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Characterizations of Quantales by the Properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$ -Fuzzy (Subquantales) Ideals

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Abstract. The notion of quantale, which designates a complete lattice equipped with an associative binary operation distributing over arbitrary joins, was used for the first time by Mulvey in 1986. In this paper, we present (α, β)-$f$uzzy (subquantales) ideals in quantales, where α, β may be one of these $\in_\gamma, q_\delta, \in_\gamma \vee q_\delta$ and $\in_\gamma \wedge q_\delta$. Special attension is considered to $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy (subquantales) ideals. Some characterizations about $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy prime and semi-prime ideals are also proved.

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1. Introduction

The notion of quantale, was used for the first time by C. J. Mulvey, [28] in 1 986. The connection between quantale theory and linear logic was established by Yetter, in 1990, [49]. However, multiplicative ordered structures were studied already in the form of lattices of ideals of a ring. During the previous two decades, quantales have found their application in areas of algebraic theory [23], rough set theory [24, 33, 34, 36, 47, 48], topological theory [11], theoretical computer science [38] and linear logic [9]. Theory of fuzzy quantale is a generalization of classical quantale theory.

Fuzzy set theory, initially proposed by Zadeh [53], has given a valuable scientific and mathematical tool for illustrating the behaviors of those systems which are excessively intricate or indeterminate. The idea of fuzziness is generally utilized in the theory of formal languages, automata and many more. Numerous scientists utilized this idea for the generalization of algebraic structures. Certain Characterization of m-Polar Fuzzy Graphs by Level Graphs were discussed by Akram and Shahzadi., [2]. In 1993, Ahsan et al. [1], proposed fuzzy semirings and fuzzy subgroups were defined by Rosenfeld. Fahmi et al. suggested Weighted Average Rating Method for Solving Group Decision Making Problem Using Triangular Cubic Fuzzy Hybrid Aggregation operator [8]. Certain Properties of Bipolar Fuzzy Soft Topology Via Q-Neighborhood were introduced by Riaz and Tehrim [39]. For further applications of fuzzy sets see [3, 4, 10, 16, 17, 22, 27, 29, 30, 31, 50, 51, 52, 54, 55]. There are several authors who applied the theory of fuzzy sets to quantale, for instance, Luo and Wang applied the fuzzy set theory to quantales [24]. They defined fuzzy prime, fuzzy semi-prime and fuzzy primary ideals of quantales. They also introduced the notions of rough fuzzy (prime, semi-prime, primary) ideals of quantales. Generalized rough fuzzy ideals in quantales were introduced by Qurashi and Shabir [33].

The significance of fuzzy algebraic structures can be seen by utilizing the thought of belongingness and quasi-coincidence with a fuzzy set. Ming and Ming [32] presented the idea of quasi-coincidence of a fuzzy point with a fuzzy subset. Davvaz in [5] investigated the properties of $(\in, \in \vee q)$-fuzzy sub-nearrings. The idea of (α, β)-fuzzy ideals of hemirings was explored by Dudek et al., [7]. The ordered semigroups in terms of $(\in, \in \vee q)$-fuzzy interior ideals were examined by Khan et al., [19]. The generalization of fuzzy interior ideals of semigroup was presented by Jun and Song [15]. Also, in [12], the concept of (α, β)-fuzzy subalgebras (ideals) of a BCK/BCI algebra and related results were discussed by Jun. $(\in, \in \vee q_k)$-fuzzy ideals of ternary semigroups were studied by Shabir and Noor [44]. Jun et al., discussed the general form of (α, β)-fuzzy ideals of hemirings [13]. In [43], Shabir et al., characterized semigroups by the properties of $(\in, \in \vee q_k)$-fuzzy ideals (fuzzy bi-ideals) and $(\in, \in \vee q_k)$-fuzzy quasi-ideals. Zulfiqar and Shabir [56], characterized $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideals in BCK-algebras. Ma et al., discussed $(\in, \in \vee q)$-fuzzy filters of RO-algebras [25, 26]. For more details see [3, 4, 6, 10, 14, 15, 18, 20, 21, 29, 30, 31, 37, 41, 42, 50, 51, 54, 55].

In the present paper, we deal with a generalization of the paper of Qurashi and Shabir [35], we discuss more new types of $(\in, \in \vee q)$-$f$uzzy (subquantales) ideals of Quantales. We introduce the concepts of (α, β)-$f$uzzy (subquantales) ideals and some related properties are examined. Special consideration is given to $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy (subquantales) ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy prime (semi-prime) ideals, and some interesting results are obtained. Furthermore, subquantale, prime, semi-prime and $f$uzzy subquantale, $f$uzzy prime ideals, $f$uzzy semi-prime ideals of the types $(\in_\gamma, \in_\gamma \vee q_\delta)$ are linked by using level subsets.

2. Preliminaries

This section gives the fundamental definitions and starter results, concerning quantales, fuzzy ideals in quantales and concept of belongingness which are valuable for our consequent sections. All through this paper, we will utilize Q_t for quantale, unless stated otherwise.

Definition 2.1. [40] *A quantale Q_t is a complete lattice equipped with an associative, binary operation $\otimes$ distributing over arbitrary joins. In other words, for any $y \in Q_t$ and*

$\{z_i\} \subseteq Q_t$, $i \in I$, *it holds:*

$$y \otimes (\vee_{i\in I} z_i) = \vee_{i\in I} (y \otimes z_i);$$
$$(\vee_{i\in I} y_i) \otimes z = \vee_{i\in I} (y_i \otimes z).$$

Let $X_i, X, Y \subseteq Q_t$, *we define the followings*;

$$\vee_{i\in I} X_i = \{\vee_{i\in I} x_i \mid x_i \in X_i\};$$
$$X \vee Y = \{x \vee y \mid x \in X,\ y \in Y\};$$
$$X \otimes Y = \{x \otimes y \mid x \in X, y \in Y\}.$$

Throughout the paper, the symbol $\top$ *will denote the top element and* $\bot$ *will stand for the bottom one for quantale, unless stated otherwise.*

The following definition is about the ideal in quantales. Prime and semi-prime ideals will also be discussed in this section.

Definition 2.2. [45, 46] *A subset* $\emptyset \neq I_d$ *of quantale* Q_t *is said to be an ideal of* Q_t *if the conditions below are satisfied:*

(1) If $w, x \in I_d$ implies $w \vee x \in I_d$;
(2) for all $w, x \in Q_t$ and $x \in I_d$ such that $w \leqslant x$ implies $w \in I_d$;
(3) for all $w \in Q_t$ and $x \in I_d$ implies $w \otimes x \in I_d$ and $x \otimes w \in I_d$.

Let I_d be an ideal of Q_t. Then, I_d is said to be a prime ideal if $w \otimes y \in I_d$ implies $w \in I_d$ or $y \in I_d$, $\forall\, w, y \in Q_t$. An ideal I_d is said to be a semi prime ideal if $w \otimes w \in I_d$ implies $w \in I_d$ for each $w \in Q_t$.

Example 2.3. *Let* $Q_t = \{\bot, c, \top\}$. *Then* Q_t *is a quantale and the Fig.*1 *and Table* 1, *represent the partial order and binary operation* $\otimes$, *respectively.*

Table 1. Binary operation $\otimes$ subject to Q_t.

$\otimes$	$\bot$	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
c	$\bot$	c	c
$\top$	$\bot$	c	$\top$

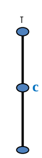

FIGURE 1. Illustration of Q_t.

It is widely known in the $fuzzy$ set theory given by Zadeh [53], a $fuzzy$ subset, g of a non-empty set Q_t is a mapping from Q_t to $[0,1]$. All through this paper, we will utilize inf for infimum and sup for suprimum in $[0,1]$, except if expressed something else while $\wedge$ and $\vee$ will symbolize the respective infimum and supremum for the elements of Q_t.

From here onward, for our convenience, for $f$uzzy subset, left ideal, right ideal, fuzzy subquantale, fuzzy ideal, fuzzy prime and fuzzy semi-prime ideal, the following shortened forms f-subset, LI, RI, FS, FI, FPI and $FSPI$, respectively, will be utilized.

Definition 2.4. [35] *Let g be a f-subset of quantale Q_t. Then, g is a FS of Q_t if,*

(1) $g(\vee_{i\in I} z_i) \geq \underset{i\in I}{inf}\ g(z_i)$;
(2) $g(w \otimes z) \geq inf(g(w), g(z), \forall \{z_i\} \subseteq Q_t\ (i \in I)$ and $\forall\ z, w \in Q_t$.

Definition 2.5. [24] *A non-empty f-subset g of Q_t is called an FI of Q_t, if the conditions below are satisfied:*
(FI_3) $z_1 \leq z_2 \Longrightarrow g(z_2) \leq g(z_1)$;
(FI_4) $inf\{g(z_1), g(z_2)\} \leq g(z_1 \vee z_2)$;
(FI_5) $sup\{g(z_1), g(z_2)\} \leq g(z_1 \otimes z_2)\ \forall\ z_1, z_2 \in Q_t$.

