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Bayesian Inference of Mixture of two Rayleigh Distributions: A New Look

Tabassum Naz Sindhu
Department of Statistics,
Quaid-i-Azam University 45320, Islamabad 44000, Pakistan
Department of Statistics,
Riphah International University, Islamabad, Pakistan
Email: sindhuqau@gmail.com

Navid Feroze
Department of Statistics,
Government Post Graduate College, Muzaffarabad, Azad Kashmir, Pakistan
Department of Statistics,
Riphah International University, Islamabad, Pakistan
Email: navidferoz@gmail.com

Muhammad Aslam
Department of Statistics,
Riphah International University, Islamabad, Pakistan

Anum Shafiq
Department of Mathematics,
Preston University Kohat, Islamabad campus, Pakistan

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Abstract. There are several papers which discuss the Bayesian analysis of the mixture models under type I singly censored samples. This paper considers a new methodology for Bayesian analysis of mixture models under doubly censored samples. We have discussed the Bayesian estimation of parameters of the two-component mixture of Rayleigh distribution under square root gamma, Maxwell and half normal priors using two loss functions. Further, we analyzed some simulated and real data sets in order to investigate the performance of the estimators for the parameters of the proposed mixture model.

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1 Introduction

In survival analysis, data are subject to censoring. The most common type of censoring is right censoring, in which the survival time is larger than the observed right censoring time. In some cases, however, data are subject to left, as well as, right censoring. When left censoring occurs, the only information available to an analyst is that the survival time is less than or equal to the observed left censoring time. A more complex censoring scheme is found when both initial and final times are interval-censored. This situation is referred as double censoring, or the data with both right and left censored observations are known as doubly censored data.

Analysis of doubly censored data for simple (single) distribution has been studied by many authors. Fernandez [1] investigated maximum likelihood prediction based on type II doubly censored exponential data. Fernandez [2] has discussed Bayesian estimation based on trimmed samples from Pareto populations. Khan et al. [3] studied predictive inference from a two-parameter Rayleigh life model given a doubly censored sample. Kim and Song [4] have discussed Bayesian estimation of the parameters of the generalized exponential distribution from doubly censored samples. Khan et al. [5] studied sensitivity analysis of predictive modeling for responses from the three-parameter Weibull model with a follow-up doubly censored sample of cancer patients. Pak et al. [6] has proposed the estimation of Rayleigh scale parameter under doubly type-II censoring from imprecise data.

In statistics, a mixture distribution is signified as a convex fusion of other probability distributions. It can be used to model a statistical population with subpopulations, where constituent of mixture probability densities are the densities of the subpopulations. Mixture distribution may appropriately be used for certain data set where the subsets of the whole data set possess different properties that can best be modeled separately. They can be more mathematically manageable, because the individual mixture components are dealt with more ease than the overall mixture density. The families of mixture distributions have a wider range of applications in different fields such as fisheries, agriculture, botany, economics, medicine, psychology, electrophoresis, finance, communication theory, geology and zoology.

Soliman [7] derived estimators for the finite mixture of Rayleigh model based on progressively censored data. Sultan et al. [8] have discussed some properties of the mixture of two inverse Weibull distributions. Saleem and Aslam [9] presented a comparison of the Maximum Likelihood (ML) estimates with the Bayes estimates assuming the Uniform and the Jeffreys priors for the parameters of the Rayleigh mixture. Kundu and Howalder [10] considered the Bayesian inference and prediction of the inverse Weibull distribution for type-II censored data. Saleem et al. [11] considered the Bayesian analysis of the mixture of Power function distribution using the complete and the censored sample. Shi and Yan [12] studied the case of the two parameter exponential distribution under type I censoring to get empirical Bayes estimates. Eluebaly and Bougila [13] have presented a Bayesian approach to analyze finite generalized Gaussian mixture models which incorporate several standard mixtures, widely used in signal and image processing applications, such as Laplace and Gaussian. Sultan and Al.-Moisheer [14] developed approximate Bayes estimation of the parameters and reliability function of mixture of two inverse Weibull distributions under Type-II censoring.

The article is outlined as follows. In the section 2, we define the mixture model, sampling and likelihood function of Rayleigh model and the posterior distributions are derived under different priors. Expressions for the Bayes estimators and corresponding posterior risks are derived in the section 3. The elicitation of hyperparameters via prior predictive approach is discussed in the section 4. Simulation study and comparison of the estimates are given in the section 5. Real data sets to illustrate the methodology of the proposed mixture model are discussed in the section 6. Some concluding remarks are given in the section 7.

1 The Proposed Mixture Model And The Likelihood Function

The probability density function (pdf) of the Rayleigh distribution with rate parameter λ_i is:

$$f_i\left(x_{ij}\right) = 2x_{ij}\lambda_i^2 \exp\left(-x_{ij}^2\lambda_i^2\right),\ 0 < x_{ij} < \infty\ \lambda_i^2 > 0, i = 1, 2, and\ j = 1, 2, ..., n_i. \qquad (1.\ 1)$$

The cumulative distribution function (CDF) of the distribution is:

$$F\left(x_{ij}\right) = 1 - \exp\left(-x_{ij}^2\lambda_i^2\right),\ 0 < x_{ij} < \infty\ \lambda_i^2 > 0, i = 1, 2, and\ j = 1, 2, ..., n_i. \qquad (1.\ 2)$$

A density function for mixture of two components densities with mixing weights $(p_1, 1 - p_1)$ is:

$$f\left(x\right) = p_1 f_1\left(x\right) + \left(1 - p_1\right) f_2\left(x\right),\ 0 < p_1 < 1. \qquad (1.\ 3)$$

The cumulative distribution function for the mixture model is:

$$F\left(x\right) = p_1 F_1\left(x\right) + \left(1 - p_1\right) F_2\left(x\right). \qquad (1.\ 4)$$

