Abstract. Let $D(V, A)$ be a digraph of order $p$ and size $q$. For an integer $k \geq 1$ and for $v \in V(D)$, let $w_k(v) = \sum_{e \in E_k(v)} f(e)$, where $E_k(v)$ is the set containing all arcs which are at distance at most $k$ from $v$. The digraph $D$ is said to be $E_k$-regular with regularity $r$ if and only if $|E_k(e)| = r$ for some integer $r \geq 1$ and for all $e \in A(D)$. A $V_k$-super vertex out-magic labeling ($V_k$-SVOML) is an one-to-one onto function $f: V(D) \cup A(D) \rightarrow \{1, 2, \ldots, p+q\}$ such that $f(V(D)) = \{1, 2, \ldots, p\}$ and there exists a positive integer $M$ such that $f(v) + w_k(v) = M$, $\forall v \in V(D)$. A digraph that admits a $V_k$-SVOML is called $V_k$-super vertex out-magic ($V_k$-SVOM). This paper contains several properties of $V_k$-SVOML in digraphs. We characterized the digraphs which are $V_k$-SVOM. Also, the magic constant for $E_k$-regular graphs has been obtained. Further, we characterized the unidirectional cycles and union of unidirectional cycles which are $V_2$-SVOM.

AMS (MOS) Subject Classification Codes: 05C78

Key Words: $V_k$-super vertex out-magic labeling, $E_k$-regular graphs, digraph labeling.

1. INTRODUCTION

Let $D(V, A)$ be a digraph of order $p$ and size $q$. For a vertex $v \in V(D)$, the out-neighborhood of $v$ is defined by $O(v) = \{u : (v, u) \in A(D)\}$. The out-degree of $v$ is defined by $deg^+(v) = |O(v)|$. For basic definition and results we follow [3].

A graph labeling is an assignment of integers (usually positive or non-negative integers), which assigned to vertices /or edges /or both into a set of numbers. Lot of labelings have been defined and studied by many authors and an excellent survey of graph labeling can be found in [5].

The magic labeling in graphs was introduced by Sedlák [11]. A magic labeling is an one-to-one onto function $f$ from $E(G)$ onto $\{1, 2, \ldots, q\}$ such that for all $v \in V(G)$, $\sum_{u \in N(v)} f(uv) = M$ for some positive integer $M$. 

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In 2002, MacDougall et al. [8] introduced the notion of vertex magic total labeling (VMTL) in graphs. Let $G(V, E)$ be a graph with $|V(G)| = p$ and $|E(G)| = q$. A one-to-one map $f$ from $E(G) \cup V(G)$ onto the integers $\{1, 2, \ldots, p+q\}$ is a VMTL if there is a constant $M$ so that for every vertex $x \in V(G)$, $f(x) + \sum f(xy) = M$, where the sum is taken over all vertices $y$ adjacent to $x$.

In 2004, MacDougall et al. [9] defined the super vertex-magic total labeling (SVMTL) in graphs. They call a VMTL is super if $f(V(G)) = \{1, 2, \ldots, p\}$. In this labeling, the smallest labels are assigned to the vertices.

Another labeling parameter called ‘Super Edge magic total labeling’ with different meaning has been defined and studied in [1, 6, 7, 10].

In 2008, Bloom et al. [2] extended the idea of magic labeling to digraphs.

The $V$-super vertex out-magic total labeling ($V$-SVOMTL) in digraph was introduced by Durga Devi et al. [4]. A $V$-SVOMTL is an one-to-one onto function $f : V(D) \cup A(D) \rightarrow \{1, 2, \ldots, p+q\}$ such that $f(V(D)) = \{1, 2, \ldots, p\}$ and for every $v \in V(D)$, $f(v) + \sum_{u \in O(v)} f((u, v)) = M$ for some positive integer $M$.