From (FI_3) and (FI_4) in Definition 2.5 it is observed that $g(z_1 \vee z_2) = inf\{g(z_1), g(z_2)\}$, $\forall\ z_1, z_2 \in Q_t$. Thus, a f-subset g of Q_t is a FI of Q_t if and only if $g(z_1 \vee z_2) = inf\{g(z_1), g(z_2)\}$ and $g(z_1 \otimes z_2) \geq sup\{g(z_1), g(z_2)\}$, $\forall\ z_1, z_2 \in Q_t$.
The details of FPI and $FSPI$ are as follows.

Definition 2.6. [24] *A non-constant FI, g of a quantale Q_t is called an FPI of Q_t if it satisfies*

$$(FI_6)\ g(z_1 \otimes z_2) = g(z_1)\ or\ g(z_1 \otimes z_2) = g(z_2)\ \forall z_1, z_2 \in Q_t.$$

Definition 2.7. [24] *Let g be a FI of a quantale Q_t. Then g is called an $FSPI$ of Q_t if the assertion below is satisfied:*

$$(FI_7)\ g(z^2) = g(z)\ \forall z \in Q_t.$$

The next discussion is about the idea of belongingness and quasi-coincidence of a fuzzy point with an f-subset.

An f-subset g of a set Q_t is of the form $g(y) = \begin{cases} p(\neq 0), & \text{if } y = z \\ 0, & \text{otherwise} \end{cases}\ \forall\ y \in Q_t$ is said to be a fuzzy point with support z and value $p \in (0,1]$ and is denoted by z_p (see [32]). For a fuzzy point z_p and an f-subset in a set Q_t, we say that

(a) If $g(z) \geq p$, then it conveys that z_p belongs to g and is denoted as $z_p \in g$.
(b) If $g(z) + p > 1$, then z_p is called quasi-coincident with g and is represented as $z_p q g$.
(c) If $g(z) \geq p$ or $g(z) + p > 1$, then it means that z_p belongs to g or z_p is quasi-coincident with g and is denoted as $z_p\ (\in \vee q)g$. Likewise, $z_p \in g$ and $z_p q g$ is represented by $z_p\ (\in \wedge q)g$.

If one of $z_p \in g$, $z_p q g$ and $z_p\ (\in \vee q)g$ does not satisfy, then we communicate as $z_p \overline{\in} g$, $z_p\ \overline{q}\ g$ and $z_p\ \overline{(\in \vee q)}\ g$, respectively. Thus, $z_p \overline{\alpha} g$ conveys that $z_p \alpha g$ does not hold. Each

f-subset g defined on Q_t can be characterized by its level subsets, i.e., by the sets of the form $g_v = \{x \in Q_t : g(x) \geq v\}$, where $v \in [0,1]$. A vital part is played by the support of g, i.e., the set $g_\circ = \{x \in Q_t : g(x) > 0\}$

3. (α, β)-Fuzzy Subquantales (Ideals) of Quantale

In this section, we present some new connections between fuzzy points and f-subsets, and investigate (α, β)-FS and (α, β)-FI of quantales.

Throughout the remaining paper $\gamma, \delta \in [0,1]$, where $\gamma < \delta$ and $\alpha, \beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$. For a fuzzy point z_p and an f-subset g of Q_t, we say that

(1) $z_p \in_\gamma g$ if $g(z) \geq p > \gamma$.
(2) $z_p q_\delta g$ if $g(z) + p > 2\delta$.
(3) $z_p(\in_\gamma \vee q_\delta)g$ if $z_p \in_\gamma g$ or $z_p q_\delta g$.
(4) $z_p(\in_\gamma \wedge q_\delta)g$ if $z_p \in_\gamma g$ and $z_p q_\delta g$.
(5) $z_p \overline{\alpha} g$ if $z_p \alpha g$ does not hold for α where α is one of $\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta$.

Note that the case when $\alpha = \in_\gamma \wedge q_\delta$ is left out. Suppose that g is an f-subset of a quantale Q_t such that $g(z) \leq \delta \ \forall \ z \in Q_t$. Suppose $z \in Q_t$ and $p \in [0,1]$ be such that $z_p(\in_\gamma \wedge q_\delta)g$. Then it follows that $g(z) \geq p > \gamma$ and $g(z) + p > 2\delta$. Hence, $2\delta < g(z) + p \leq g(z) + g(z) = 2g(z)$, that is $g(z) > \delta$. This means that $\{z_p : z_p(\in_\gamma \wedge q_\delta)g\} = \emptyset$. Therefore, we are not taking the case when $\alpha = \in_\gamma \wedge q_\delta$.

Table 2. Binary operation $\otimes'$ subject to Q'_t.

$\otimes'$	$\bot$	i	j	k	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
i	$\bot$	i	$\bot$	i	i
j	$\bot$	$\bot$	j	j	j
k	$\bot$	i	j	k	k
$\top$	$\top$	i	j	k	$\top$

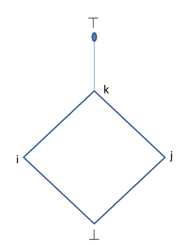

Figure 2. Illustration of Q'_t.

From here onward, we will utilize abbreviated forms like (α,β)-FS, (α,β)-FI, $(\in_\gamma,\in_\gamma\vee q_\delta)$-$FI$, $(\in_\gamma,\in_\gamma\vee q_\delta)$-$FLI$, $(\in_\gamma,\in_\gamma\vee q_\delta)$-$FRI$ and $(\in_\gamma,\in_\gamma\vee q_\delta)$-$FS$ instead of (α,β)-fuzzy subquantale, (α,β)-fuzzy ideal, $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy ideal, $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left ideal, $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy right ideal and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy subquantale.

Definition 3.1. *An f-subset g of a quantale Q_t is called an (α,β)-FS of Q_t, if*

(F_1) $(z_i)_{p_i}\alpha g\longrightarrow(\vee_{i\in I}z_i)_{\underset{i\in I}{inf}p_i}\beta g$;
(F_2) $z_p\alpha g,\, w_v\alpha g\longrightarrow(z\otimes w)_{inf(p,v)}\beta g\;\forall\, z,w,z_i\in Q_t,(i\in I),\forall\, p_i\in(0,1]$.

Theorem 3.2. *Let g be a non-zero (α,β)-FS of Q_t and $2\delta=1+\gamma$. Then $g_\gamma=\{y\in Q_t\mid g(y)>\gamma\}$ is a subquantale of Q_t.*

Proof. Let $y_i\in g_\gamma$ for $i\in I$. Then $g(y_i)>\gamma\;\forall\, i\in I$. Let $g(\vee_{i\in I}y_i)\le\gamma$. If $\alpha\in\{\in_\gamma,\in_\gamma\vee q_\delta\}$, then $(y_i)_{g(y_i)}\alpha g\;\forall\, i\in I$ but $g(\vee_{i\in I}y_i)\le\gamma<\underset{i\in I}{inf}\;g(y_i)$ and $g(\vee_{i\in I}y_i)+\underset{i\in I}{inf}\;g(y_i)\le\gamma+\underset{i\in I}{inf}g(y_i)\le\gamma+1=2\delta$. So $(\vee_{i\in I}y_i)_{\underset{i\in I}{inf}g(y_i)}\overline{\beta}g$ for every β where β is one of $\in_\gamma,q_\delta,\in_\gamma\wedge q_\delta$ and $\in_\gamma\vee q_\delta$. Thus we obtain a contradiction. Hence $g(\vee_{i\in I}y_i)>\gamma$, i.e., $\vee_{i\in I}y_i\in g_\gamma$. If $\alpha=q_s$ then $(y_i)_1q_\delta g\;\forall\, i\in I$ because $g(y_i)+1>1+\gamma=2\delta$, but $(\vee_{i\in I}y_i)_1\overline{\beta}g$ for every β where β is one of $\in_\gamma,q_\delta,\in_\gamma\wedge q_\delta$ and $\in_\gamma\vee q_\delta$ because $g(\vee_{i\in I}y_i)\le\gamma$, so $(\vee_{i\in I}y_i)_1\overline{\in}_\gamma g$ and $g(\vee_{i\in I}y_i)+1\le\gamma+1=2\delta$, so $(\vee_{i\in I}y_i)_1\overline{q}_\delta g$. Hence $g(\vee_{i\in I}y_i)>\gamma$, that is $\vee_{i\in I}y_i\in g_\gamma$. Thus g_γ is closed under arbitrary join. The proof is similar for g_γ to be closed under $\otimes$. This shows that g_γ is a subquantale of Q_t. □

Definition 3.3. *An f-subset g of a quantale Q_t is said to be an (α,β)-FLI (FRI) of Q_t, if*

(1) $z_p\alpha g,w_v\alpha g\longrightarrow(z\vee w)_{inf(p,v)}\beta g$;
(2) $z_v\alpha g$ and $w\le z\longrightarrow w_v\beta g$;
(3) $z_v\alpha g,w\in Q_t\longrightarrow(w\otimes z)_v\beta g,((z\otimes w)_v\beta g)\;\forall\, z,w\in Q_t$ and $p,v\in(0,1]$.