Consider a random sample of size n from Rayleigh distribution, and let $x_r, x_{r+1}, ..., x_s$ be the ordered observations that can only be observed. The remaining $r-1$ smallest observations and the $n-s$ largest observations have been assumed to be censored. Now based on causes of failure, the failed items are assumed to come either from subpopulation 1 or from subpopulation 2; so the $x_{1r_1}, ..., x_{1s_1}$ and $x_{2r_2}, ..., x_{2s_2}$ failed items come from first and second subpopulations respectively. The rest of the observations which are less than x_r and greater than x_s have been assumed to be censored from each component. Where $x_s = \max(x_{1,s_1}, x_{2,s_2})$ and $x_r = \min(x_{1,r_1}, x_{2,r_2})$. Therefore, $m_1 = s_1 - r_1 + 1$ and $m_2 = s_2 - r_2 + 1$ number of failed items can be observed from first and second subpopulations respectively. The remaining $n-(s-r+2)$ items are assumed to be censored observations, and $s-r+2$ are the uncensored items. Where $r = r_1 + r_2$, $s = s_1 + s_2$ and $m = m_1 + m_2$. Then the likelihood function for the Type II doubly censored sample $\mathbf{x} = \{(x_{1r_1}, ..., x_{1s_1}), (x_{2r_2}, ..., x_{2s_2})\}$, assuming the causes of the failure of the left censored items are identified, can be written as:

$$L(\lambda_1, \lambda_2, p_1|\mathbf{x}) \propto p_1^{s_1}(1-p_1)^{s_2}\{F_1(x_{r_1}, \lambda_1)\}^{r_1-1}\{F(x_{r_2}, \lambda_2)\}^{r_2-1}\{1-F_2(x_s, \lambda_1, \lambda_2)\}^{n-s} \left\{{}_{i=r_1}^{s_1} f_1(x_{1(i)}, \lambda_1)\right\}\left\{{}_{i=r_2}^{s_2} f_2(x_{2(i)}, \lambda_2)\right\} \quad (1.5)$$

$$L(\lambda_1, \lambda_2, p_1|\mathbf{x}) \propto \sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2} r_1 - 1k_1r_2 - 1k_2n - sk_3p_1^{n-s-k_3+s_1}(1-p_1)^{s_2+k_3} \lambda_1^{2m_1}\lambda_2^{2m_2}\exp\left\{-\lambda_1^2(\Omega(x_{1j}))\right\}\exp\left\{-\lambda_2^2(\Omega(x_{2j}))\right\} \quad (1.6)$$

where $\Omega(x_{1j}) = \sum_{i=r_1}^{s_1} x_{1(i)}^2 + (n-s-k_3)x_s^2 + kx_{(r_1)}^2$, $\Omega(x_{2j}) = \sum_{i=r_2}^{s_2} x_{2(i)}^2 + (k_3)x_s^2 + kx_{(r_2)}^2$, $m_1 = s_1 - r_1 + 1$, and $m_2 = s_2 - r_2 + 1$.

2 Bayes Estimation

For the Bayesian estimation, let us assume that the parameters λ_i and p_1 for $i = 1, 2$, are independent random variables. Further, we considered the different priors for the Bayes estimation of the parameters.

2.1 *Bayesian estimation using square root gamma prior*

The prior for the rate parameters λ_i $(for\ i = 1, 2)$ is assumed to be the square root gamma distribution, with the hyperparameters α_i and β_i and is given by

$$f_{\lambda_i}(\lambda_i) = \frac{2\beta_i^{\alpha_i}}{\Gamma(\alpha_i)}\lambda_i^{2\alpha_i-1}\exp\left(-\lambda_i^2\beta_i\right), \quad \alpha_i, \beta_i > 0. \quad (2.7)$$

The prior for p_1 is assumed to be the beta distribution, whose density is given by

$$f_{p_1}(p_1) = \frac{\Gamma(c_4+d_4)}{\Gamma(c_4)\Gamma(d_4)}p_1^{c_4-1}(1-p_1)^{d_4-1}, \quad c_4, d_4 > 0. \quad (2.8)$$

From equations (2.7) – (2.8), we proposed the following joint prior density of the vector $\Theta = (\lambda_1, \lambda_2, p_1)$:

$$g(\Theta) \propto \lambda_i^{2\alpha_i-1}\exp\left(-\lambda_i^2\beta_i\right)p_1^{c_4-1}(1-p_1)^{d_4-1}, \ 0 < p_1 < 1, \alpha_i, \beta_i > 0, c_4, d_4 > 0. \quad (2.9)$$

From (2.9) and (1.6), the joint posterior density for the vector Θ given the data becomes:

$$\pi(\Theta|\mathbf{x}) \propto \sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s} {}_{i=1}^{2}(-1)^{k_1+k_2} r_1 - 1k_1r_2 - 1k_2n - sk_3p_1^{n-s-k_3+s_1+c_4-1}(1-p_1)^{s_2+k_3+d_4-1} \lambda_i^{2(\alpha_i+m_i)-1}\exp\left\{-\lambda_i^2(\beta_i + \Omega(x_{ij}))\right\} \quad (2.10)$$

Marginal distributions of λ_i and p_1 for $i = 1, 2$, can be obtained by integrating the nuisance parameters.

2.2 Bayesian Estimation using Maxwell prior

The prior for the rate parameters λ_i $(for\ i = 1,\ 2)$ is assumed to be the Maxwell prior, with the hyperparameters w_i, and is given by

$$f_{\lambda_i}\left(\lambda_i\right) = \sqrt{\frac{2}{\pi}}\frac{\lambda_i^2}{w_i^3}\exp\left(\frac{-\lambda_i^2}{2w_i^2}\right), \quad w_i > 0. \qquad (\ 2.\ 11\ )$$

The prior for p_1 is assumed to be the beta distribution, whose density is given by

$$f_{p_1}\left(p_1\right) = \frac{\Gamma\left(c_5 + d_5\right)}{\Gamma\left(c_5\right)\Gamma\left(d_5\right)}p_1^{c_5-1}\left(1 - p_1\right)^{d_5-1}, \quad c_5, d_5 > 0. \qquad (\ 2.\ 12\ )$$

From equations (2.11) – (2.12), we propose the following joint prior density of the vector $\Theta = \left(\lambda_1, \lambda_2, p_1\right)$:

$$g\left(\Theta\right) \propto \lambda_i^2 \exp\left(\frac{-\lambda_i^2}{2w_i^2}\right) p_1^{c_5-1}\left(1 - p_1\right)^{d_5-1}, \ 0 < p_1 < 1, w_i > 0, c_5, d_5 > 0. \qquad (\ 2.\ 13\ )$$

By multiplying Equation (2.13) with Equation (1.6), the joint posterior density for the vector Θ given the data becomes:

$$\pi\left(\Theta|\,\mathbf{x}\right) \propto \sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}{}_{i=1}^{2}\left(-1\right)^{k_1+k_2} r_1 - 1k_1r_2 - 1k_2n - sk_3p_1^{n-s-k_3+s_1+c_5-1}\left(1 - p_1\right)^{s_2+k_3+d_5-1}$$
$$\lambda_i^{2m_i+2}\exp\left\{-\lambda_i^2\left(\frac{1}{2w_i^2} + \Omega\left(x_{ij}\right)\right)\right\} \qquad (\ 2.\ 14\ )$$

