j_m} \sum_{k \in I_n} x_{jk}, \tag{1.5}$$

where $j_m = [m - \lambda_m + 1, m]$ and $I_n = [n - v_n + 1, n]$. This is a generalization of classical de la Vallée Poussin mean to the double case. $x = (x_{j,k})$ is strongly (V, λ, v)_summable to L if

$$\lim_{m,n} \frac{1}{\lambda_m v_n} \sum_{j \in j_m} \sum_{k \in I_n} |x_{jk} - L| = 0. \tag{1.6}$$

$[V, \lambda, v]$ denotes the set of all double strongly (V, λ, v)_summable sequences. If $\lambda_m = m$ and $v_n = n$ for all m, n, then strongly (V, λ, v)_summability is reduced to strongly Cesàro summability and $[V, \lambda, v] = [C, 1, 1]$, where $[C, 1, 1]$ is the space of all strongly Cesàro summable double sequences. De la Vallée Poussin mean summability is stronger and general than Cesàro summability. In the classical sense, these summabilities are studied only for special case of time scales. Our generalization gives the opportunity to study on different time scales and to make different interpretations in different spaces. Here, we remind the concept of (λ, v) _statistical convergence for double sequences.

Definition 1.7. [26] The sequence $x = (x_{j,k})$ is (λ, v) _statistically convergent to L if $\delta_{\lambda,v}(E) = 0$, where $E = \{j \in j_m, k \in I_n : |x_{j,k} - L| \geq \varepsilon\}$,i.e., and for $\forall \varepsilon > 0$,

$$P\text{\_}\lim_{m,n} \frac{1}{\lambda_m v_n} |\{m - \lambda_m + 1 \leq j \leq m, n - v_n + 1 \leq k \leq n : |x_{j,k} - L| \geq \varepsilon\}| = 0. \tag{1.7}$$

In this case, $st_{\lambda,v}\text{\_}\lim x_{j,k} = L$ and $S_{\lambda,v}$ indicates the set of all (λ, v) _statistically convergent double sequences.

2. Time Scale Calculus

Here, our aim is to define all above concepts on a product time scale. But, we firstly need to give a brief about the historical improvement of time scale calculus. The idea of time scale calculus was given by Hilger in his doctoral dissertation in 1988 [18]. Later Guseinov [17] was constructed measure theory on time scales and then further studies were made by Bohner, Peterson [7] and Bohner, Svetlin [8]. In the following years, many important results are obtained by many authors in different areas about time scale calculus (see [1], [6], [10], [16]). A time scale $\mathbb{T}$ is an arbitrary, nonempty, closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ can be expressed by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, respectively. A semi closed interval on $\mathbb{T}$ is defined by $[a, b)_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t < b$ where $a, b \in \mathbb{T}\}$. Open and closed intervals can be defined similarly on time scales.

It is necessary to generalize the geometric concept of *length* defined for intervals and generalization is called *measure,* specifically *delta measure* on time scales. The following definition gives Δ_measures of single point set and different types of intervals respectively, established by Guseinov.

Definition 2.1. [17] Let B denotes the family of semi closed intervals $[a,b)_{\mathbb{T}} \in \mathbb{T}$ and $m : B \to [0,\infty)$ be a set function on B such that $m([a,b)_{\mathbb{T}}) = b-a$. Then, the set function m is a countably additive measure on B. Now, the Caratheodory extension of m associated with family B is called Lebesque Δ_measure on $\mathbb{T}$ and is denoted by μ_Δ. The properties of μ_Δ_measure are expressed as follows

$i)$ If $\{a\} \in \mathbb{T}\backslash\{\max\mathbb{T}\}$, then single point set $\{a\}$ is Δ_measurable and $\mu_\Delta(a) = \sigma(a) - a$,

$ii)$ If $a,b \in \mathbb{T}$ and $a \le b$, then $\mu_\Delta([a,b)_{\mathbb{T}}) = b-a$ and $\mu_\Delta((a,b)_{\mathbb{T}}) = b - \sigma(a)$.