A f-subset g of a quantale Q_t is called an (α,β)-FI of Q_t if it is both an (α,β)-FRI and (α,β)-FLI of Q_t.

Theorem 3.4. *Let $2\delta=1+\gamma$ and g be a non-zero (α,β)-FLI (FRI) of Q_t. Then $g_\gamma=\{y\in Q_t\mid g(y)>\gamma\}$ is a LI (RI) of Q_t.*

Proof. Let g be a nonzero (α,β)-FLI of Q_t. Let $y,z\in g_\gamma$. Then $g(y)>\gamma$ and $g(z)>\gamma$. Let $\gamma\ge g(y\vee z)$. If $\alpha\in\{\in_\gamma,\in_\gamma\vee q_\delta\}$, then $(y)_{g(y)}\alpha g$ and $(z)_{g(z)}\alpha g$ but $(y\vee z)_{inf(g(y),g(z))}\overline{\beta}g$ for every β where β is one of $\in_\gamma,q_\delta,\in_\gamma\wedge q_\delta$ and $\in_\gamma\vee q_\delta$ (because $g(y\vee z)\le\gamma<inf(g(y),g(z))$ so $(y\vee z)_{inf(g(y),g(z))}\overline{\in}_\gamma g$ and $g(y\vee z)+inf(g(y),g(z))\le\gamma+inf(g(y),g(z))\le\gamma+1=2\delta$, so $(y\vee z)_{inf(g(y),g(z))}\overline{q}_\delta g$, a contradiction. Hence $g(y\vee z)>\gamma$, that is $y\vee z\in g_\gamma$. If $\alpha=q_\delta$ then $y_1q_\delta g$ and $z_1q_\delta g$ (because $g(y)+1>1+\gamma=2\delta$ and $g(z)+1>1+\gamma=2\delta$) but $(y\vee z)_1\overline{\beta}g$ for every β but β is one of $\in_\gamma,q_\delta$, $\in_\gamma\wedge q_\delta$ and $\in_\gamma\vee q_\delta$, (because $g(y\vee z)\le\gamma$, so $(y\vee z)_1\overline{\in}_\gamma g$ and $g(y\vee z)+1\le1+\gamma=2\delta$), a contradiction. Hence $g(y\vee z)>\gamma$, that is $y\vee z\in g_\gamma$. Thus g_γ is closed under join.

Let $y,z\in Q_t$ and $y\le z$. If $z\in g_\gamma$, then $g(z)>\gamma$. Assume that $g(y)\le\gamma$. If $\alpha\in\{\in_\gamma,\in_\gamma\vee q_\delta\}$, then $(z)_{g(z)}\alpha g$ but $(y)_{g(y)}\overline{\beta}g$ for every β where β is one of $\in_\gamma,q_\delta$,

$\in_\gamma \wedge q_\delta$ and $\in_\gamma \vee q_\delta$, a contradiction. Also $z_1 q g$ but $y_1 \overline{\beta} g$ for every β where β is one of the following $\in_\gamma, q_\delta, \in_\gamma \wedge q_\delta, \in_\gamma \vee q_\delta$ (because $g(y) \leq \gamma$ so $y_1 \overline{\in}_\gamma g$ and $g(y)+1 \leq \gamma+1 = 2\delta$, so $y_1 \overline{q}_\delta g$). Hence $g(y) > \gamma$, i.e., $y \in g_\gamma$.

Let $y \in g_\gamma$ and $z \in Q_t$. Then $g(y) > \gamma$. We want to show that $g(z \otimes y) > \gamma$. Suppose that $g(z \otimes y) \leq \gamma$ and let $\alpha \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$. Then $(y)_{g(y)} \alpha g$ but $(z \otimes y)_{g(y)} \overline{\beta} g$ for every β where β will be one of $\in_\gamma, q_\delta$, $\in_\gamma \wedge q_\delta$ and $\in_\gamma \vee q_\delta$, this is a contradiction again. Also $y_1 q_\delta g$ but $(z \otimes y)_1 \overline{\beta} g$ where β is one of $\in_\gamma, q_\delta$, $\in_\gamma \wedge q_\delta$ and $\in_\gamma \vee q_\delta$, a contradiction. Therefore $g(z \otimes y) > \gamma$ and so $z \otimes y \in g_\gamma$. Thus, g_γ is a LI of Q_t. □

Theorem 3.5. *Let $2\delta = 1 + \gamma$ and $\emptyset \neq C \subseteq Q_t$. Then C is an LI (RI) of Q_t if and only if the f-subset g of Q_t defined by*

$$g(w) = \begin{cases} \geq \delta \; if \; w \in C \\ \gamma \; otherwise \end{cases} .$$

is an $(\alpha, \in_\gamma \vee q_\delta)$-$FLI$ (FRI) of Q_t.

Proof. Let C be a LI of Q_t.

(**a**) Let $w, z \in Q_t$ and $p, v \in (\gamma, 1]$ be such that $w_p \in_\gamma g$ and $z_v \in_\gamma g$. Then $g(w) \geq p > \gamma$ and $g(z) \geq v > \gamma$. Hence $g(w) \geq \delta$ and $g(z) \geq \delta$. Thus $w, z \in C$ and so $w \vee z \in C$, that is $g(w \vee z) \geq \delta$. If $inf\{p, v\} \leq \delta$, then $g(w \vee z) \geq \delta \geq inf\{p, v\} > \gamma$. Hence $(w \vee z)_{inf(p,v)} \in_\gamma g$. If $inf\{p, v\} > \delta$, then $g(w \vee z) + inf\{p, v\} > \delta + \delta = 2\delta$ and so $(w \vee z)_{inf(p,v)} q_\delta g$. Therefore $(w \vee z)_{inf(p,v)} (\in_\gamma \vee q_\delta) g$.

Let $w, z \in Q_t$, $w \leq z$ and $v \in (\gamma, 1]$ be such that $z_v \in_\gamma g$. Then $g(z) \geq v > \gamma$. Thus $z \in C$ and since C is a LI so $w \in C$, that is $g(w) \geq \delta$. If $v \leq \delta$, then $g(w) \geq \delta \geq v > \gamma$. Hence $w_v \in_\gamma g$. If $v > \delta$, then $g(w) + v > \delta + \delta = 2\delta$ and so $w_v q_\delta g$. It follows that $w_v (\in_\gamma \vee q_\delta) g$.

Now let $w, z \in Q_t$ and $p \in (\gamma, 1]$ be such that $w_p \in_\gamma g$. Then $g(w) \geq p > \gamma$, which implies $w \in C$, and so $z \otimes w \in C$, $\forall\, z \in Q_t$. Consequently $g(z \otimes w) \geq \delta$. If $p \leq \delta$, then $g(z \otimes w) \geq \delta \geq p > \gamma$. Hence $(z \otimes w)_p \in_\gamma g$. If $p > \delta$, then $g(z \otimes w) + p > \delta + \delta = 2\delta$ and so $(z \otimes w)_p q_\delta g$. Thus $(z \otimes w)_p (\in_\gamma \vee q_\delta) g$. Hence g is an $(\in_\gamma \vee q_\delta)$-$FLI$ of Q_t.

(**b**) Let $w, z \in Q_t$ and $p, v \in (\gamma, 1]$ be such that $w_p q_\delta g$ and $z_v q_\delta g$. Then, $g(w) + p > 2\delta$ and $g(z) + v > 2\delta$, and hence $g(w) > 2\delta - p \geq 2\delta - 1 = \gamma$ and $g(z) > 2\delta - v \geq 2\delta - 1 = \gamma$, it follows that $g(w) \geq \gamma$ and $g(z) \geq \gamma$, i.e., $w, z \in C$. Since C is a LI so $w \vee z \in C$, hence we have $g(w \vee z) \geq \delta$. If $inf\{p, v\} \leq \delta$, then $g(w \vee z) \geq \delta \geq inf\{p, v\} > \gamma$. Hence $(w \vee z)_{inf(p,v)} \in_\gamma g$. If $inf\{p, v\} > \delta$, then $g(w \vee z) + inf\{p, v\} > \delta + \delta = 2\delta$ and so $(w \vee z)_{inf(p,v)} q_\delta g$. Therefore $(w \vee z)_{inf(p,v)} (\in_\gamma \vee q_\delta) g$.

Let $w, z \in Q_t$, $w \leq z$ and $v \in (\gamma, 1]$ be such that $z_v q_\delta g$. Then $g(z) + v > 2\delta$ so $g(z) > 2\delta - v \geq 2\delta - 1 = \gamma$. Thus $z \in C$ and since C is a LI so $w \in C$, that is $g(w) \geq \delta$. If $v \leq \delta$, then $g(w) \geq \delta \geq v > \gamma$. Hence $w_v \in_\gamma g$. If $v > \delta$, then $g(w) + v > \delta + \delta = 2\delta$ and so $w_v q_\delta g$. It follows that $w_v (\in_\gamma \vee q_\delta) g$.