2.3 Bayesian Estimation using half normal prior

The prior for the rate parameters λ_i (for $i = 1,\ 2$) is assumed to be the half normal prior, with the hyperparameter g_i and is given by

$$f_{\lambda_i}\left(\lambda_i\right) = \sqrt{\frac{2}{\pi}}\frac{1}{g_i}\exp\left(\frac{-\lambda_i^2}{2g_i^2}\right), \ g_i > 0. \qquad (\ 2.\ 15\ )$$

The prior for p_1 is assumed to be the beta distribution, whose density is given by

$$f_{p_1}\left(p_1\right) = \frac{\Gamma\left(c_6 + d_6\right)}{\Gamma\left(c_6\right)\Gamma\left(d_6\right)}p_1^{c_6-1}\left(1 - p_1\right)^{d_6-1}, \quad c_6, d_6 > 0. \qquad (\ 2.\ 16\ )$$

From equations (2.15) – (2.16), we propose the following joint prior density of the vector $\Theta = \left(\lambda_1, \lambda_2, p_1\right)$:

$$g\left(\Theta\right) \propto \exp\left(\frac{-\lambda_i^2}{2g_i^2}\right) p_1^{c_6-1}\left(1 - p_1\right)^{d_6-1}, \ 0 < p_1 < 1, g_i > 0, c_6, d_6 > 0. \qquad (\ 2.\ 17\ )$$

By multiplying Equation (2.17) with Equation (1.6), the joint posterior density for the vector Θ given the data becomes

$$\pi\left(\Theta|\,\mathbf{x}\right) \propto \sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}{}_{i=1}^{2}\left(-1\right)^{k_1+k_2} r_1 - 1k_1r_2 - 1k_2n - sk_3p_1^{n-s-k_3+s_1+c_6-1}\left(1 - p_1\right)^{s_2+k_3+d_6-1}$$
$$\lambda_i^{2m_i}\exp\left\{-\lambda_i^2\left(\frac{1}{2g_i^2} + \Omega\left(x_{ij}\right)\right)\right\} \qquad (\ 2.\ 18\ )$$

Marginal distributions of λ_i and p_1 for $i = 1, 2$, can be obtained by integrating the nuisance parameters.

3 Bayes Estimation of the Vector of Parameters Θ

The Bayesian point estimation is connected to a loss function in general, signifying the loss induced when the estimate $\hat{\theta}$ differ from true parameter θ. Since there is no specific rule that helps us to identify the appropriate loss function to be used, squared error loss is used in this article as it serve as standard loss. It is well known that under the squared error loss function, the Bayes estimator of a function of the parameters is the posterior mean of the function and risk is the posterior variance. The most popular loss function, squared error loss function (SELF), is defined as $l\left(\hat{\theta},\theta\right)=\left(\theta-\hat{\theta}\right)^2$. It was originally used in estimation problems when the unbiased estimator of θ was being considered. Another reason for its popularity is due to its relationship to least squares theory. The use of SELF makes the calculations simpler. K-Loss function (KLF) which is particularized as $l\left(\hat{\theta},\theta\right)=\left(\theta-\hat{\theta}\right)^2/\hat{\theta}\theta$ is proposed by Wasan [15] is well fitted for a measure of inaccuracy for an estimator of a scale parameter of a distribution defined on $R^+=(0,\infty)$.Under K-Loss function the Bayes estimates and posterior risks are defined as $\hat{\theta}=\sqrt{E\left(\theta|\,\mathbf{x}\right)/E\left(\theta^{-1}|\,\mathbf{x}\right)}$,and $\rho\left(\hat{\theta}\right)=2\left\{E\left(\theta|\,\mathbf{x}\right)/E\left(\theta^{-1}\big|\,\mathbf{x}\right)-1\right\}$.

In this section, the respective marginal distribution of each parameter has been used to derive the Bayes estimators and posterior risks for λ_1, λ_2 and p_1 under squared error loss function (SELF) assuming square root gamma prior are given as:

Let

$$\sum K_i=\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}r_1-1k_1r_2-1k_2n-sk_3, A=n-s-k_3+s_1+c_4, B=s_2+k_3+d_4.$$

Then the Bayes estimators of λ_1, λ_2 and p_1 are:

$$\hat{\lambda}_{1(SELF)}=S^{-1}\left\{\sum K_i\frac{B\left(A,B\right)\Gamma\left(\alpha_1+m_1+1/2\right)\Gamma\left(\alpha_2+m_2\right)}{2\left\{\beta_1+\Omega\left(x_{1j}\right)\right\}^{(\alpha_1+m_1+1/2)}2\left\{\beta_2+\Omega\left(x_{2j}\right)\right\}^{(\alpha_2+m_2)}}\right\}.$$

$$\hat{\lambda}_{2(SELF)}=S^{-1}\left\{\sum K_i\frac{B\left(A,B\right)\Gamma\left(\alpha_1+m_1\right)\Gamma\left(\alpha_2+m_2+1/2\right)}{2\left\{\beta_1+\Omega\left(x_{1j}\right)\right\}^{(\alpha_1+m_1+1/2)}2\left\{\beta_2+\Omega\left(x_{2j}\right)\right\}^{(\alpha_2+m_2)}}\right\}.$$

$$\hat{p}_{1(SELF)}=S^{-1}\left\{\sum K_i\frac{B\left(A+1,B\right)\Gamma\left(\alpha_1+m_1\right)\Gamma\left(\alpha_2+m_2\right)}{2\left\{\beta_1+\Omega\left(x_{1j}\right)\right\}^{(\alpha_1+m_1)}2\left\{\beta_2+\Omega\left(x_{2j}\right)\right\}^{(\alpha_2+m_2)}}\right\}.$$