This paper generalizes the definition of $V$-SVOMTL and defines a new labeling called $V_k$-super vertex out-magic labeling ($V_k$-SVOML). For an integer $k \geq 1$, let $E_k(u) = \{(u, v) : d(u, v) \leq k\}$ and $w_k(u) = \sum_{e \in E_k(u)} f(e)$. Note that if $(u, v)$ is an directed edge, then $d(u, v) = 1$. A $V_k$-SVOML is an one-to-one onto function $f : V(D) \cup A(D) \rightarrow \{1, 2, \ldots, p+q\}$ such that $f(V(D)) = \{1, 2, \ldots, p\}$ and there exists a positive integer $M$ such that $f(v) + w_k(v) = M$, $\forall v \in V(D)$. A digraph that admits a $V_k$-SVOML is called $V_k$-super vertex out-magic ($V_k$-SVOM). If $x_1$ and $y_1$ are vertices of a digraph $D$ then the distance from $x_1$ to $y_1$ in $D$, is the minimum length of a directed $x_1 - y_1$ path if $y_1$ is reachable from $x_1$, and otherwise it is taken as infinity.

Let $k$ be an integer such that $1 \leq k \leq \text{diam}(D) + 1$. For $e = (u, v) \in A(D)$, we define $E_k(e) = \{w \in V(D) : 1 \leq d(u, w) \leq k\}$. The digraph $D$ is said to be $E_k$-regular with regularity $r$ if and only if $|E_k(e)| = r$ for some integer $r \geq 1$ and for all $e \in A(D)$. Consider the digraph $D$ (see Figure 1). In $D$, $E_2(v_2) = \{e_2, e_3, e_8, e_9\}$, $E_2(v_7) = \{e_7, e_4\}$, $E_2(e_1) = \{v_2, v_3, v_8\}$ and $E_2(e_7) = \{v_4, v_5\}$.

![Figure 1: D](image)

**Observation 1.1.** Let $D$ be a digraph of order $p \geq 2$. If $E_k(x_1) = E_k(x_2)$ for $x_1, x_2 \in V(D)$ ($x_1 \neq x_2$), then $D$ is not $V_k$-SVOM.
Proof. Let $D$ be a digraph of order $p \geq 2$. Suppose $E_k(x_1) = E_k(x_2)$ for a pair of vertices $x_1$ and $x_2$ $(x_1 \neq x_2)$ of $D$. Then $f(x_1) + w_k(x_1) \neq f(x_2) + w_k(x_2)$ for any $V_k$-SVOML $f$ of $D$ (since $f$ is one to one). In this case, $D$ does not admit $V_k$-SVOML.

A digraph $D$ is said to be strongly connected if every pair of vertices are mutually reachable.

**Remark 1.2.** Let $D$ be a strongly connected digraph. If $k \geq \text{diam}(D) + 1$, then $E_k(u) = A(D)$ for every $u \in V(D)$.

**Proof.** Let $u \in V(D)$. Then $E_k(u) \subseteq A(D)$. Let $(x, y) \in A(D)$. Since $D$ is strongly connected, there exist directed $u-x$ path and $u-y$ path in $D$ with length $\leq \text{diam}(D)$. Then there exist a directed $u-x-y$ path of length $\leq \text{diam}(D) + 1 \leq k$ and so $(x, y) \in E_k(u)$. Thus $A(D) \subseteq E_k(u)$. □

### 2. $V_k$-SVOML in Digraphs

This section will explore the basic properties of $V_k$-SVOML.

**Theorem 2.1.** Let $D$ be a digraph and $f_1$ be an one-to-one function from $A(D)$ onto $\{p+1, p+2, \ldots, p+q\}$. Then $f_1$ can be extended to a $V_k$-SVOML of $D$ if and only if $\{w_k(v) : v \in V(D)\}$ is a set of $p$ successive integers.