Now, let $w, z \in Q_t$ and $p \in (\gamma, 1]$ be such that $w_p q_\delta g$, which implies that $g(w) + p > 2\delta$. Thus $w \in C$ and so $z \otimes w$ is in C. This means that $g(z \otimes w) \geq \delta$. If $p \leq \delta$, then $g(z \otimes w) \geq \delta \geq p > \gamma$. Hence $(z \otimes w)_p \in_\gamma g$. If $p > \delta$, then $g(z \otimes w) + p > \delta + \delta = 2\delta$ and so $(z \otimes w)_p q_\delta g$. Thus $(z \otimes w)_p (\in_\gamma \vee q_\delta) g$. Hence g is an $(q_\delta, \in_\gamma \vee q_\delta)$-$FLI$ of Q_t.

(**c**) Let $w, z \in Q_t$ and $p, v \in (\gamma, 1]$ be such that $w_p \in_\gamma g$ and $z_v q_\delta g$. Then $g(w) \geq p > \gamma$ and $g(z) + v > 2\delta$. Since $w, z \in C$ implies that $w \vee z \in C$. Hence $g(w \vee z) \geq \delta$. In a similar way we obtain $(w \vee z)_{inf(p,v)} \in_\gamma g$ for $inf\{p, v\} \leq \delta$ and $(w \vee z)_{inf(p,v)} q_\delta g$ for $inf\{p, v\} > \delta$. Thus $(w \vee z)_{inf(p,v)}(\in_\gamma \vee q_\delta)g$. The rest follows from parts (a) and (b).

Conversely, suppose that g is an $(\alpha, \in_\gamma \vee q_\delta)$-$FLI$ of Q_t. It is simple to verify that $C = g_\gamma$. Hence, from Theorem 3.4, C is an LI of Q_t. □

The next Theorem can be obtained in a similar way.

Theorem 3.6. *Let $2\delta = 1 + \gamma$ and $\emptyset \neq C \subseteq Q_t$. Then C is a subquantale of Q_t if and only if the f-subset g of Q_t defined by*

$$g(w) = \begin{cases} \geq \delta \; if \; w \in C \\ \gamma \; otherwise \end{cases} .$$

is an $(\alpha, \in_\gamma \vee q_\delta)$-$FS$ of Q_t.

4. $(\in_\gamma, \in_\gamma \vee \mathbf{q}_\delta)$- Fuzzy Suquantales (Ideals) of Quantale

Here, we start to establish a new sort of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t and research some of their properties.

Definition 4.1. *An f-subset g of Q_t is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t, if*

(F_1) $(z_i)_{p_i} \in_\gamma g \longrightarrow (\vee_{i \in I} z_i)_{\underset{i \in I}{inf} p_i}(\in_\gamma \vee q_\delta)g$;

(F_2) $z_p \in_\gamma g$ and $w_v \in_\gamma g \longrightarrow (z \otimes w)_{inf(p,v)}(\in_\gamma \vee q_\delta)g$, $\forall \{z_i\} \subseteq Q_t, z, w \in Q_t$, $p_i, p, v \in (\gamma, 1], (i \in I)$.

Theorem 4.2. *Let g be an f-subset of Q_t. If g is a $(q_\delta, \in_\gamma \vee q_\delta)$-$FS$ of Q_t, then conditions below hold:*

(1) $sup \, \{g(\vee_{i \in I} z_i), \gamma\} \geq inf \, \{\underset{i \in I}{inf} g(z_i), \delta\})$

(2) $sup \, \{g(z \otimes y), \gamma\} \geq inf \, \{g(z), g(y), \delta\} \; \forall \, \{z_i\} \subseteq Q_t, (i \in I), z, y \in Q_t$.

Proof. Let g be a $(q_\delta, \in_\gamma \vee q_\delta)$-$FS$ of Q_t. Assume that there exist $z_i \in Q_t$ such that $sup\{g(\vee_{i \in I} z_i), \gamma\} < inf \, \{\underset{i \in I}{inf} g(z_i), \delta\}$. Then $\forall \, \gamma < v \leq 1$ such that

$$2\delta - sup\{g(\vee_{i \in I} z_i), \gamma\} > v \geq 2\delta - inf\{\underset{i \in I}{inf} g(z_i), \delta\}$$

and so

$$2\delta - g(\vee_{i \in I} z_i) \geq 2\delta - sup\{g(\vee_{i \in I} z_i), \gamma\} > v \geq sup\{2\delta - \underset{i \in I}{inf} g(z_i), \delta\}$$

Thus,

$$\underset{i \in I}{inf} g(z_i) + v > 2\delta, \; g(\vee_{i \in I} z_i) + v < 2\delta$$

and $g(\vee_{i \in I} z_i) < \delta < v$. Hence $(z_i)_v q_\delta g \, \forall \, i \in I$. But $(\vee_{i \in I} z_i)_v \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Therefore $sup\{g(\vee_{i \in I} z_i), \gamma\} \geq inf \, \{\underset{i \in I}{inf} g(z_i), \delta\}$.

Let $z, y \in Q_t$ be such that $sup\{g(z\otimes y),\gamma\} < inf\ \{g(z), g(y), \delta\}$. Then $\forall\ \gamma < t \leq 1$ such that

$$2\delta - sup\{g(z\otimes y),\gamma\} > t \geq 2\delta - inf\{g(z), g(y), \delta\}$$

we have

$$2\delta - g(z\otimes y) \geq 2\delta - sup\{g(z\otimes y),\gamma\} > t \geq sup\{2\delta - g(z), 2\delta - g(y), \delta\}$$

and so

$$g(z) + t > 2\delta,\ g(y) + t > 2\delta,\ g(z\otimes y) + t < 2\delta$$

and $g(z\otimes y) < \delta < t$. Hence $z_t q_\delta g,\ y_t q_\delta g$ but $(z\otimes y)_t \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Therefore, $sup\{g(z\otimes y),\gamma\} \geq inf\{g(z), g(y), \delta\}\ \forall\ z, y \in Q_t$. □

Theorem 4.3. *An f-subset g of Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t if and only if the conditions below hold:*

(1) $sup\ \{g(\vee_{i\in I} z_i), \gamma\} \geq inf\{\underset{i\in I}{inf} g(z_i), \delta\}$;

(2) $sup\ \{g(z\otimes y), \gamma\} \geq inf\{g(z), g(y), \delta\}, \forall\ \{z_i\} \subseteq Q_t, (i \in I), z, y \in Q_t$.

Proof. Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t. Let there exist some $z_i \in Q_t$ and $v \in (\gamma, 1]$ such that $sup\{g(\vee_{i\in I} z_i), \gamma\} < v \leq inf\{\underset{i\in I}{inf} g(z_i), \delta\}$. Then $g(z_i) \geq v > \gamma\ \forall\ i \in I$, $g(\vee_{i\in I} z_i) < v$ and $g(\vee_{i\in I} z_i) + v < 2v \leq 2\delta$, *i.e.*, $(z_i)_v \in_\gamma g\ \forall\ i \in I$ but $(\vee_{i\in I} z_i)_v \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Thus, $sup\{g(\vee_{i\in I} z_i), \gamma\} \geq inf\ \{\underset{i\in I}{inf} g(z_i), \delta\}\ \forall$ $z_i \in Q_t$.

Let there exist $z, y \in Q_t$ and $v \in (\gamma, 1]$ such that $sup\{g(z\otimes y), \gamma\} < v \leq inf$ $\{g(z), g(y), \delta\}$. Then $g(z) \geq v > \gamma$, $g(y) \geq v > \gamma$, $g(z\otimes y) < v$ and $g(z\otimes y) + v < 2v \leq 2\delta$, *i.e.*, $z_v \in_\gamma g$, $y_v \in_\gamma g$ but $(z\otimes y)_v \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Thus, $sup\{g(z\otimes y), \gamma\} \geq inf\ \{g(z), g(y), \delta\}\ \forall\ z, y \in Q_t$.

Conversely, suppose that the above two conditions are true. We show that g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t. Let $z_i \in Q_t$ and $v_i \in (\gamma, 1]$ such that $(z_i)_{v_i} \in_\gamma g$ but $(\vee_{i\in I} z_i)_{\underset{i\in I}{inf} v_i} \overline{(\in_\gamma \vee q_\delta)} g$. Then $g(z_i) \geq v_i\ \forall\ i \in I$, $g(\vee_{i\in I} z_i) < \underset{i\in I}{inf} v_i$ and $g(\vee_{i\in I} z_i) + \underset{i\in I}{inf} v_i \leq 2\delta$. It follows that $g(\vee_{i\in I} z_i) < \delta$ and so $sup\ \{g(\vee_{i\in I} z_i), \gamma\} < inf\ \{\underset{i\in I}{inf} g(z_i), \delta\}$, a contradiction. Hence $(\vee_{i\in I} z_i)_{\underset{i\in I}{inf} p_i} (\in_\gamma \vee q_\delta) g$. Similarly, it can be shown that if $z_p \in_\gamma g$, and $w_v \in_\gamma g$ then $g(z\otimes w)_{inf(p,v)} (\in_\gamma \vee q_\delta) g$. □

Example 4.4. *Let $(Q'_t, \otimes')$ be a quantale, where Q'_t is delineated in Fig.2 and the binary operation $\otimes'$ on Q'_t is shown in the Table 2. Taking $g = \frac{0.9}{\perp} + \frac{0.5}{i} + \frac{0.5}{j} + \frac{0.5}{k} + \frac{0.6}{\top}$. Then by routine calculations g is an $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FS$ of Q'_t.*

The following Propositions are obvious.