The Posterior risks of λ_1, λ_2 and p_1 are:

$$\rho\left(\hat{\lambda}_{1(SELF)}\right)=S^{-1}\left\{\sum K_i\frac{B\left(A,B\right)\Gamma\left(\alpha_1+m_1+1\right)\Gamma\left(\alpha_2+m_2\right)}{2\left\{\beta_1+\Omega\left(x_{1j}\right)\right\}^{(\alpha_1+m_1)}2\left\{\beta_2+\Omega\left(x_{2j}\right)\right\}^{(\alpha_2+m_2)}}\right\}-\left(\hat{\lambda}_{1(SELF)}\right)^2.$$

$$\rho\left(\hat{\lambda}_{2(SELF)}\right)=S^{-1}\left\{\sum K_i\frac{B\left(A,B\right)\Gamma\left(\alpha_1+m_1\right)\Gamma\left(\alpha_2+m_2+1\right)}{2\left\{\beta_1+\Omega\left(x_{1j}\right)\right\}^{(\alpha_1+m_1)}2\left\{\beta_2+\Omega\left(x_{2j}\right)\right\}^{(\alpha_2+m_2)}}\right\}-\left(\hat{\lambda}_{2(SELF)}\right)^2.$$

$$\rho\left(\hat{p}_{1(SELF)}\right)=S^{-1}\left\{\sum K_i\frac{B\left(A+2,B\right)\Gamma\left(\alpha_1+m_1\right)\Gamma\left(\alpha_2+m_2\right)}{2\left\{\beta_1+\Omega\left(x_{1j}\right)\right\}^{(\alpha_1+m_1)}2\left\{\beta_2+\Omega\left(x_{2j}\right)\right\}^{(\alpha_2+m_2)}}\right\}-\left(\hat{p}_{1(SELF)}\right)^2.$$

where S^{-1} is formulized as

$$S^{-1}=\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}r_1-1k_1r_2-1k_2n-sk_3B\left(n-s-k_3+s_1+c_4,s_2+k_3+d_4\right)\frac{\Gamma\left(\alpha_1+m_1\right)\Gamma\left(\alpha_2+m_2\right)}{2\left\{\beta_1+\Omega\left(x_{1j}\right)\right\}^{(\alpha_1+m_1)}2\left\{\beta_2+\Omega\left(x_{2j}\right)\right\}^{(\alpha_2+m_2)}}.$$

Similarly, expressions for Bayes estimators and their posterior risks under the rest of the priors using KLF can be obtained with little modifications.

4 Elicitation

In Bayesian analysis the elicitation of opinion is a crucial step. It helps to make it easy for us to understand what the experts believe in, and what their opinions are. In statistical inference, the characteristics of a certain predictive distribution proposed by an expert determine the hyperparameters of a prior distribution. In this article, we focused on a method of elicitation based on prior predictive distribution. The elicitation of hyperparameter from the prior $p(\lambda)$ is a difficult task. The prior predictive distribution is used for the elicitation of the hyperparameters which is compared with the experts' judgment about this distribution and then the hyperparameters are chosen in such a way so as to make the judgment agree closely as possible with the given distribution. Readers desiring more detail may refer to: Grimshaw et al. [16], O'Hagan et al. [17], Jenkinson [18] and Leon et al. [19]. According to Aslam [20], the method of elicitation is to compare the prior predictive distribution with experts' assessment about this distribution and then to choose the hyperparameters that make the assessment agree closely with the member of the family. The prior predictive distributions under all the priors are derived using the following formula:

$$p(y) =_{\Theta} p(y|\,\Theta)\, p(\Theta)\, d\Theta.$$

4.1 *Elicitation under square root gamma*

The prior predictive distribution using square root gamma prior is:

$$p(y) = 2(\beta_1)^{\alpha_1} \frac{y\alpha_1 c_4}{(c_4+d_4)(y^2+\beta_1)^{(\alpha_1+1)}} + 2(\beta_2)^{\alpha_2} \frac{y\alpha_2 d_4}{(c_4+d_4)(y^2+\beta_2)^{(\alpha_2+1)}}, y > 0. \quad (4.\ 19)$$

For the elicitation of the six hyperparameters, six different intervals are considered. From Equation (19), the experts' probabilities/assessments are supposed to be 0.10 for each case. The six integrals for equation (19) are considered with the following limits of the values of random variable Y : (0, 10), (10, 20), (20, 30), (30, 40), (40, 50) and (50, 60)respectively. For the elicitation of hyperparameters, α_1, α_2, β_1, β_2, c_4, and d_4. these six integrals are solved simultaneously through computer program developed in SAS package using the command of PROC SYSLIN. Thus the values of hyperparameters obtained by applying this methodology are: 2.32587, 1.17440, 0.023281, 0.0558, 0.040125 and 0.6508.

4.2 *Elicitation under Maxwell prior*

The prior predictive distribution using Maxwell prior is:

$$p(y) = \left(\frac{1}{2w_1^2}\right)^{3/2} \frac{3yc_5}{(c_5+d_5)\left(y^2+\frac{1}{2w_1^2}\right)^{(5/2)}} + \left(\frac{1}{2w_2^2}\right)^{3/2} \frac{3yd_5}{(c_5+d_5)\left(y^2+\frac{1}{2w_2^2}\right)^{(5/2)}}, y > 0. \quad (4.\ 20)$$

Now, we have to elicit four hyperparameters, so we have to consider the four integrals. The expert probabilities are assumed to 0.15 for each integral with the following limits of the values of random variable $Y : (0,\ 15), (15,\ 30), (30,\ 45)$ and $(45,\ 60)$. Using the similar kind of program, as discussed above, we have the following values of hyperparameters $w_1 = 3.14798$, $w_2 = 4.52436$, $c_5 = 0.6235$, and $d_5 = 0.9325$.

4.3 *Elicitation under half normal prior*

he prior predictive distribution using half normal prior is:

$$p(y) = \frac{yc_6}{\sqrt{2}g_1(c_6+d_6)\left(y^2+\frac{1}{2g_1^2}\right)^{(3/2)}} + \frac{yd_6}{\sqrt{2}g_2(c_6+d_6)\left(y^2+\frac{1}{2g_2^2}\right)^{(3/2)}}, y > 0. \quad (4.\ 21)$$

Again, we have to elicit four hyperparameters, so we have to consider the four integrals. The expert probabilities are assumed to 0.15 for each integral with the following limits of the values of random variable $Y : (0,\ 15), (15,\ 30), (30,\ 45)$ and $(45,\ 60)$. Using the similar kind of program, as discussed above, we attained the following values of hyperparameters $g_1 = 3.92568$, $g_2 = 3.56890$, $c_6 = 0.98256$, and $d_6 = 0.3256$.