**Proof.** Assume that $\{w_k(v) : v \in V(D)\}$ is a set of $p$ successive integers. Let $t = \min \{w_k(v) : v \in V(D)\}$. Define $f_2 : V(D) \cup A(D) \to \{1, 2, \ldots, p+q\}$ as $f_2((u, v)) = f_1((u, v))$ for $(u, v) \in A(D)$ and $f_2(v) = t + p - w_k(v)$ for $v \in V(D)$. Since $\{w_k(v) - t : v \in V(D)\}$ is a set of successive integers, $f_2(V(D)) = \{1, 2, \ldots, p\}$. Also $f_2(A(D)) = \{p+1, p+2, \ldots, p+q\}$. Hence $f_2$ is $V_k$-SVOML of $D$ with magic constant $M = t + p$.

Conversely, assume that $f_1$ can be extended to a $V_k$-SVOML $f_2$ of $D$. Let $M$ be the magic constant. Since $f_2(v) + w_k(v) = M$ for every $v \in V(D)$, $\{w_k(v) : v \in V(D)\} = \{M-p, M-p+1, \ldots, M-1\}$ is a set of $p$ successive integers. □

**Lemma 2.2.** If a digraph $D(p, q)$ is $V_k$-SVOM and $E_k$-regular with regularity $r$, then the magic constant $M = \frac{p-1}{2} + rq + \frac{r(q+1)}{2}$.

**Proof.** Let $f$ be a $V_k$-SVOML of $D$ and $M$ be the magic constant. Note that $M = f(x) + w_k(x)$ for all $x \in V(D)$. Summing over all $x \in V(D)$, we get $pM = \sum_{x \in V(D)} f(x) + \sum_{x \in V(D)} w_k(x) = \sum_{x \in V(D)} f(x) + \sum_{x \in V(D)} \sum_{e \in E_k(x)} f(e)$ (since each edge is counted exactly $r$ times in the sum).

Since $f(V(D)) = \{1, 2, \ldots, p\}$ and $f(A(D)) = \{p+1, p+2, \ldots, p+q\}$, $pM = \frac{p(p+1)}{2} + r(pq) + \frac{r(q+1)}{2}$ and so $M = \frac{p+1}{2} + rq + \frac{r(q+1)}{2}$. □

Lemma 2.2 gives the magic constant for $E_k$-regular graphs which are $V_k$-SVOM for $k \geq 1$. In 2017, Durga Devi et al. [4] obtained the following result which gives the magic constant for all digraphs which admit $V$-SOMTL.

**Lemma 2.3.** [4] If a non-trivial digraph $D$ is $V$-SOMT, then the magic constant $M$ is given by $M = q + \frac{p+1}{2} + \frac{r(q+1)}{2p}$. 


When $k = 1$, we have $r = |E_1(v)| = 1$ for all $v \in A(D)$. The above result is a corollary of Lemma 2.2 when $k = 1$.

**Theorem 2.4.** For $k \geq 2$, trees are not $V_k$-SVOM.

**Proof.** Suppose there is a tree that is $V_k$-SVOM. Let $V(D) = \{v_1, v_2, \ldots, v_p\}$. Since $D$ is a tree, $q = p - 1$ and so at least one vertex of $D$ has out-degree zero, let it be $v_n$. Then by Observation 1.1, $v_n$ is the only vertex of $D$ with out-degree zero. Since $w_k(v_n) = 0$, by Theorem 2.1, $\{w_k(v) : v \in V(D)\} = \{0, 1, 2, \ldots, p - 1\}$, which is impossible since $f(A) = \{p, p + 1, p + 2, p + 3, \ldots, 2p - 1\}$. \hfill $\square$

**Remark 2.5.** From Theorem 2.4, we observe that a connected digraph is not $V_k$-SVOM ($k \geq 2$) when $q = p - 1$. Thus if a connected digraph is $V_k$-SVOM ($k \geq 2$), then $q \geq p$.