Proposition 4.5. *Every $(\in_\gamma \vee q_\delta), \in_\gamma \vee q_\delta)$-$FS$ of Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t.*

Proposition 4.6. *Every $(\in_\gamma, \in_\gamma)$-FS of Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t.*

The example below demonstrates that the converses of Propositions 4.5 and 4.6 are not valid.

Example 4.7. *Consider the quantale Q'_t as defined in Example 4.4 and taking $g = \frac{0.9}{\perp} + \frac{0.7}{i} + \frac{0.65}{j} + \frac{0.54}{k} + \frac{0.31}{\top}$.*

Then

(1) It is simple to confirm that g is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$-$FS$ of Q'_t.
(2) g is not an $(\in_{0.3}, \in_{0.3})$-FS of Q'_t, since $i_{0.68} \in_{0.3} g$ and $j_{0.61} \in_{0.3} g$ but $(i \vee j)_{inf(0.68,0.61)} = k_{0.61}\overline{\in}_{0.3}g$.
(3) g is not an $(\in_{0.3} \vee q_{0.6}, \in_{0.3} \vee q_{0.6})$-$FS$ of Q'_t, since $i_{0.68}(\in_{0.3} \vee q_{0.6})g$ and $j_{0.59}(\in_{0.3} \vee q_{0.6})g$ but $(i \vee j)_{inf(0.6,0.59)} = k_{0.59}\overline{(\in_{0.3} \vee q_{0.6})}g$.

Definition 4.8. *An f-subset g of Q_t is said to be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FLI$ (FRI) of Q_t, if*

(F_3) $z_p \in_\gamma g, w_v \in_\gamma g \longrightarrow (z \vee w)_{inf(p,v)}(\in_\gamma \vee q_\delta)g$;
(F_4) $z_v \in_\gamma g$ and $w \leq z \longrightarrow w_v(\in_\gamma \vee q_\delta)g$;
(F_5) $z_v \in_\gamma g, w \in Q_t \longrightarrow (w \otimes z)_v(\in_\gamma \vee q_\delta)g$, $((z \otimes w)_p(\in_\gamma \vee q_\delta)g)$, $\forall\, z, w \in Q_t$ and $p, v \in (\gamma, 1]$.

If an f-subset g of Q_t is both an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FRI$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FLI$ of Q_t, then it is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t.

Theorem 4.9. *Let g be an f-subset of Q_t such that g be a $(q_\delta, \in_\gamma \vee q_\delta)$-$FLI$ (FRI) of Q_t. Then the conditions below are satisfied:*

(1) $sup\{g(z \vee w), \gamma\} \geq inf\ \{g(z), g(w), \delta\}$);
(2) $sup\{g(w), \gamma\} \geq inf\ \{g(z), \delta\}$ with $w \leq z$);
(3) $sup\{g(w \otimes z), \gamma\} \geq inf\ \{g(z), \delta\})$, $(sup\ \{g(z \otimes w), \gamma\} \geq inf\ \{g(z), \delta\})$, $\forall\, z, w \in Q_t$.

Proof. If there exist $z, w \in Q_t$ such that $sup\{g(z \vee w), \gamma\} < inf\ \{g(z), g(w), \delta\}$. Then $\forall\, \gamma < v \leq 1$ such that

$$2\delta - sup\{g(z \vee w), \gamma\} > v \geq 2\delta - inf\{g(z), g(w), \delta\}$$

Thus, we have

$$2\delta - g(z \vee w) \geq 2\delta - sup\{g(z \vee w), \gamma\} > v \geq sup\{2\delta - g(z), 2\delta - g(w), \delta\}$$

and so,

$$g(z) + v > 2\delta,\ g(w) + v > 2\delta,\ g(z \vee w) + v < 2\delta$$

and $g(z \vee w) < \delta < v$. Hence $w_v q_\delta g$, $z_v q_\delta g$ but $(z \vee w)_v\overline{(\in_\gamma \vee q_\delta)}g$, a contradiction. Therefore

$$sup\{g(z \vee w), \gamma\} \geq inf\{g(z), g(w), \delta\}\ \forall z, y \in Q_t.$$

Let $z, y \in Q_t$ be such that $sup\{g(w \otimes z), \gamma\} < inf\{g(z), \delta\}$. Then $\forall\, \gamma < p \leq 1$ such that

$$2\delta - sup\{g(w \otimes z), \gamma\} > p \geq 2\delta - inf\{g(z), \delta\}$$

we have

$$2\delta - g(w \otimes z) \geq 2\delta - sup\{g(w \otimes z), \gamma\} > p \geq sup\{2\delta - g(z), \delta\}$$

and so

$$g(z) + p > 2\delta,\ g(w \otimes z) + p < 2\delta$$

and $g(w \otimes z) < \delta < p$. Hence $z_p q_\delta g$ but $(w \otimes z)_p \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Therefore $sup\{g(w \otimes z), \gamma\} \geq inf\{g(z), \delta\}$ $\forall$ $z, y \in Q_t$. Similarly, we can prove that sup $\{g(w), \gamma\} \geq inf$ $\{g(z), \delta\}$ with $w \leq z$ $\forall$ $z, y \in Q_t$. □

Theorem 4.10. *An f-subset g of Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FRI$ (FLI) of Q_t if and only if the following conditions are satisfied:*

(1) sup $\{g(z \vee w), \gamma\} \geq inf$ $\{g(z), g(w), \delta\}$);
(2) sup $\{g(w), \gamma\} \geq inf$ $\{g(z), \delta\}$ with $w \leq z$);
(3) sup $\{g(w \otimes z), \gamma\} \geq inf$ $\{g(z), \delta\}$), (sup $\{g(z \otimes w), \gamma\} \geq inf$ $\{g(z), \delta\}$), $\forall$ $z, w \in Q_t$.

Proof. The proof is a routine verification and hence can be omitted. □

Proposition 4.11. *Every $(q_\delta, \in_\gamma \vee q_\delta)$-$FI$ of Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t.*

The converse of above Proposition is not valid in general, as explained in example below.

Example 4.12. *Consider the quantale as given in Example 4.4 and define an f-subset g of Q'_t as follows:*

$$g = \frac{1}{\perp} + \frac{0.75}{i} + \frac{0.67}{j} + \frac{0.54}{k} + \frac{0.32}{\top}.$$

Then g is an $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FI$ of Q'_t, but it is not a $(q_{0.6}, \in_{0.3} \vee q_{0.6})$-$FI$, since $i_{0.68} q_{0.6} g$ and $j_{0.61} q_{0.6} g$ but $(i \vee j)_{inf(0.68,0.61)} = k_{0.61} \overline{(\in_{0.3} \vee q_{0.6})} g$.

For any $g \in \mathcal{F}(Q_t)$, where $\mathcal{F}(Q_t)$ denotes the set of all f-subsets of Q_t, we define

$$g_v = \{y \in Q_t \mid y_v \in_\gamma g\} \text{ for all } v \in (\gamma, 1];$$

$$g_v^\delta = \{y \in Q_t \mid y_v q_\delta g\} \text{ for all } v \in (\gamma, 1];$$

and

$$[g]_v^\delta = \{y \in Q_t \mid y_v (\in_\gamma \vee q_\delta) g\} \text{ for all } v \in (\gamma, 1].$$

It follows that $[g]_v^\delta = g_v \Cup g_v^\delta$.

The following theorem gives the connection between $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ and crisp subquantale of Q_t.

Theorem 4.13. *For any an f-subset g of quantale Q_t, the following are equivalent:*

(F_6) g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t;
(F_7) $g_v (\neq \emptyset)$ is a subquantale of Q_t $\forall$ $v \in (\gamma, \delta]$.