5 Simulation Study and Comparisons

A simulation study is carried out in order to obtain and investigate the performance of Bayes estimators under tenfold choice of the parameters, different sample sizes, and the different values of the mixing proportion. We take random samples of sizes $n = 20$, 40, and 80 from the two component mixture of Rayleigh distributions with tenfold choice of parameters $(\lambda_1, \lambda_2) \in \{(0.1, 0.12), (1, 1.2), (10, 12), (0.1, 12), (10, 0.12)\}$, To generate a mixture data we make use of probabilistic mixing with probabilities $p_1$and $(1 - p_1)$. A uniform number is generated n times and if $u < p_1$ the observation is taken randomly from (the Rayleigh distribution with parameter) otherwise from (from the Rayleigh distribution with parameter). The choice of the censoring time is made in such a way that the censoring rate in the resultant sample is approximately 20%. To implement censored samplings, the $(x_{1r_1}, ..., x_{1s_1})$ and $(x_{2r_2}, ..., x_{2s_2})$ failed items come from first and second subpopulations respectively. The rest of the observations which are less than x_r and greater than x_s have been assumed to be censored from each component. Where $x_s = \max(x_{1,s_1}, x_{2,s_2})$ and $x_r = \min(x_{1,r_1}, x_{2,r_2})$. The simulated data sets have been obtained using following steps:

Step 1 : Draw samples of size n from the mixture model.

Step 2 : Generate a uniform random no. u for each observation.

Step 3 : If $u < p_1$, then take the observation from first subpopulation otherwise from the second subpopulation.

Step 4 : Determine the test termination points on left and right, that is, determine the values of x_r and x_s.

Step 5 : The observations which are less than x_r and greater than x_s have been considered to be censored from each component.

Step 6 : Use the remaining observations from each component for the analysis.

To avoid an extreme sample, we simulate 10,000 data sets each of size n. The Bayes estimates and posterior risks (in parenthesis) are computed using Mathematica 8.0. The average of these estimates and corresponding risks are reported in Tables 1 − 18. The abbreviations used in the tables are: B.Es: Bayes estimators; P.Rs: Posterior risks; SRGP: Square root gamma prior; MP: Maxwell prior; HNP: Half normal prior.

Table 1: B.Es and P.Rs under SRGP.

	$(\lambda_1, \lambda_2, p_1) = (0.1, 0.12, 0.45)$			$(\lambda_1, \lambda_2, p_1) = (0.1, 0.12, 0.60)$		
n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	0.120706 (0.000497)	0.132832 (0.000563)	0.464593 (0.013072)	0.112661 (0.000313)	0.144087 (0.000910)	0.63509 (0.012135)
40	0.111567 (0.000267)	1.25739 (0.000292)	0.460969 (0.007267)	0.101895 (0.000138)	0.138704 (0.000465)	0.624409 (0.006512)
80	0.10510 (0.000135)	0.125075 (0.000171)	0.460047 (0.004025)	0.090782 (0.000055)	0.133123 (0.000217)	0.605373 (0.003267)
k-loss function						
20	0.11909 (0.067778)	0.12714 (0.064218)	0.44605 (0.150025)	0.10786 (0.047157)	0.14139 (0.093525)	0.627623 (0.068832)
40	0.10436 (0.039444)	0.123017 (0.036847)	0.44663 (0.073654)	0.101514 (0.026479)	0.132128 (0.053417)	0.61704 (0.034743)
80	0.09609 (0.02017)	0.12139 (0.019164)	0.45725 (0.035277)	0.09693 (0.015321)	0.130471 (0.032079)	0.610776 (0.017124)

Table 2: B.Es and P.Rs under SRGP.

n	$(\lambda_1,\lambda_2,p_1)=(1,1.2,0.45)$			$(\lambda_1,\lambda_2,p_1)=(1,1.2,0.60)$		
	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	1.15428 (0.046817)	1.33157 (0.056282)	0.468062 (0.013109)	1.10334 (0.030669)	1.45572 (0.091339)	0.636734 (0.012070)
40	1.08208 (0.024728)	1.28959 (0.029089)	0.464498 (0.007205)	1.00912 (0.014183)	1.39332 (0.047100)	0.625084 (0.006495)
80	1.00965 (0.011394)	1.24678 (0.014543)	0.460567 (0.003786)	0.951208 (0.006421)	1.39106 (0.024736)	0.620353 (0.003334)
k-loss function						
20	1.13857 (0.067054)	1.31824 (0.065609)	0.452844 (0.144884)	1.08006 (0.047371)	1.36529 (0.092071)	0.62433 (0.070195)
40	1.09099 (0.039933)	1.24577 (0.036265)	0.45189 (0.076001)	1.00903 (0.025546)	1.328802 (0.050858)	0.61398 (0.033716)
80	1.06134 (0.022527)	1.23182 (0.019924)	0.45044 (0.039414)	1.00147 (0.014568)	1.24980 (0.026828)	0.605815 (0.018213)

Table 3: B.Es and P.Rs under SRGP.

n	$(\lambda_1,\lambda_2,p_1)=(0.1,0.12,0.45)$			$(\lambda_1,\lambda_2,p_1)=(0.1,0.12,0.60)$		
	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	10.41330 (3.290550)	9.06778 (2.193840)	0.4462268 (0.013211)	10.24480 (2.419790)	8.50795 (2.693530)	0.608873 (0.012913)
40	10.26100 (1.988450)	10.34920 (1.746410)	0.449363 (0.007349)	10.02120 (1.390050)	9.66687 (2.227730)	0.60719 (0.007061)
80	9.52746 (1.05752)	11.33630 (1.319780)	0.463751 (0.003947)	9.877580 (0.726008)	11.21010 (1.819510)	0.603190 (0.003641)
k-loss function						
20	10.42290 (0.062339)	8.94662 (0.054246)	0.42824 (0.160945)	10.16083 (0.046858)	8.40797 (0.077781)	0.59731 (0.081019)
40	10.34801 (0.037338)	10.25652 (0.032344)	0.441125 (0.080060)	10.09712 (0.027719)	9.54778 (0.048165)	0.611588 (0.040346)
80	10.07110 (0.020534)	11.56313 (0.018385)	0.45093 (0.039732)	9.54455 (0.013685)	11.51973 (0.028994)	0.610531 (0.017381)

Table 4: B.Es and P.Rs under SRGP.

n	$(\lambda_1,\lambda_2,p_1)=(0.1,12,0.45)$			$(\lambda_1,\lambda_2,p_1)=(0.1,12,0.60)$		
	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	0.10377 (0.000275)	9.98217 (2.26845)	0.509886 (0.012077)	0.103314 (0.000207)	9.43784 (2.77973)	0.66224 (0.010810)
40	0.0984041 (0.000135)	11.5072 (1.58232)	0.505165 (0.006461)	0.097614 (0.000099)	11.1057 (2.05988)	0.66043 (0.005763)
80	0.0909124 (0.000061)	12.4429 (0.949157)	0.502642 (0.003347)	0.090735 (0.000045)	11.96180 (1.23449)	0.656484 (0.002981)
k-loss function						
20	0.10552 (0.051517)	9.86413 (0.047385)	0.49660 (0.108431)	0.102636 (0.039356)	9.21708 (0.066197)	0.653142 (0.056105)
40	0.09818 (0.028245)	11.3872 (0.024333)	0.49377 (0.054859)	0.097629 (0.02109)	10.90890 (0.034369)	0.64976 (0.027924)
80	0.09167 (0.014836)	12.1267 (0.012331)	0.45921 (0.027599)	0.096647 (0.010936)	12.66150 (0.017516)	0.63179 (0.013926)

Table 5: B.Es and P.Rs under SRGP.