$iii)$ If $a,b \in \mathbb{T}\backslash\{\max\mathbb{T}\}$ and $a \le b$, then $\mu_\Delta((a,b]_{\mathbb{T}}) = \sigma(b) - \sigma(a)$ and $\mu_\Delta([a,b]_{\mathbb{T}}) = \sigma(b) - a$

Let us now express forward and backward jump operators, graininess function and Δ_measure for multivariable case. Suppose that $n \in \mathbb{N}$ is a fixed and $\mathbb{T}_j$ are time scales for $j = \overline{1,n}$. Furthermore, σ_j, ρ_j and μ_j are forward and backward jump operators and graininess function on $\mathbb{T}_j$, respectively. Let us set

$$\Psi^n = \mathbb{T}_1 \times \mathbb{T}_2 \times ... \times \mathbb{T}_n = \{t = (t_1, t_2, ..., t_n) : t_j \in \mathbb{T}_j \text{ for all } j = 1,2,...,n\}. \quad (2.1)$$

Denote the collection of all rectangular parallelepipeds in Ψ^n by $\Im$ of the form

$$V = [a_1,b_1)\times[a_2,b_2)\times...\times[a_n,b_n) = \{t = (t_1,t_2,...,t_n) \in \Psi^n : a_j \le t_j \le b_j,\ j = 1,2,...,n\}, \quad (2.2)$$

with $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n) \in \Psi^n$. Let $\mathfrak{m} : \Im \to [0,\infty)$ be the set function that assigns to each parallelepiped $V = [a,b)$. Then, it is not difficult to verify that $\Im$ is a semiring of subsets of Ψ^n and $\mathfrak{m}$ is a $\sigma-$additive measure on $\Im$. By μ_Δ we denote the Caratheodory extension of the measure $\mathfrak{m}$ defined on the semiring $\Im$ and call μ_Δ the Lebesque Δ_measure on Ψ^n (see [4]). When $n = 2$, we can give following explanations for two variable case.

Definition 2.2. [19] Let $\Psi^2 = \mathbb{T}_1 \times \mathbb{T}_2 = \{t = (t_1, t_2) : t_j \in \mathbb{T}_j \text{ for all } j = 1,2\}$ be a time scale. The forward jump operator $\sigma : \Psi^2 \to \Psi^2$ can be defined by

$$\sigma(t) = (\sigma_1(t_1), \sigma_2(t_2)), \quad (2.3)$$

where $\sigma_j(t_j)$ represents the forward jump operator of $t_j \in \mathbb{T}_j$ on the time scale $\mathbb{T}_j$ for all $1 \le j \le 2$. The backward jump operator $\rho : \Psi^2 \to \Psi^2$ by

$$\rho(t) = (\rho_1(t_1), \rho_2(t_2)), \quad (2.4)$$

where $\rho_j(t_j)$ represents the backward jump operator of $t_j \in \mathbb{T}_j$ on the time scale $\mathbb{T}_j$ for all $1 \le j \le 2$. Eventually, the graininess function $\mu : \Psi^2 \to \mathbb{R}^2$ by

$$\mu(t) = (\mu_1(t_1), \mu_2(t_2)), \quad (2.5)$$

where $\mu_j(t_j)$ represents the graininess function of $t_j \in \mathbb{T}_j$ on the time scale $\mathbb{T}_j$ for all $1 \le j \le 2$.

Theorem 2.3. [4] Let $\mathbb{T}_j^0 = \mathbb{T}_j - \{\max \mathbb{T}_j\}, j = 1, 2$. For each point $t = (t_1, t_2) \in \mathbb{T}_1^0 \times \mathbb{T}_2^0$, the single point set $\{t\}$ is Δ_measurable and its Δ_measure is given by

$$\mu_\Delta(\{t\}) = \prod_{j=1}^{2} \mu_j(t_j). \tag{2.6}$$

Furthermore, if $t_j < \sigma_j(t_j)$ for all $j = 1, 2$, $\{t\} = [t_1, \sigma_1(t_1)) \times [t_2, \sigma_2(t_2)) \in \Im$, and

$$\mu_\Delta(\{t\}) = \mathfrak{m}\left([t_1, \sigma_1(t_1)) \times [t_2, \sigma_2(t_2))\right) = \prod_{j=1}^{2} \left(\sigma_j(t_j) - t_j\right). \tag{2.7}$$