Proof. $(\mathbf{F}_6) \Longrightarrow (\mathbf{F}_7)$. Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t. Let $z_i \in Q_t$ for some $i \in I$ and $v \in (\gamma, \delta]$ be such that $z_i \in g_v$ $\forall$ $i \in I$. Then $(z_i)_v \in_\gamma g$ $\forall$ $i \in I$ and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t, therefore $(\vee_{i \in I} z_i)_v (\in_\gamma \vee q_\delta) g$. If $(\vee_{i \in I} z_i)_v \in_\gamma g$, then $\vee_{i \in I} z_i \in g_v$ and if $(\vee_{i \in I} z_i)_v q_\delta g$, then $g(\vee_{i \in I} z_i) > 2\delta - v \geq v > \gamma$; that is, $\vee_{i \in I} z_i \in g_v$. Let $w, z \in Q_t$ be such that $w, z \in g_v$ for some $v \in (\gamma, \delta]$. Then $z_v \in_\gamma g$ and $w_v \in_\gamma g$, and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t, therefore $(z \otimes w)_v (\in_\gamma \vee q_\delta) g$. If

$(z \otimes w)_v \in_\gamma g$, then $z \otimes w \in g_v$ and if $(z \otimes w)_v q_\delta g$, then $g(z \otimes w) > 2\delta - v \geq v > \gamma$; that is, $z \otimes w \in g_v$. Therefore g_v is a subquantale of Q_t.

$(\mathbf{F}_7) \Longrightarrow (\mathbf{F}_6)$. Assume that $g_v(\neq \emptyset)$ is a subquantale of $Q_t \; \forall \; v \in (\gamma, \delta]$. Suppose that there exist $z_i \in Q_t$ for some $i \in I$ such that $sup\{g(\vee_{i \in I} z_i), \gamma\} < inf\{\underset{i \in I}{inf} g(z_i), \delta\}$; then there exist $v \in (\gamma, \delta]$ such that $sup\ \{g(\vee_{i \in I} z_i), \gamma\} < v \leq inf\ \{\underset{i \in I}{inf} g(z_i), \delta\}$; this shows that $(z_i)_v \in_\gamma g \; \forall \; i \in I$; that is, $z_i \in g_v \; \forall \; i \in I$ but $(\vee_{i \in I} z_i) \overline{\in} g_v$, a contradiction. Therefore, $sup\{g(\vee_{i \in I} z_i), \gamma\} \geq inf\ \{\underset{i \in I}{inf} g(z_i), \delta\} \; \forall \; z_i \in Q_t, (i \in I)$. Let $z, w \in Q_t$ and $sup\{g(z \otimes w), \gamma\} < inf\{g(z), g(w), \delta\}$; then $sup\{g(z \otimes w), \gamma\} < v \leq inf$ $\{g(z), g(w), \delta\}$ for some $v \in (\gamma, \delta]$. This implies that $z \in g_v$ and $w \in g_v$ but $(z \otimes w) \overline{\in} g_v$, a contradiction. Therefore, $\{g(z \otimes w), \gamma\} \geq inf\ \{g(z), g(w), \delta\}$. By Theorem 4.3, g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t. □

Theorem 4.14. *Let $2\delta = 1 + \gamma$. Then an f-subset g of Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-FS if and only if $g_v^\delta(\neq \emptyset)$ is a subquantale of $Q_t \; \forall \; v \in (\delta, 1]$.*

Proof. Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-FS of Q_t. Let $z_i \in Q_t$ for some $i \in I$ and $v \in (\delta, 1]$ be such that $z_i \in g_v^\delta \; \forall \; i \in I$. Then $(z_i)_v q_\delta g \; \forall \; i \in I$; that is $g(z_i) > 2\delta - v \geq 2\delta - 1 = \gamma$. Thus, $g(z_i) > \gamma$. Since $v \in (\delta, 1]$,we have $2\delta - v < \delta < v$. By hypothesis, we have,

$$\begin{aligned} sup\{g(\vee_{i \in I} z_i), \gamma\} &\geq inf\{\underset{i \in I}{inf} g(z_i),\ \delta\}; \\ g(\vee_{i \in I} z_i) &\geq inf\{2\delta - v,\ \delta\}; \\ &= 2\delta - v. \end{aligned}$$

that is, $g(\vee_{i \in I} z_i) \geq 2\delta - v$. Hence $\vee_{i \in I} z_i \in g_v^\delta$.

Let $w, z \in Q_t$ be such that $w, z \in g_v^\delta$ for some $v \in (\delta, 1]$. Then $z_v q_\delta g$ and $w_v q_\delta g$, that is $g(z) > 2\delta - v \geq 2\delta - 1 = \gamma$, $g(w) > 2\delta - v \geq 2\delta - 1 = \gamma$ and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-FS of Q_t, therefore,

$$\begin{aligned} sup\{g(z \otimes w), \gamma\} &\geq inf\{g(z), g(w), \delta\} \\ &\geq inf\{2\delta - v, 2\delta - v, \delta\} \\ &= 2\delta - v; \end{aligned}$$

that is, $g(z \otimes w) \geq 2\delta - v$. Hence $z \otimes w \in g_v^\delta$. So, g_v^δ is a subquantale of Q_t.

Conversely, assume that $g_v(\neq \emptyset)$ is a subquantale of $Q_t \; \forall \; v \in (\delta, 1]$. Suppose that there exist $z_i \in Q_t$ for some $i \in I$ such that $sup\ \{g(\vee_{i \in I} z_i), \gamma\} < v = inf\{\underset{i \in I}{inf} g(z_i), \delta\}$. This shows that $z_i \in g_v^\delta \; \forall \; i \in I$ but $\vee_{i \in I} z_i \overline{\in} g_v^\delta$, a contradiction. Therefore, $sup\{g(\vee_{i \in I} z_i), \gamma\}$ $\geq inf\ \{\underset{i \in I}{inf} g(z_i), \delta\} \; \forall \; z_i \in Q_t, (i \in I)$. Let $z, y \in Q_t$ and $sup\{g(z \otimes y), \gamma\} < v = inf$ $\{g(z), g(y), \delta\}$; this implies that $z \in g_v^\delta$ and $y \in g_v^\delta$ but $(z \otimes y) \overline{\in} g_v^\delta$, a contradiction. Therefore, $\{g(z \otimes y), \gamma\} \geq inf\ \{g(z), g(y), \delta\}$. Hence g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t by Theorem 4.3. □

The following Theorem is similarly obtained from Theorem 4.13 and 4.14.

Theorem 4.15. *An f-subset g of a quantale Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ if and only if $[g]_v^\delta$ $(\neq \emptyset)$ is a subquantale of $Q_t \; \forall \; v \in (\gamma, 1]$.*

Corollary 4.16. *Let $\gamma, \gamma', \delta, \delta' \in [0,1]$ be such that $\gamma < \delta, \gamma' < \delta'$, and $\delta' < \delta$. Then every $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t is an $(\in_{\gamma'}, \in_{\gamma'} \vee q_{\delta'})$-$FS$ of Q_t.*

The example below demonstrates that the converse of Corollary 4.16 is not valid in general.

Example 4.17. *Let Q'_t be a quantale and g be an f-subset as discussed in Example 4.7. Then g is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$-$FS$ of Q'_t but not an $(\in_{0.3}, \in_{0.3} \vee q_{0.9})$-$FS$ of Q'_t.*

We have the following Theorem, if we take $\gamma = 0$ and $\delta = 0.5$ in Theorem 4.13.

Theorem 4.18. [35] *An f-subset, g of Q_t is an $(\in, \in \vee q)$-FS of Q_t if and only if each $\emptyset \neq U(g; p)$ is a subquantale of Q_t $\forall\, p \in (0, 0.5]$.*

Theorem 4.19. *Let $g \in \mathcal{F}(Q_t)$. Then*

(1) g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t if and only if $\emptyset \neq g_v$ is an ideal of Q_t $\forall\, v \in (\gamma, \delta]$.
(2) If $2\delta = 1 + \gamma$, then g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ if and only if $\emptyset \neq g_v^\delta$ is an ideal of Q_t $\forall\, v \in (\delta, 1]$.
(3) If $2\delta = 1 + \gamma$, then g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ if and only if $\emptyset \neq [g]_v^\delta$ is an ideal of Q_t $\forall\, v \in (\gamma, 1]$.

Proof. (1). Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t. Let $z, w \in Q_t$ with $w \leq z$ and $v \in (\gamma, \delta]$ be such that $z \in g_v$. Then $z_v \in_\gamma g$ and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t, so $w_v(\in_\gamma \vee q_\delta)g$. If $w_v \in_\gamma g$, then $w \in g_v$ and if $w_v q_\delta g$, then $g(w) > 2\delta - v > v > \gamma$, that is, $w \in g_v$.

Now we have to show that $z \vee w \in g_v, \forall\, z, w \in g_v$. Let $z, w \in Q_t$ be such that $z, w \in g_v$ for some $v \in (\gamma, \delta]$. Then $w_v \in_\gamma g$ and $z_v \in_\gamma g$, and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t, therefore $(w \vee z)_v(\in_\gamma \vee q_\delta)g$. If $(w \vee z)_v \in_\gamma g$, then $(w \vee z) \in g_v$ and if $(w \vee z)_v q_\delta g$, then $g(w \vee z) > 2\delta - v > v > \gamma$, that is, $w \vee z \in g_v$. Let $z \in Q_t$ and $z' \in g_v$ for some $v \in (\gamma, \delta]$. Then $z'_v \in_\gamma g$ and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t, therefore $(z' \otimes z)_v(\in_\gamma \vee q_\delta)g$ and $(z \otimes z')_v(\in_\gamma \vee q_\delta)g$. If $(z' \otimes z)_v \in_\gamma g$, then $(z' \otimes z) \in g_v$ and if $(z' \otimes z)_v q_\delta g$, then $g(z' \otimes z) > 2\delta - v > v > \gamma$, that is, $z' \otimes z \in g_v$. Similarly, $z \otimes z' \in g_v$. Thus, g_v is an ideal of Q_t.