	$(\lambda_1, \lambda_2, p_1) = (10, 0.12, 0.45)$			$(\lambda_1, \lambda_2, p_1) = (10, 0.12, 0.60)$		
n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	11.2501 (3.14529)	0.119784 (0.000333)	0.408332 (0.011676)	11.2244 (2.41444)	0.123373 (0.000493)	0.560671 (0.011905)
40	11.2342 (1.74962)	0.11980 (0.000152)	0.419038 (0.006198)	11.0642 (1.27824)	0.116013 (0.000228)	0.58528 (0.006374)
80	11.0286 (0.892522)	0.119832 (0.000078)	0.439410 (0.003197)	0.7419 (0.625749)	0.112827 (0.000110)	0.59652 (0.003304)
k-loss function						
20	11.93501 (0.051517)	0.12524 (0.047391)	0.392169 (0.168089)	11.01450 (0.039356)	0.12275 (0.066257)	0.54879 (0.087515)
40	11.25863 (0.028246)	0.123957 (0.024332)	0.397631 (0.085606)	10.92320 (0.021096)	0.116409 (0.034372)	0.552175 (0.044088)
80	10.97556 (0.014838)	0.121662 (0.012332)	0.42899 (0.043218)	10.83856 (0.010942)	0.119932 (0.017517)	0.586638 (0.022132)

Table 6: B.Es and P.Rs under MP.

	$(\lambda_1, \lambda_2, p_1) = (0.1, 0.12, 0.45)$			$(\lambda_1, \lambda_2, p_1) = (0.1, 0.12, 0.60)$		
n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	0.113966 (0.000400)	0.139049 (0.000585)	0.481209 (0.012546)	0.104844 (0.000279)	0.152321 (0.000894)	0.643315 (0.011279)
40	0.099319 (0.000199)	0.133566 (0.000297)	0.480382 (0.006939)	0.101979 (0.000122)	0.142803 (0.000460)	0.640778 (0.006283)
80	0.097455 (0.000099)	0.131050 (0.000154)	0.477876 (0.003703)	0.090527 (0.000053)	0.138028 (0.000210)	0.635618 (0.003181)
k-loss function						
20	0.114641 (0.062409)	0.137384 (0.060515)	0.464293 (0.130816)	0.10578 (0.040851)	0.149191 (0.084289)	0.633306 (0.0632242)
40	0.105818 (0.038223)	0.127759 (0.035059)	0.463465 (0.070529)	0.097698 (0.025689)	0.139323 (0.047663)	0.631429 (0.032065)
80	0.099601 (0.022422)	0.123343 (0.020955)	0.457442 (0.036285)	0.097092 (0.012866)	0.134879 (0.026283)	0.621645 (0.015780)

Table 7: B.Es and P.Rs under MP.

	$(\lambda_1, \lambda_2, p_1) = (1, 1.2, 0.45)$			$(\lambda_1, \lambda_2, p_1) = (1.2, 0.60)$		
n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	1.12023 (0.046314)	1.38720 (0.057887)	0.482344 (0.012579)	1.05022 (0.028276)	1.45773 (0.084368)	0.640831 (0.011435)
40	1.04842 (0.023762)	1.29744 (0.028962)	0.473538 (0.007055)	0.994683 (0.013322)	1.448260 (0.046732)	0.631640 (0.006207)
80	1.02093 (0.011733)	1.23058 (0.014295)	0.467743 (0.003755)	0.920867 (0.005678)	1.35693 (0.021973)	0.621143 (0.003241)
k-loss function						
20	1.03792 (0.069236)	1.43180 (0.061310)	0.475992 (0.12391)	1.02462 (0.048256)	1.48727 (0.084806)	0.635992 (0.062309)
40	1.02197 (0.040629)	1.295520 (0.034276)	0.464927 (0.070164)	0.98544 (0.025728)	1.41889 (0.047455)	0.63347 (0.032091)
80	0.0919916 (0.015209)	1.28041 (0.012235)	0.450126 (0.027052)	0.99331 (0.012429)	1.35388 (0.024408)	0.615295 (0.015480)

Table 8: B.Es and P.Rs under MP.

	$(\lambda_1, \lambda_2, p_1) = (10, 12, 0.45)$			$(\lambda_1, \lambda_2, p_1) = (10, 12, 0.60)$		
n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	8.33679 (2.06278)	11.73370 (3.683220)	0.488803 (0.012593)	8.70392 (1.676340)	11.45390 (4.78946)	0.641304 (0.011518)
40	8.9368 (1.39434)	11.79210 (2.13679)	0.476869 (0.007029)	8.96134 (0.942712)	12.99770 (3.21575)	0.634746 (0.006244)
80	9.54378 (0.869606)	12.57510 (1.39548)	0.474653 (0.003762)	9.47886 (0.563675)	12.85280 (1.978860)	0.620224 (0.003281)
k-loss function						
20	8.12639 (0.060951)	11.55610 (0.055257)	0.475007 (0.124962)	8.39057 (0.045291)	11.14625 (0.077415)	0.632324 (0.064082)
40	8.65511 (0.034259)	11.90350 (0.031770)	0.473682 (0.066413)	8.96032 (0.024691)	11.76524 (0.045818)	0.613973 (0.032918)
80	9.81083 (0.018226)	12.35396 (0.018009)	0.467550 (0.033605)	9.52009 (0.012878)	12.38625 (0.026114)	0.604746 (0.016327)

Table 9: B.Es and P.Rs under MP.