To fill the gap between time scale calculus and summability theory, Seyyidoğlu and Tan [33] introduced the statistical convergence on time scales and some new concepts such that Δ_convergence, Δ_Cauchy by using Δ_density and relations between them in 2012. Turan and Duman ([37], [38]) continued on this subject and extended the idea of statistical convergence of Δ_measurable real- valued functions to an arbitrary time scale and they expressed some methods for convergence in 2013. Altin *et al.*[2] studied the uniform statistical convergence on time scale in 2014. Afterwards, Seyyidoğlu and Tan [34] gave a generalization of statistical limit points on time scale in 2015. Eventually, Yilmaz and his coworkers [40] defined λ_statistical convergence and strongly λ_p_Cesaro summability on time scales in 2016. In 2017, Turan and Duman [39] defined Lacunary statistical convergence on time scales. As far as we know, this issue is quite new and striking. Therefore, we focus on moving some topics in summability theory to time scale calculus.

The remaining part of this study is arranged as follows: In section 3, we define (λ, v) _density and (λ, v) _statistical convergence on a product time scale. Moreover, we define some inconclusion relations, strongly (V, λ, v)_summability, statistical limit inferior and superior on a product time scale. Finally, we expressed a few conclusions about the result of this study in section 4.

3. Main Results

In this section, we define (λ, v) _density and (λ, v) _statistical convergence on a product time scale. Then, we define de la Vallée Poussin mean and $[V, \lambda, v]_{\Psi^2}^p$ summability on this product time scale. For these generalizations, we need to construct the structure of a time scale for multivariable case. Let us set

$$\Psi^2 = \mathbb{T} \times \mathbb{T} = \{t = (t_1, t_2) : t_i \in \mathbb{T} \text{ for all } i = 1, 2\}. \tag{3.1}$$

Ψ^2 is called product (or $2-$dimensional) time scale where $\mathbb{T}$ is a time scale. Ψ^2 is a complete metric space with the metric d defined by (see [3], [5])

$$d(t, r) = \left(\sum_{i=1}^{2} |t_i - r_i|^2\right)^{\frac{1}{2}} \quad \text{for } t, r \in \Psi^2. \tag{3.2}$$

Asymptotic density is an important tool to define statistical convergence. Statistical convergence is an area of active research. Many mathematicians studied properties of statistical convergence and applied this concept in various areas such as measure theory, trigonometric series, approximation theory, locally convex spaces, finitely additive set functions and

Banach spaces. Since density and statistical convergence have many applications to real life problems their relations and generalizations to product time scales are important. Here, we introduce the (λ, v) _density and (λ, v) _statistical convergence for double sequences on Ψ^2.

Definition 3.1. Let Ω be a Δ_measurable subset of Ψ^2. Then, $\Omega_{(\lambda,v)}(t,r)$ is defined by

$$\Omega_{(\lambda,v)}(t,r) = \{(s,u) \in [t-\lambda_t+t_0,t]_{\mathbb{T}} \times [r-v_r+r_0,r]_{\mathbb{R}} : (s,u) \in \Omega\}, \qquad (3.3.)$$

for $(t,r) \in \Psi^2$ where $t_0, r_0 \in \mathbb{T}$. Thus, the (λ, v) _density of Ω on Ψ^2 is defined by

$$\delta_{\Psi^2}^{(\lambda,v)}(\Omega) = \lim_{(t,r)\to\infty} \frac{\mu_\Delta\left(\Omega_{(\lambda,v)}(t,r)\right)}{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}} \times [r-v_r+r_0,r]_{\mathbb{T}}\right)}, \qquad (3.4.)$$

if the above limit exists.