Conversely, suppose that $\emptyset \neq g_v$ is an ideal of Q_t for all $v \in (\gamma, \delta]$. Let $z, w \in Q_t$ with $w \leq z$ and $sup\{g(w), \gamma\} < inf\{g(z), \delta\}$; then there be $v \in (\gamma, \delta]$ such that $sup\{g(w), \gamma\} < v \leq inf\{g(z), \delta\}$. This shows that $z_v \in_\gamma g$; that is $z \in g_v$ but $w \overline{\in} g_v$, a contradiction. Hence, $sup\{g(w), \gamma\} \geq inf\ \{g(z), \delta\}\ \forall\, z, w \in Q_t$ with $w \leq z$. Let $z, w \in Q_t$ and $sup\ \{g(z \vee w), \gamma\} < inf\ \{g(z), g(w), \delta\}$; then $sup\ \{g(z \vee w), \gamma\} < v \leq inf\ \{g(z), g(w), \delta\}$ for some $v \in (\gamma, \delta]$. This implies that $z \in g_v$ and $w \in g_v$ but $(z \vee w) \overline{\in} g_v$, a contradiction. Therefore, $sup\{g(z \vee w), \gamma\} \geq inf\{g(z), g(w), \delta\}$.

Similarly, we can show that $sup\ \{g(y \otimes z), \gamma\} \geq inf\{g(z), \delta\}$, [respectively, ($sup$ $\{g(z \otimes y), \gamma\} \geq inf\ \{g(z), \delta\}$)] $\forall z, y \in Q_t$. Consequently, g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t.

The proof of parts 2 and 3 are a routine and similar verification and hence can be omitted. □

Corollary 4.20. *Let $\gamma, \gamma', \delta, \delta' \in [0,1]$ be such that $\gamma < \delta, \gamma' < \delta'$, and $\delta' < \delta$. Then every $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t is an $(\in_{\gamma'}, \in_{\gamma'} \vee q_{\delta'})$-$FI$ of Q_t.*

The Example below demonstrates that above Corollary is not valid in general

Example 4.21. *Consider the quantale Q'_t and f-subset g as discussed in Example 4.12. Then g is an $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FI$ of Q'_t but not an $(\in_{0.3}, \in_{0.3} \vee q_{0.95})$-$FI$ of Q'_t.*

If we take $\gamma = 0$ and $\delta = 0.5$ in Theorem 4.19 we have,

Theorem 4.22. [35] *Let g be a f-subset of Q_t. Then g is an $(\in, \in \vee q)$-FI of Q_t if and only if each $\emptyset \neq U(g;p)$ is an ideal of Q_t $\forall\, p \in (0, 0.5]$.*

The following Propositions are straightforward.

Proposition 4.23. *Every $(\in_\gamma, \in_\gamma)$-FI of Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t.*

Proposition 4.24. *Every $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-FI of Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-FI of Q_t.*

Converses of Propositions 4.23 and 4.24 donot hold in general as given in the Example below.

Example 4.25. *Consider the quantale Q'_t as discussed in Example 4.4 and take $g = \frac{0.9}{\perp} + \frac{0.7}{i} + \frac{0.65}{j} + \frac{0.54}{k} + \frac{0.31}{\top}$.*

Then

(1) It is simple to confirm that g is an $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FI$ of Q'_t.
(2) g is not an $(\in_{0.3}, \in_{0.3})$-FI of Q'_t, since $i_{0.68} \in_{0.3} g$ and $j_{0.61} \in_{0.3} g$ but $(i \vee j)_{inf(0.68,0.61)} = k_{0.61} \overline{\in}_{0.3} g$.
(3) g is not an $(\in_{0.3} \vee q_{0.6}, \in_{0.3} \vee q_{0.6})$-$FI$ of Q'_t, since $i_{0.68}(\in_{0.3} \vee q_{0.6})g$ and $j_{0.59}(\in_{0.3} \vee q_{0.6})g$ but $(i \vee j)_{inf(0.6,0.59)} = k_{0.59}\overline{(\in_{0.3} \vee q_{0.6})}g$.

Lemma 4.26. *If C is an ideal of Q_t, then K_C of Q_t is an $(\in_\gamma, \in_\gamma)$-FI of Q_t.*

Proof. Let $w, z \in Q_t$ and $p, v \in (\gamma, 1]$ be such that $w_p \in_\gamma K_C$ and $z_v \in_\gamma K_C$. Then $K_C(w) \geq p > \gamma$ and $K_C(z) \geq v > \gamma$, which imply that $K_C(w) = K_C(z) = 1$. As C is an ideal and $w, z \in C$, so $w \vee z \in C$. It follows that $K_C(w \vee z) = 1 \geq inf\{p, v\} > \gamma$ so that $(w \vee z)_{inf(p,v)} \in_\gamma K_C$. Now let $b, z \in Q_t$ and $p \in (\gamma, 1]$ be such that $b_p \in_\gamma K_C$. Then $K_C(b) \geq p > \gamma$, and so $K_C(b) = 1$, i.e., $b \in C$. As C is an ideal of Q_t, we obtain $b \otimes z, z \otimes b \in C$ and hence $K_C(b \otimes z) = K_C(z \otimes b) = 1 \geq p > \gamma$. Therefore $(b \otimes z)_p \in_\gamma K_C$ and $(z \otimes b)_p \in_\gamma K_C$. Let $w, z \in Q_t$, $z_p \in_\gamma K_C$ with $w \leq z$. Then $K_C(z) \geq p > \gamma$, and so $K_C(z) = 1$, i.e., $z \in C$. Since C is a lower set, we have $w \in C$ and so $K_C(w) = 1 \geq p > \gamma$. Therefore $w_p \in_\gamma K_C$ and consequently K_C is an $(\in_\gamma, \in_\gamma)$-FI of Q_t. □

Proposition 4.27. *Let $\emptyset \neq C \subseteq Q_t$. Then C is an ideal of Q_t if and only if K_C is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t.*

Proof. Consider $w, z \in C$ and $p, v \in (\gamma, 1]$. Let K_C be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t, Then $w_1 \in_\gamma K_C$ and $z_1 \in_\gamma K_C$ which show that $(w \vee z)_1 = (w \vee z)_{inf(1,1)}\ (\in_\gamma \vee q_\delta)K_C$. Hence $K_C(w \vee z) > \gamma$, and so $w \vee z \in C$. Let $w, z \in Q_t$ with $w \leq z$ and $z \in C$. Then $K_C(z) = 1$, and thus $z_1 \in_\gamma K_C$. Since K_C is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$, so we have $w_1 \in_\gamma K_C$. Thus $K_C(w) = 1$. Hence $w \in C$. Now let $w \in Q_t$ and $z \in C$. Then $K_C(z) = 1$, and thus $z_1 \in_\gamma K_C$. Since K_C is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$, it follows that $(z \otimes w)_1 \in_\gamma K_C$ so that $K_C(z \otimes w) = 1$. Hence $z \otimes w \in C$ and similarly, $w \otimes z \in C$. Thus, C is an ideal of Q_t.

Conversely, suppose C is an ideal of Q_t, then by lemma 4.26, K_C is an $(\in_\gamma, \in_\gamma)$-FI of Q_t. Thus, K_C is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t by Proposition 4.23. □

5. $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Prime (Semi-Prime) Ideals of Quantale

In this section, we define the $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ $(FSPI)$ of Quantale. We also discuss the relationship between prime (semi-prime) and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ $(FSPI)$ of Quantale as well.

Definition 5.1. *An (α, β)-FI, g of a quantale Q_t is called an (α, β)-FPI of Q_t if $\forall\, p \in (\gamma, 1]$ and $z, w \in Q_t$, $(z \otimes w)_p \alpha g \longrightarrow z_p \beta g$ or $w_p \beta g$. An (α, β)-FI g of a quantale Q_t is called an (α, β)-FSPI of Q_t if $\forall\, z \in Q_t$ and $p \in (\gamma, 1]$, $(z \otimes z)_p \alpha g \longrightarrow z_p \beta g$.*

Proposition 5.2. *An f-subset g of a quantale Q_t is a FPI if and only if g is an $(\in_\gamma, \in_\gamma)$-FPI.*

Proof. Let g be an FPI. Then $g(w) = g(w \otimes z)$ or $g(z) = g(w \otimes z)$ $\forall\, z, w \in Q_t$. Let $(w \otimes z)_p \in_\gamma g$ for some $p \in (\gamma, 1]$. Then $g(w \otimes z) \geq p > \gamma$. Thus $g(w) = g(w \otimes z) \geq p > \gamma$ or $g(z) = g(w \otimes z) \geq p > \gamma$. This implies that $w_p \in_\gamma g$ or $z_p \in_\gamma g$. Therefore g is an $(\in_\gamma, \in_\gamma)$-FPI.