	$(\lambda_1, \lambda_2, p_1) = (0.10, 12, 0.45)$			$(\lambda_1, \lambda_2, p_1) = (0.10, 12, 0.60)$		
n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	0.104237 (0.000299)	12.80440 (3.307201)	0.516808 (0.011585)	0.099400 (0.000205)	11.91620 (4.291850)	0.662751 (0.010368)
40	0.093458 (0.000128)	12.61541 (1.878070)	0.508961 (0.006318)	0.095131 (0.000098)	12.86970 (2.72645)	0.660457 (0.005635)
80	0.093040 (0.000065)	12.59690 (0.992395)	0.504634 (0.003309)	0.099615 (0.000048)	12.78067 (1.612500)	0.65870 (0.002946)
k-loss function						
20	0.099271 (0.056305)	11.92880 (0.045977)	0.504302 (0.100419)	0.101388 (0.042091)	11.85390 (0.063488)	0.654071 (0.053432)
40	0.09396 (0.029627)	12.9666 (0.023955)	0.502381 (0.052733)	0.095064 (0.021857)	12.72910 (0.033628)	0.63098 (0.027724)
80	0.091209 (0.015210)	12.29546 (0.012235)	0.501256 (0.027052)	0.093928 (0.011143)	12.31176 (0.017323)	0.62133 (0.013755)

Table 10: B.Es and P.Rs under MP.

	$(\lambda_1, \lambda_2, p_1) = (10, 0.12, 0.45)$			$(\lambda_1, \lambda_2, p_1) = (10, 0.12, 0.60)$		
n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	9.14349 (2.251210)	0.120925 (0.000331)	0.419513 (0.011297)	9.60061 (1.880070)	0.124249 (0.000485)	0.56546 (0.011399)
40	9.23950 (1.520270)	0.11989 (0.000157)	0.42516 (0.006093)	9.82321 (1.119880)	0.120417 (0.000215)	0.56834 (0.006227)
80	10.48410 (0.828461)	0.118993 (0.000072)	0.439733 (0.003169)	10.76850 (0.640213)	0.119417 (0.000103)	0.58529 (0.003264)
k-loss function						
20	9.09923 (0.056305)	0.123842 (0.045980)	0.40440 (0.152289)	9.36116 (0.042091)	0.121491 (0.063497)	0.554234 (0.081808)
40	9.98107 (0.029628)	0.122645 (0.023954)	0.421879 (0.081346)	10.43161 (0.02186)	0.12135 (0.033619)	0.554956 (0.042589)
80	10.51401 (0.015211)	0.123335 (0.012233)	0.439321 (0.042109)	10.17339 (0.011144)	0.120943 (0.017329)	0.555275 (0.021762)

Table 11: B.Es and P.Rs under HNP.

	$(\lambda_1, \lambda_2, p_1) = (0.1, 0.12, 0.45)$			$(\lambda_1, \lambda_2, p_1) = (0.1, 0.12, 0.60)$		
n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	0.105918 (0.000459)	0.137145 (0.000602)	0.509557 (0.012672)	0.098450 (0.000284)	0.14013 (0.000899)	0.668245 (0.011127)
40	0.100613 (0.000229)	0.12871 (0.000288)	0.488336 (0.007064)	0.098591 (0.000125)	0.138617 (0.000463)	0.664190 (0.006120)
80	0.099884 (0.000122)	0.12823 (0.000153)	0.477906 (0.003767)	0.099875 (0.000066)	0.13183 (0.000223)	0.65722 (0.000330)
k-loss function						
20	0.103914 (0.079860)	0.132883 (0.065613)	0.492997 (0.117630)	0.098706 (0.053801)	0.143959 (0.096716)	0.661045 (0.056740)
40	0.100417 (0.042904)	0.126631 (0.036344)	0.479637 (0.066377)	0.096926 (0.026903)	0.137768 (0.050705)	0.656704 (0.030449)
80	0.090126 (0.020845)	0.120206 (0.019280)	0.479205 (0.034129)	0.095139 (0.013585)	0.129998 (0.027754)	0.65398 (0.015298)

Table 12: B.Es and P.Rs under HNP.

	$(\lambda_1, \lambda_2, p_1) = (1, 1.2, 0.45)$			$(\lambda_1, \lambda_2, p_1) = (1, 1.2, 0.60)$		
n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	1.057620 (0.047775)	1.284300 (0.053174)	0.504983 (0.012746)	1.026430 (0.029657)	1.38323 (0.086231)	0.66699 (0.011204)
40	1.002990 (0.021704)	1.26675 (0.02980)	0.489816 (0.007035)	1.00297 (0.014489)	1.37417 (0.046812)	0.65957 (0.006215)
80	0.986514 (0.010515)	1.25707 (0.014785)	0.479164 (0.003809)	0.973682 (0.006896)	1.33346 (0.021813)	0.65331 (0.003328)
k-loss function						
20	1.03329 (0.079696)	1.27340 (0.066334)	0.492286 (0.118614)	0.97426 (0.052452)	1.43169 (0.095498)	0.66226 (0.05629)
40	0.99149 (0.042656)	1.27047 (0.037025)	0.481961 (0.065629)	0.95028 (0.027101)	1.34313 (0.051853)	0.65690 (0.030575)
80	0.94038 (0.022797)	1.19721 (0.020553)	0.475805 (0.035469)	0.95276 (0.014515)	1.29623 (0.028519)	0.63943 (0.015539)

Table 13: B.Es and P.Rs under HNP.

	$(\lambda_1, \lambda_2, p_1) = (10, 12, 0.45)$			$(\lambda_1, \lambda_2, p_1) = (10, 12, 0.60)$		
n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	8.57913 (2.610720)	9.88622 (2.823820)	0.500944 (0.012932)	8.89466 (2.008291)	9.35271 (3.59721)	0.655973 (0.011749)
40	9.06921 (1.610440)	10.74931 (1.926290)	0.481933 (0.007203)	9.24511 (1.196021)	10.50582 (2.687681)	0.648517 (0.006553)
80	9.705820 (1.001070)	11.53641 (1.250031)	0.47178 (0.003893)	9.83573 (0.619937)	11.64982 (1.821741)	0.634880 (0.003427)
k-loss function						
20	8.32958 (0.071460)	9.85215 (0.059791)	0.48777 (0.122091)	8.86194 (0.052413)	8.99842 (0.087606)	0.64393 (0.063768)
40	9.15985 (0.038853)	10.75050 (0.033029)	0.473142 (0.068366)	9.28602 (0.027532)	10.45031 (0.049513)	0.624652 (0.033815)
80	9.35556 (0.021005)	11.71821 (0.018572)	0.46751 (0.036733)	9.92177 (0.013671)	12.23450 (0.027962)	0.621553 (0.015927)

Table 14: B.Es and P.Rs under HNP.