**Corollary 2.6.** Let $D$ be a connected $E_k$-regular digraph ($k \geq 2$) with regularity $r$. If $D$ is $V_k$-SVOM, then $M \geq \frac{p+1}{2} + \frac{r(3p+1)}{2}$.

**Proof.** When $q = p - 1$, $D$ is a tree. By Theorem 2.4, $D$ is not $V_k$-SVOM. Assume that $q \geq p$. Then by Lemma 2.2, $M \geq \frac{p+1}{2} + \frac{r(3p+1)}{2}$. \hfill $\square$

**Remark 2.7.** From Corollary 2.6, we get $M = \frac{p+1}{2} + \frac{r(3p+1)}{2}$ when $p = q$. For example consider the following digraph $C_7$.

![Figure 2: $V_2$-SVOM of $C_7$](image)

The unidirectional cycle $C_7$ is $E_2$-regular with regularity $r = 2$ and $V_2$-SVOM with magic constant $M = \frac{p+1}{2} + \frac{r(3p+1)}{2} = 26$.

3. $V_2$-SVOML OF UNIDIRECTIONAL CYCLES AND UNION OF UNIDIRECTIONAL CYCLES

Durga Devi et al. [4] proved that all the unidirectional cycles admit $V$-SVOMTL. When $k = 2$, not all the unidirectional cycles admit $V_2$-SVOML. The next result characterize the unidirectional cycles which are $V_2$-SVOM.

**Theorem 3.1.** Let $n(\geq 3)$ be an integer. Then the unidirectional cycle $C_n$ is $V_2$-SVOM if and only if $n$ is an odd integer.

**Proof.** Suppose there exists a $V_2$-SVOML $f$ of $C_n$. Since $|E_2(e)| = r = 2$ for all $e \in A(C_n)$, by taking $k = 2$, $p = q = n$ and $r = 2$ in Lemma 2.2, we get $M = \frac{7n+3}{2}$. If $n$ is an even integer, then $M$ is not an integer, a contradiction. Thus $n$ must be odd.

Conversely, assume that $n$ is odd and $n \geq 3$. Let $V(C_n) = \{a_i : 1 \leq i \leq n\}$ and $A(C_n) = \{(a_i, a_{i+1}) : 1 \leq i \leq n\}$, where the operation $\oplus_n$ stands for addition modulo $n$. Define $f : V(D) \cup A(D) \rightarrow \{1, 2, \ldots, 2n\}$ as follows:
Suppose there exists a function $f$ such that $f((a_i, a_{i+1})) = \frac{n+3}{2}$ when $i$ is odd and $f((a_i, a_{i+1})) = \frac{3n+1}{2} + \frac{j}{2}$ when $i$ is even. Then the set $\{w_2(a_1), w_2(a_2), \ldots, w_2(a_n - 1)\} = \left\{\frac{5n+3}{2}, \frac{5n+5}{2}, \ldots, \frac{7n+1}{2}\right\}$ is a set of $n$ successive integers. Thus by Theorem 2.1, $C_n$ is $V_2$-SVOM.

**Theorem 3.2.** Let $m \geq 1$ be an integer. Then $mC_n$ is $V_2$-SVOM if and only if $m$ and $n$ are odd integers.

**Proof.** Suppose there exists a $V_2$-SVOM $f$ of $mC_n$. Since $|E_E(e)| = r = 2$ for all $e \in A(mC_n)$, by taking $k = 2, p = q = mn$ and $r = 2$ in Lemma 2.2, we get $M = \frac{7mn+3}{2}$. Either $m$ or $n$ is even then $M$ is not an integer, a contradiction. Thus both $m$ and $n$ are odd integers.