Definition 3.2. Let $f : \Psi^2 \to \mathbb{R}$ be a Δ_measurable function. Then, f is (λ, v) _statistically convergent to L on Ψ^2 if

$$\lim_{(t,r)\to\infty} \frac{\mu_\Delta\left((s,u) \in [t-\lambda_t+t_0,t]_{\mathbb{T}} \times [r-v_r+r_0,r]_{\mathbb{T}} : |f(s,u)-L| \geq \varepsilon\right)}{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}} \times [r-v_r+r_0,r]_{\mathbb{T}}\right)} = 0, \qquad (3.5)$$

for $\forall\varepsilon > 0$ where $t_0, r_0 \in \mathbb{T}$. In this case, we write $st_{\Psi^2}^{(\lambda,v)}\text{-}\lim_{(t,r)\to\infty} f(t,r) = L$. If $\mathbb{T} = \mathbb{N}$ and $\lambda_t = t$, $v_r = r$, we get classical (λ, v) _statistical convergence. Therewithal, one can obtain statistical convergence for double sequences only when $\lambda_t = t$, $v_r = r$. (λ, v) _asymptotic density is an important tool to define (λ, v) _statistical convergence similarly to classical density and statistical convergence.

Proposition 3.3. The $st_{\Psi^2}^{(\lambda,v)}$_limit of function $f : \Psi^2 \to \mathbb{R}$ is unique.

Theorem 3.4. Let $f, g : \Psi^2 \to \mathbb{R}$ be two Δ_measurable functions and $c \in \mathbb{R}$. If

$$st_{\Psi^2}^{(\lambda,v)}\text{-}\lim_{(t,r)\to\infty} f(t,r) = L_1 \text{ and } st_{\Psi^2}^{(\lambda,v)}\text{-}\lim_{(t,r)\to\infty} g(t,r) = L_2,$$

then the following statements hold:

$i)$ $st_{\Psi^2}^{(\lambda,v)}\text{-}\lim_{(t,r)\to\infty} (f(t,r)+g(t,r)) = L_1 + L_2$,

$ii)$ $st_{\Psi^2}^{(\lambda,v)}\text{-}\lim_{(t,r)\to\infty} (cf(t,r)) = cL_1$.

Proof. The proof is obvious in case of $c = 0$. Assume that $c \neq 0$, then the proof of $i)$ and $ii)$ follows from

$$\frac{\mu_\Delta\left(\{(s,u) \in [t-\lambda_t+t_0,t]_{\mathbb{T}} \times [r-v_r+r_0,r]_{\mathbb{T}} : |cf(s,u)-cL| \geq \varepsilon\}\right)}{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)\mu_\Delta\left([r-v_r+r_0,r]_{\mathbb{T}}\right)}$$

$$= \frac{\mu_\Delta\left(\left\{(s,u) \in [t-\lambda_t+t_0,t]_{\mathbb{T}} \times [r-\mu_r+r_0,r]_{\mathbb{T}} : |f(s,u)-L| \geq \frac{\varepsilon}{|c|}\right\}\right)}{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)\mu_\Delta\left([r-v_r+r_0,r]_{\mathbb{T}}\right)}$$

and

$$\frac{\mu_\Delta\left(\left\{(s,u)\in[t-\lambda_t+t_0,t]_{\mathbb{T}}\times[r-v_r+r_0,r]_{\mathbb{T}}:\left|f(s,u)+g(s,u)-(L_1+L_2)\right|\geq\varepsilon\right\}\right)}{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)\mu_\Delta\left([r-v_r+r_0,r]_{\mathbb{T}}\right)}$$
$$\leq\frac{\mu_\Delta\left(\left\{(s,u)\in[t-\lambda_t+t_0,t]_{\mathbb{T}}\times[r-v_r+r_0,r]_{\mathbb{T}}:\left|f(s,u)-L_1\right|\geq\frac{\varepsilon}{2}\right\}\right)}{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)\mu_\Delta\left([r-v_r+r_0,r]_{\mathbb{T}}\right)}$$
$$+\frac{\mu_\Delta\left(\left\{(s,u)\in[t-\lambda_t+t_0,t]_{\mathbb{T}}\times[r-v_r+r_0,r]_{\mathbb{T}}:\left|f(s,u)-L_2\right|\geq\frac{\varepsilon}{2}\right\}\right)}{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)\mu_\Delta\left([r-v_r+r_0,r]_{\mathbb{T}}\right)},$$

respectively.