Conversely, let g be an $(\in_\gamma, \in_\gamma)$-FPI. Let $z, w \in Q_t$ and $g(w \otimes z) = v$ where $v \in (\gamma, 1]$. Then $g(w \otimes z) \geq v$. This shows that $(z \otimes w)_v \in_\gamma g$. This gives $w_v \in_\gamma g$ or $z_v \in_\gamma g$. So $g(w) \geq v > \gamma$ or $g(z) \geq v > \gamma$. Thus we have, $g(w \otimes z) = g(w)$ or $g(w \otimes z) = g(z)$. Hence, g is an FPI. □

Proposition 5.3. *An $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ g of a quantale Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-FPI if and only if $sup\{g(z), g(w), \gamma\} \geq inf\{g(z \otimes w), \delta\}$ $\forall\, w, z \in Q_t$ and $\forall\, v \in (\gamma, \delta]$.*

Proof. Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-FPI of a quantale Q_t. We want to show that $sup\{g(z), g(w), \gamma\} \geq inf\{g(z \otimes w), \delta\}$ $\forall\, w, z \in Q_t$. Let there be $y, z \in Q_t$ and $v \in (\gamma, \delta]$ such that $sup\{g(z), g(y), \gamma\} < v \leq inf\{g(z \otimes y), \delta\}$. Then $g(z \otimes y) \geq v > \gamma$, $g(z) < v$, $g(y) < v$ and $g(z) + v < 2v \leq 2\delta$, $g(y) + v < 2v \leq 2\delta$. This means that $(z \otimes y)_v \in_\gamma g$. But $y_v \overline{(\in_\gamma \vee q_\delta)} g$ or $z_v \overline{(\in_\gamma \vee q_\delta)} g$. This gives a contradiction. Hence we have, $sup\{g(z), g(w), \gamma\} \geq inf\{g(z \otimes w), \delta\}$ $\forall\, w, z \in Q_t$.

Conversely, suppose that the condition $sup\{g(z), g(w), \gamma\} \geq inf\{g(z \otimes w), \delta\}$ $\forall\, w, z \in Q_t$ is hold. Let $w, z \in Q_t$ and $v \in (\gamma, \delta]$ such that $(w \otimes z)_v \in_\gamma g$ but $w_v \overline{(\in_\gamma \vee q_\delta)} g$ and $z_v \overline{(\in_\gamma \vee q_\delta)} g$, then $g(w \otimes z) \geq v > \gamma$, $g(w) < v$ and $g(w) + v < 2\delta$, similarly, $g(z) < v$ and $g(z) + v < 2\delta$. It follows that $g(w) < \delta$, $g(z) < \delta$ and so $sup\{g(z), g(w), \gamma\} < inf\{g(z \otimes w), \delta\}$, a contradiction. Therefore g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-FPI of Q_t. □

Theorem 5.4. *Let g be a f-subset of a quantale Q_t. Then g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-FPI if and only if g_v is a PI of Q_t $\forall\, v \in (\gamma, \delta]$.*

Proof. Suppose g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-FPI of Q_t. Let $y, z \in Q_t$ and $v \in (\gamma, \delta]$ be such that $y \otimes z \in g_v$. Then $(y \otimes z)_v \in_\gamma g$. Also since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-FPI of Q_t, hence $y_v(\in_\gamma \vee q_\delta)g$ or $z_v(\in_\gamma \vee q_\delta)g$. If $y_v \in_\gamma g$ then $y \in g_v$ and if $y_v q_\delta g$, then $g(y) > 2\delta - v \geq v > \gamma$; that is, $y \in g_v$. Similarly $z \in g_v$. Hence g_v is a PI of Q_t.

Conversely, suppose that g_v is a PI of Q_t $\forall\, v \in (\gamma, \delta]$ and assume that the condition $sup\{g(z), g(w), \gamma\} \geq inf\{g(z \otimes w), \delta\}$ is not true, then there be some $a, c \in Q_t$

such that $sup\{g(a), g(c), \gamma\} < inf\{g(a \otimes c), \delta\}$; then there exists $v \in (\gamma, \delta]$ such that $sup\{g(a), g(c), \gamma\} < v \leq inf\{g(a \otimes c), \delta\}$. This implies that $(a \otimes c)_v \in_\gamma g$; that is $a \otimes c \in g_v$. Since g_v is a PI of Q_t, we have $a \in g_v$ or $c \in g_v$, i.e., $g(a) \geq v$ or $g(c) \geq v$, which contradicts the condition. Hence we have $sup\{g(z), g(w), \gamma\} \geq inf\{g(z \otimes w), \delta\}$. Consequently g is an $(\in, \in \vee q)$-FPI of Q_t by Proposition 5.3. □

Proposition 5.5. *Let $\emptyset \neq C \subseteq Q_t$. Then C is a PI of Q_t if and only if K_C is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ of Q_t.*

Proof. Let K_C be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-FPI of Q_t. Then K_C is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t. By Proposition 4.27 C is an ideal of Q_t . Let $w, z \in Q_t$ such that $w \otimes z \in C$. Then $K_C(w \otimes z) = 1$. Hence $sup\{K_C(w), K_C(z), \gamma\} \geq inf\{K_C(w \otimes z), \delta\} = inf\{1, \delta\} = \delta \longrightarrow sup\{K_C(w), K_C(z), \gamma\} \geq \delta \longrightarrow K_C(w) \geq \delta$ or $K_C(z) \geq \delta \longrightarrow K_C(w) = 1$ or $K_C(z) = 1 \longrightarrow w \in C$ or $z \in C$. Thus $w \otimes z \in C \longrightarrow w \in C$ or $z \in C$. Hence C is a PI of Q_t.

Conversely, let C be a PI of Q_t. Then $(K_C)_v = \{z \in Q_t : K_C(z) \geq v\} = C$, $\forall v \in (\gamma, 1]$. This shows that $(K_C)_v$ is a PI. K_C is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ of Q_t, by Theorem 5.4. □

Proposition 5.6. *An $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$, g of a quantale Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$ if and only if $sup\{g(z), \gamma\} \geq inf\{g(z \otimes z), \delta\}$ $\forall z \in Q_t$.*

Proof. Proof is obtained in a similar way from Proposition 5.3. □

Proposition 5.7. *Let g be an f-subset of a quantale Q_t. Then g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$ if and only if g_v is a SPI of Q_t $\forall v \in (\gamma, \delta]$.*

Proof. Consider g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$. Let $(y \otimes y) \in g_v$. Then $g(y \otimes y) \geq v$. Thus by Proposition 5.6, we have $sup\{g(z), \gamma\} \geq inf\{g(z \otimes z), \delta\} \geq inf\{v, \delta\} = v$. So, $g(z) \geq v$. Thus $z \in g_v$. Hence g_v is a SPI of Q_t.

Conversely, suppose that g_v is a SPI of Q_t $\forall v \in (\gamma, \delta]$ and assume that condition $sup\{g(z), \gamma\} \geq inf\{g(z \otimes z), \delta\}$ does not hold, then there be some $c \in Q_t$ such that $sup\{g(c), \gamma\} < inf\{g(c \otimes c), \delta\}$ and we take $v \in (\gamma, \delta]$ such that $sup\{g(c), \gamma\} < v \leq inf\{g(c \otimes c), \delta\}$. This implies that $(c \otimes c) \in g_v$. Since g_v is a SPI of Q_t, we have $c \in g_v$, i.e., $g(c) \geq v$, which contradicts the condition. Hence we must have $sup\{g(z), \gamma\} \geq inf\{g(z \otimes z), \delta\}$ $\forall z \in Q_t$. Consequently g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$ of Q_t by Proposition 5.6. □

The following proposition is similarly obtained from Proposition 5.5.

Proposition 5.8. *Let $\emptyset \neq C \subseteq Q_t$. Then K_C is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$ of Q_t if and only if C is a SPI of Q_t.*

6. Conclusion

Due to the significant role of Quantales and their different characterizations in several applied fields such as topological theory, linear logic, theoretical computer science, algebraic theory and rough set theory, the latest research has been carried out in the last few decades by considering various characterizations of Quantales in terms of different types of fuzzy ideals. In the present paper, a more generalized form of Qurashi and Shabir [35] approach of fuzzy (subquantales) ideals are introduced and established $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy (subquantales) ideals. Further several characterization theorems of Quantales in terms of these notions are provided. The relationship between ordinary (subquantales) ideals and $f$uzzy (subquantales) ideals (Prime, Semi-prime) of the type $(\in_\gamma, \in_\gamma \vee q_\delta)$ is also constructed.

In future, the following examinations may be completed:

(1) $(\in_\gamma, \in_\gamma \vee q_\delta)$-weakly prime and weakly semi prime fuzzy ideals in quantales.
(2) $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy sub-nearrings and ideals.
(3) $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy prime and irreducible ideals in BCK-algebras.

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