Conversely, assume that $m$ and $n$ are odd integers. Let $V(mC_n) = V_1 \cup V_2 \cup \ldots \cup V_m$, where $V_i = \{v_i^1, v_i^2, \ldots, v_i^n\}$ for $i = 1, 2, \ldots, m$. Let $A(mC_n) = A_1 \cup A_2 \cup \ldots \cup A_m$, where $A_i = \{e_i^1, e_i^2, \ldots, e_i^n\}$ with $e_i^j = (v_i^j, v_i^{j+1})$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Define $f : V(D) \cup A(D) \rightarrow \{1, 2, \ldots, 2mn\}$ as follows:

For $1 \leq i \leq \frac{m+1}{2}$,

$$f(v_i^j) = \begin{cases} (n-j)m + 1 - 2i & \text{for } j = 1, 2, \ldots, n - 2 \\ i & \text{for } j = n - 1 \\ \frac{2mn-2m+1}{2} + i & \text{for } j = n \end{cases}$$

$$f(e_i^j) = \begin{cases} \frac{(j-1)m}{2} + i + mn & \text{for } j = 1, 3, \ldots, n - 2 \\ \frac{(n+1)m+1}{2} + i + nm & \text{for } j = 2, 4, \ldots, n - 1 \\ \frac{(n+2)m}{2} + 2i + nm & \text{for } j = n. \end{cases}$$

For $\frac{m+1}{2} \leq i \leq m$,

$$f(v_i^j) = \begin{cases} (n+1-j)m + 1 - 2i & \text{for } j = 1, 2, \ldots, n - 2 \\ i & \text{for } j = n - 1 \\ \frac{(2n-3)m+1}{2} + i & \text{for } j = n \end{cases}$$

$$f(e_i^j) = \begin{cases} \frac{(j-1)m}{2} + i + nm & \text{for } j = 1, 3, \ldots, n - 2 \\ \frac{(n+1)m+1}{2} + i + nm & \text{for } j = 2, 4, \ldots, n - 1 \\ \frac{(n+2)m}{2} + 2i + nm & \text{for } j = n. \end{cases}$$

To prove $f(v) + w_2(v) = \frac{7mn+3}{2}$ for every vertex $v \in V(mC_n)$. Let $v \in V(mC_n)$.

Then $f(v_i^j) + w_2(v_i^j) = f(v_i^j) + f((v_i^j, v_i^{j+1})) + f((v_i^{j+1}, v_i^j)) = f(v_i^j) + f(e_i^j) + f(e_i^{j+1})$.

**Case 1:** Suppose $1 \leq i \leq \frac{m-1}{2}$ and $j \geq 5$.

If $j$ is even, then $f(v_i^j) + f(e_i^j) + f(e_i^{j+1}) = (n-j)m + 1 - 2i + \frac{(n+j)m+1}{2} + i + nm + \frac{m}{2} = nm - jm + 1 + nm + \frac{m}{2} + \frac{m}{2} + \frac{m}{2} = \frac{7mn+3}{2}.$

If $j$ is odd, then $f(v_i^j) + f(e_i^j) + f(e_i^{j+1}) = (n-j)m + 1 - 2i + \frac{(j-1)m}{2} + i + nm + \frac{(n+j+1)m+1}{2} + i + nm = 3nm - jm + 1 + \frac{m}{2} + \frac{m}{2} + \frac{m}{2} = \frac{7mn+3}{2}.$

**Case 2:** Suppose $\frac{m+1}{2} \leq i \leq m$ and $j \geq 5$.

If $j$ is even, then $f(v_i^j) + f(e_i^j) + f(e_i^{j+1}) = (n-j)m + 1 - 2i + \frac{(n+j-2)m}{2} + \frac{m}{2} = \frac{7mn+3}{2}.$

If $j$ is odd, then $f(v_i^j) + f(e_i^j) + f(e_i^{j+1}) = (n-j)m + 1 - 2i + \frac{(n+j-2)m}{2} + \frac{m}{2} = \frac{7mn+3}{2}.$
The unidirectional crown digraph
Let
The above result has been illustrated through an example. Consider the

Define
Similarly,

Similarly,

Similarly,

Similarly,

In the next result, we find the magic constant for unidirectional crown digraph which is not \( E_k \)-regular for all \( k \geq 2 \).