Theorem 3.5. $st_{\Psi^2}\subset st_{\Psi^2}^{(\lambda,v)}$ if and only if

$$\lim_{(t,r)\to\infty}\inf\frac{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\times[r-v_r+r_0,r]_{\mathbb{T}}\right)}{\mu_\Delta\left([t_0,t]\times[r_0,r]\right)}\geq 0. \tag{3.6}$$

Proof. For a given $\varepsilon>0$, we have

$$\left\{(s,u)\in[t-\lambda_t+t_0,t]_{\mathbb{T}}\times[r-v_r+r_0,r]_{\mathbb{T}}:\left|f(s,u)-L\right|\geq\varepsilon\right\}$$
$$\subset\left\{(s,u):[t_0,t]\times[r_0,r]:\left|f(s,u)-L\right|\geq\varepsilon\right\}$$

and

$$\mu_\Delta\left(\left\{(s,u)\in[t-\lambda_t+t_0,t]_{\mathbb{T}}\times[r-v_r+r_0,r]_{\mathbb{T}}:\left|f(s,u)-L\right|\geq\varepsilon\right\}\right)$$
$$\leq\mu_\Delta\left(\left\{(s,u):[t_0,t]\times[r_0,r]:\left|f(s,u)-L\right|\geq\varepsilon\right\}\right)$$

Therefore

$$\frac{\mu_\Delta\left(\left\{(s,u):[t_0,t]\times[r_0,r]:\left|f(s,u)-L\right|\geq\varepsilon\right\}\right)}{\mu_\Delta\left([t_0,t]\times[r_0,r]\right)}$$
$$\geq\frac{\mu_\Delta\left(\left\{(s,u)\in[t-\lambda_t+t_0,t]_{\mathbb{T}}\times[r-v_r+r_0,r]_{\mathbb{T}}:\left|f(s,u)-L\right|\geq\varepsilon\right\}\right)}{\mu_\Delta\left([t_0,t]\times[r_0,r]\right)}$$
$$=\frac{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)}{\mu_\Delta\left([t_0,t]\right)}\frac{\mu_\Delta\left([r-v_r+r_0,r]_{\mathbb{T}}\right)}{\mu_\Delta\left([r_0,r]\right)}$$
$$\frac{\mu_\Delta\left(\left\{(s,u)\in[t-\lambda_t+t_0,t]_{\mathbb{T}}\times[r-v_r+r_0,r]_{\mathbb{T}}:\left|f(s,u)-L\right|\geq\varepsilon\right\}\right)}{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\right)\mu_\Delta\left([r-v_r+r_0,r]_{\mathbb{T}}\right)}.$$

Hence, by using (3.6) and taking the limit as $t,r\to\infty$, we get $f(s,u)\overset{st_{\Psi^2}^{(\lambda,v)}}{\to}L$.

Definition 3.6. The de la Vallée Poussin mean on Ψ^2 is defined by

$$\frac{1}{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\times[r-v_r+r_0,r]_{\mathbb{T}}\right)}\iint_{\Psi^2}f(s,u)\,\Delta u\Delta s. \tag{3.7}$$

f is strongly (V,λ,v) -summable to a number L on Ψ^2, if

$$\lim_{(t,r)\to\infty}\frac{1}{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\times[r-v_r+r_0,r]_{\mathbb{T}}\right)}\iint_{\Psi^2}\left|f(s,u)-L\right|\Delta u\Delta s=0. \tag{3.8}$$

We indicate the set of all strongly (V,λ,v) -summable functions on Ψ^2 by $[V,\lambda,v]_{\Psi^2}$.

Definition 3.7. Let $f:\Psi^2\to\mathbb{R}$ be a Δ_measurable function, $\lambda,v\in\Lambda$ and $0<p<\infty$. The function f is $[V,\lambda,v]_{\Psi^2}^p$ _summable to L on Ψ^2, if

$$\lim_{(t,r)\to\infty}\frac{1}{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\times[r-v_r+r_0,r]_{\mathbb{T}}\right)}\iint_{\Psi^2}|f(s,u)-L|^p\,\Delta u\Delta s=0. \quad (3.9)$$

Here, we write $[V,\lambda,v]_{\Psi^2}^p$ _$\lim f(s,u)=L$. The set of all p_strongly (V,λ,v) _summable functions on Ψ^2 will be denoted by $[V,\lambda,v]_{\Psi^2}^p$.

Lemma 3.8. Let $f:\Psi^2\to\mathbb{R}$ be a Δ_measurable and

$$\Omega_{(\lambda,v)}(t,r)=\left\{(s,u)\in[t-\lambda_t+t_0,t]_{\mathbb{T}}\times[r-v_r+r_0,r]_{\mathbb{T}}:|f(s,u)-L|\geq\varepsilon\right\}, \quad (3.10)$$

for $\varepsilon>0$. In this case, we have

$$\mu_\Delta\left(\Omega_{(\lambda,v)}(t,r)\right)\leq\frac{1}{\varepsilon}\iint_{\Omega_{(\lambda,v)}(t,r)}|f(s,u)-L|\,\Delta s\Delta u\leq\frac{1}{\varepsilon}\iint_{\substack{[t-\lambda_t+t_0,t]_{\mathbb{T}}\\ \times[r-v_r+r_0,r]_{\mathbb{T}}}}|f(s,u)-L|\,\Delta s\Delta u. \quad (3.11)$$

Theorem 3.9. Let $f:\Psi^2\to\mathbb{R}$ be a Δ_measurable function, $\lambda,v\in\Lambda$ and $0<p<\infty$. Then

$i)$ If f is $[V,\lambda,v]_{\Psi^2}^p$ _summable to L, then

$$st_{\Psi^2}^{(\lambda,v)}\text{-}\lim_{(t,r)\to\infty}f(t,r)=L,$$

$ii)$ If $st_{\Psi^2}^{(\lambda,v)}\text{-}\lim_{(t,r)\to\infty}f(t,r)=L$ and f is a bounded, then f is $[V,\lambda,v]_{\Psi^2}^p$ _summable to L.

Proof.

$i)$ Let $[V,\lambda,v]_{\Psi^2}^p\text{-}\lim_{(t,r)\to\infty}f(s,u)=L$ and $\varepsilon>0$, and

$$\Omega_{(\lambda,v)}(t,r)=\left\{(s,u)\in[t-\lambda_t+t_0,t]_{\mathbb{T}}\times[r-v_r+r_0,r]_{\mathbb{T}}:|f(s,u)-L|\geq\varepsilon\right\}, \quad (3.12)$$

on Ψ^2. Then, Lemma 3.8. yields

$$\varepsilon^p\mu_\Delta\left(\Omega_{(\lambda,v)}(t,r)\right)\leq\iint_{\Psi^2}|f(s,u)-L|^p\,\Delta u\Delta s. \quad (3.13)$$

Dividing both sides of the inequality (3.13) by $\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\times[r-v_r+r_0,r]_{\mathbb{T}}\right)$ and taking limit as $(t,r)\to\infty$, we get

$$\frac{\mu_\Delta\left(\Omega_{(\lambda,v)}(t,r)\right)}{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\times[r-v_r+r_0,r]_{\mathbb{T}}\right)}$$
$$\leq\frac{1}{\varepsilon^p}\lim_{(t,r)\to\infty}\frac{1}{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}}\times[r-v_r+r_0,r]_{\mathbb{T}}\right)}\iint_{\Psi^2}|f(s,u)-L|^p\,\Delta u\Delta s=0.$$

Then, we obtain $st_{\Psi^2}^{(\lambda,\mu)}\text{-}\lim_{(t,r)\to\infty}f(s,u)=L$.

$ii)$ Let f be bounded and (λ, v)_statistical convergent to L on Ψ^2. Then, there exists a number $M > 0$ such that $|f(s,u) - L| \leq M$ for all $(s,u) \in \Psi^2$ and since f is (λ, v) _statistically convergent to L, we have

$$\lim_{(t,r)\to\infty} \frac{\mu_\Delta\left(\Omega_{(\lambda,v)}(t,r)\right)}{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}} \times [r-v_r+r_0,r]_{\mathbb{T}}\right)} = 0,$$

where $\Omega_{(\lambda,v)}(t,r) = \{(s,u) \in [t-\lambda_t+t_0,t]_{\mathbb{T}} \times [r-v_r+r_0,r]_{\mathbb{T}} : |f(s,u) - L| \geq \varepsilon\}$.
So, the following inequality can be written as