**Theorem 3.3.** The unidirectional crown digraph \( \overrightarrow{C}_p \) is \( V_2 \)-SVOML if \( p \) is odd with magic constant \( \frac{15p+3}{2} \).

**Proof.** Let \( V(\overrightarrow{C}_p) = \{a_i : 1 \leq i \leq p\} \cup \{b_i : 1 \leq i \leq p\} \) and \( A(\overrightarrow{C}_p) = \{(a_i, a_{i \oplus 1}) : 1 \leq i \leq p\} \cup \{(b_i, a_i) : 1 \leq i \leq p\} \). Note that \( |V(\overrightarrow{C}_p)| = 2p, |A(\overrightarrow{C}_p)| = 2p \).

Define \( f : V(\overrightarrow{C}_p) \cup A(\overrightarrow{C}_p) \to \{1, 2, \ldots, 4p\} \) by

To prove \( f(b_i) + w_2(b_i) = \frac{15p+3}{2} \) for \( 1 \leq i \leq p \) and \( b_i \in V(\overrightarrow{C}_p) \).

Let \( b_i \in V(\overrightarrow{C}_p) \). Then \( w_2(b_i) = f((b_i, a_i)) + f((a_i, a_{i \oplus 1})) \).

**Case 1:** Suppose \( i \) is odd, then

**Case 2:** Suppose \( i \) is even, then

Similarly, we can prove that \( f(a_i) + w_2(a_i) = \frac{15p+3}{2} \) for \( 1 \leq i \leq p \).

**Example 3.4.** The above result has been illustrated through an example. Consider the following digraph \( D_1 = \overrightarrow{C}_5 \). Here \( V(D_1) = \{a_1, a_2, \ldots, a_5\} \cup \{b_1, b_2, \ldots, b_5\} \) and \( A(D_1) = \{(a_i, a_{i \oplus 1}) : 1 \leq i \leq 5\} \cup \{(b_i, a_i) : 1 \leq i \leq 5\} \).

![Figure 3: D1](image-url)
Here \( p = 5 \). The function \( f \) is given by \( f : V(D_1) \cup A(D_1) \to \{1, 2, \ldots, 20\} \) by \( f(b_i) = 2p - \frac{i-1}{2} = 10 - \frac{i-1}{2} \) when \( i \) is odd; \( f(b_i) = \frac{3p-1}{2} - \frac{i-2}{2} = 7 - \frac{i-2}{2} \) when \( i \) is even; and \( f(a_i) = i \) for \( 1 \leq i \leq 5 \). The arc labels are given by \( f((b_i, a_i)) = 2p + i = 10 + i, 1 \leq i \leq 5; f((a_i, a_i \oplus 1)) = \frac{3p+2}{2} - \frac{i}{2} = \frac{37}{2} - \frac{i}{2} \) when \( i \) is odd; and \( f((a_i, a_i \oplus 1)) = 4p - \frac{i-1}{2} = 20 - \frac{i-2}{2} \) when \( i \) is even. From the above Figure, we can easily see that \( f \) is \( V_2\)-SVOML with magic constant \( M = 39 \).

**Conclusion**

In this paper, we introduced a new labeling in digraphs, namely \( V_k\)-SVOM. We obtain a necessary and sufficient condition for the existence of \( V_k\)-SVOML in digraphs and the magic constant for \( E_k\)-regular digraphs. Further, we characterized the unidirectional cycles and union of unidirectional cycles which are \( V_2\)-SVOM. In future we study \( V_k\)-SVOM (\( k \geq 2 \)) for directed circulant graph and the generalized de-Bruijn digraph.

4. **Acknowledgments**

The authors thank the anonymous referees for their useful comments and suggestions which improved the quality and the readability of the paper.

**References**