$$\begin{aligned}
\iint\limits_{\substack{[t-\lambda_t+t_0,t]_{\mathbb{T}} \\ \times[r-v_r+r_0,r]_{\mathbb{T}}}} |f(s,u) - L|\,\Delta u \Delta s &= \iint\limits_{\Omega_{(\lambda,v)}(t,r)} |f(s,u) - L|^p\,\Delta u \Delta s + \iint\limits_{\Psi^2 \setminus \Omega_{(\lambda,v)}(t,r)} |f(s,u) - L|^p\,\Delta u \Delta s \\
&\leq (M+|L|)^p \iint\limits_{\Omega_{(\lambda,v)}(t,r)} \Delta u \Delta s + \varepsilon^p \iint\limits_{\Psi^2 \setminus \Omega_{(\lambda,v)}(t,r)} \Delta u \Delta s \\
&= (M+|L|)^p\,\mu_\Delta\left(\Omega_{(\lambda,v)}(t,r)\right) \\
&\quad + \varepsilon^p \mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}} \times [r-v_r+r_0,r]_{\mathbb{T}}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{(t,r)\to\infty} \frac{1}{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}} \times [r-v_r+r_0,r]_{\mathbb{T}}\right)} \iint\limits_{\Psi^2} |f(s,u) - L|^p\,\Delta u \Delta s \\
&\leq (M+|L|)^p \lim_{(t,r)\to\infty} \frac{\mu_\Delta\left(\Omega_{(\lambda,v)}(t,r)\right)}{\mu_\Delta\left([t-\lambda_t+t_0,t]_{\mathbb{T}} \times [r-v_r+r_0,r]_{\mathbb{T}}\right)} + \varepsilon^p.
\end{aligned}$$

Since ε is arbitrary, it completes the proof.

In classical sense, statistical limit inferior and limit superior and their relations were introduced by Fridy and Orhan [15]. We generalize these concepts to product time scale with following definition.

Definition 3.10. Let $f : \Psi^2 \to \mathbb{R}$ be a measurable function and set

$$\begin{aligned}
B_{(\lambda,v)}^{\Psi^2}(f) &= \left\{ b \in \mathbb{R} : \delta_{\Psi^2}^{(\lambda,v)} \{(t,r) : f(t,r) > b\} \neq 0 \right\} \\
A_{(\lambda,v)}^{\Psi^2}(f) &= \left\{ a \in \mathbb{R} : \delta_{\Psi^2}^{(\lambda,v)} \{(t,r) : f(t,r) < a\} \neq 0 \right\}.
\end{aligned}$$

Then, the statistical limit superior and inferior of f is given by

$$st_{\Psi^2}^{(\lambda,v)}\text{\_}\limsup f = \begin{cases} \sup B_{(\lambda,v)}^{\Psi^2}(f), & \text{if } B_{(\lambda,v)}^{\Psi^2}(f) \neq \emptyset \\ -\infty, & \text{if } B_{(\lambda,v)}^{\Psi^2}(f) = \emptyset \end{cases},$$

$$st_{\Psi^2}^{(\lambda,v)}\text{\_}\liminf f = \begin{cases} \inf A_{(\lambda,v)}^{\Psi^2}(f), & \text{if } A_{(\lambda,v)}^{\Psi^2}(f) \neq \emptyset \\ \infty, & \text{if } A_{(\lambda,v)}^{\Psi^2}(f) = \emptyset \end{cases},$$

respectively.

Theorem 3.11. Let $f : \Psi^2 \to \mathbb{R}$ be a measurable function.

i) $st_{\Psi^2}^{(\lambda,v)}$_$\limsup f = L$ if and only if

$$\delta_{(\lambda,v)}^{\Psi^2}\left(\{(t,r) : f(t,r) > L - \varepsilon\}\right) \neq 0$$
$$\delta_{(\lambda,v)}^{\Psi^2}\left(\{(t,r) : f(t,r) > L + \varepsilon\}\right) = 0.$$

ii) $st_{\Psi^2}^{(\lambda,v)}$_$\liminf f = \ell$ if and only if

$$\delta_{(\lambda,v)}^{\Psi^2}\left(\{(t,r) : f(t,r) < \ell + \varepsilon\}\right) \neq 0$$
$$\delta_{(\lambda,v)}^{\Psi^2}\left(\{(t,r) : f(t,r) < \ell - \varepsilon\}\right) = 0.$$

Theorem 3.12. Let $f : \Psi^2 \to \mathbb{R}$ be a measurable function. Then,
i) $st_{\Psi^2}^{(\lambda,v)}$_$\liminf f \leq st_{\Psi^2}^{(\lambda,v)}$_$\limsup f$.
ii) $\liminf f \leq st_{\Psi^2}^{(\lambda,v)}$_$\liminf f \leq st_{\Psi^2}^{(\lambda,v)}$_$\limsup f \leq \limsup f$.

Definition 3.13. A measurable function $f : \Psi^2 \to \mathbb{R}$ is (λ, v)_statistically bounded if there exists a number $M > 0$ such that

$$\delta_{(\lambda,v)}^{\Psi^2}\left(\{(t,r) : |f(t,r)| > M\}\right) = 0.$$

Theorem 3.14. Let $f : \Psi^2 \to \mathbb{R}$ be a measurable function. (λ, v)_statistically bounded measurable function f is (λ, v)_statistically convergent if and only if

$$st_{\Psi^2}^{(\lambda,v)}\text{\_}\liminf f = st_{\Psi^2}^{(\lambda,v)}\text{\_}\limsup f.$$

Proof. Let $st_{\Psi^2}^{(\lambda,v)}$_$\liminf f = st_{\Psi^2}^{(\lambda,v)}$_$\limsup f = L$. For $\varepsilon > 0$, we can write

$$\delta_{(\lambda,v)}^{\Psi^2}\left(\left\{(t,r) : f(t,r) > L + \frac{\varepsilon}{2}\right\}\right) = \delta_{(\lambda,v)}^{\Psi^2}\left(\left\{(t,r) : f(t,r) < L - \frac{\varepsilon}{2}\right\}\right) = 0.$$

Hence, $st_{\Psi^2}^{(\lambda,v)}$_$\lim f = L$. Conversely, let $st_{\Psi^2}^{(\lambda,v)}$_$\lim f = L$. Then for every $\varepsilon > 0$

$$\delta_{(\lambda,v)}^{\Psi^2}\left(\{(t,r) : |f(t,r) - L| \geq \varepsilon\}\right) = 0$$

and so

$$\delta_{(\lambda,v)}^{\Psi^2}\left(\{(t,r) : f(t,r) \geq L + \varepsilon\}\right) = 0.$$

Thus, $st_{\Psi^2}^{(\lambda,v)}$_$\limsup f \leq L$. Also

$$\delta_{(\lambda,v)}^{\Psi^2}\left(\{(t,r) : f(t,r) < L - \varepsilon\}\right) = 0.$$

Then, we can write $L \leq st_{\Psi^2}^{(\lambda,v)}$_$\liminf f$. Therefore,

$$st_{\Psi^2}^{(\lambda,v)}\text{\_}\limsup f \leq st_{\Psi^2}^{(\lambda,v)}\text{\_}\liminf f \text{ and } st_{\Psi^2}^{(\lambda,v)}\text{\_}\liminf f \leq st_{\Psi^2}^{(\lambda,v)}\text{\_}\limsup f.$$

So, we conclude that

$$st_{\Psi^2}^{(\lambda,v)}\text{\_}\liminf f = st_{\Psi^2}^{(\lambda,v)}\text{\_}\limsup f.$$

4. Conclusion

Generalization of subjects in summability theory is a precious issue in applied analysis. Many mathematicians have tried to get more general results in summability theory. Because of this importance, we have decided to generalize some concepts about (λ, v) -statistical convergence. For this purpose, we defined (λ, v) -density, (λ, v) -statistical convergence and strongly (V, λ, v) -summable functions on a product time scale. If special choices are made, we get classical (λ, v) -statistical convergence and strongly (V, λ, v) -summability. Additionally, we obtained some inclusion relations and defined statistical limit inferior, superior on a product time scale.

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