n	$(\lambda_1, \lambda_2, p_1) = (0.1, 12, 0.45)$			$(\lambda_1, \lambda_2, p_1) = (0.1, 12, 0.60)$		
	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	0.096121 (0.000286)	10.68431 (2.774320)	0.540793 (0.0116544)	0.097722 (0.000215)	9.85081 (3.31084)	0.688516 (0.010065)
40	0.097104 (0.000129)	11.90812 (1.751123)	0.501625 (0.006348)	0.098664 (0.000098)	11.33812 (2.25579)	0.67825 (0.005516)
80	0.098902 (0.000061)	12.36352 (0.953057)	0.491149 (0.003318)	(0.953057) (0.000047)	12.61030 (1.407890)	0.662638 (0.002924)
k-loss function						
20	0.0955526 (0.063445)	10.53412 (0.050620)	0.528768 (0.092002)	0.092928 (0.045959)	9.77241 (0.072687)	0.680402 (0.047985)
40	0.092924 (0.031492)	11.79810 (0.025159)	0.515174 (0.050402)	0.092224 (0.022856)	11.37572 (0.036040)	0.673924 (0.025758)
80	0.092974 (0.015686)	12.53970 (0.012541)	0.507803 (0.026438)	0.092562 (0.011396)	12.56853 (0.017941)	0.670402 (0.013366)

Table 15: B.Es and P.Rs under HNP.

n	$(\lambda_1, \lambda_2, p_1) = (10, 0.12, 0.45)$			$(\lambda_1, \lambda_2, p_1) = (10, 0.12, 0.60)$		
	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	9.528260 (2.745210)	0.11130 (0.000306)	0.442311 (0.011576)	9.80686 (2.13478)	0.11439 (0.000467)	0.590134 (0.011352)
40	9.83054 (1.64125)	0.114079 (0.000153)	0.447209 (0.006185)	9.99308 (1.249880)	0.115369 (0.000224)	0.59738 (0.006221)
80	10.62651 (0.880273)	0.119944 (0.000075)	0.448934 (0.003196)	10.63520 (0.639962)	0.117354 (0.000102)	0.59898 (0.003264)
k-loss function						
20	9.23926 (0.063445)	0.11498 (0.050625)	0.42763 (0.139727)	9.66601 (0.045959)	0.112524 (0.072722)	0.57932 (0.074658)
40	10.4172 (0.031494)	0.11583 (0.025158)	0.42932 (0.077796)	10.48332 (0.022856)	0.117519 (0.036041)	0.568093 (0.040621)
80	10.26580 (0.015690)	0.119616 (0.012545)	0.439939 (0.041166)	10.12635 (0.011396)	0.118492 (0.017940)	0.562004 (0.02123)

Table 16: B.Es and P.Rs under SRGP.

n	$(\lambda_1, \lambda_2, p_1) = (0.1, 0.12, 0.10)$			$(\lambda_1, \lambda_2, p_1) = (0.1, 0.12, 0.90)$		
	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	0.123120 (0.000537)	0.130175 (0.000540)	0.110617 (0.003112)	0.113788 (0.000304)	0.145528 (0.000983)	0.952635 (0.018203)
40	0.113798 (0.000288)	1.232242 (0.000280)	0.109755 (0.001730)	0.102914 (0.000134)	0.140091 (0.000502)	0.936614 (0.009768)
80	0.107202 (0.000146)	0.122574 (0.000164)	0.109535 (0.000958)	0.091690 (0.000053)	0.134454 (0.000234)	0.908060 (0.004901)
k-loss function						
20	0.121472 (0.073200)	0.124597 (0.061649)	0.106202 (0.035720)	0.108939 (0.045742)	0.142804 (0.101007)	0.941435 (0.103248)
40	0.106447 (0.042600)	0.120557 (0.035373)	0.106340 (0.017537)	0.102529 (0.025685)	0.133449 (0.057690)	0.925560 (0.052115)
80	0.098012 (0.021784)	0.118962 (0.018397)	0.108869 (0.008399)	0.097899 (0.014861)	0.131776 (0.034645)	0.916164 (0.025686)

Table 17: B.Es and P.Rs under MP.

	$(\lambda_1, \lambda_2, p_1) = (0.1, 0.12, 0.1)$			$(\lambda_1, \lambda_2, p_1) = (0.1, 0.12, 0.90)$		
n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	0.116245 (0.000432)	0.136268 (0.000562)	0.114574 (0.002987)	0.105892 (0.000271)	0.153844 (0.000966)	0.964973 (0.016919)
40	0.101305 (0.000215)	0.130895 (0.000285)	0.114377 (0.001652)	0.102999 (0.000118)	0.144231 (0.000497)	0.961167 (0.009425)
80	0.099404 (0.000107)	0.128429 (0.000148)	0.113780 (0.000882)	0.091432 (0.000051)	0.139408 (0.000227)	0.953427 (0.004772)
k-loss function						
20	0.116934 (0.067402)	0.134636 (0.058094)	0.110546 (0.031147)	0.106838 (0.039625)	0.150683 (0.091032)	0.949959 (0.094836)
40	0.107934 (0.041281)	0.125204 (0.033657)	0.110349 (0.016793)	0.098675 (0.024918)	0.140716 (0.051476)	0.947144 (0.048098)
80	0.101593 (0.024216)	0.120876 (0.020117)	0.108915 (0.008639)	0.098063 (0.012480)	0.136228 (0.028386)	0.932468 (0.023670)

Table 18: B.Es and P.Rs under HNP.

	$(\lambda_1, \lambda_2, p_1) = (0.1, 0.12, 0.10)$			$(\lambda_1, \lambda_2, p_1) = (0.1, 0.12, , 0.9)$		
n	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
squared error loss function						
20	0.108036 (0.000496)	0.134402 (0.000578)	0.121323 (0.003017)	0.099435 (0.000275)	0.141531 (0.000971)	1.002368 (0.016691)
40	0.102625 (0.000247)	0.126136 (0.000276)	0.116270 (0.001682)	0.099577 (0.000121)	0.140003 (0.000500)	0.996285 (0.009180)
80	0.101882 (0.000132)	0.125665 (0.000147)	0.113787 (0.000897)	0.100874 (0.000064)	0.133148 (0.000241)	0.985830 (0.000495)
k-loss function						
20	0.105992 (0.086249)	0.130225 (0.062988)	0.117380 (0.028007)	0.099693 (0.052187)	0.145399 (0.104453)	0.991568 (0.085110)
40	0.102425 (0.046336)	0.124098 (0.034890)	0.114199 (0.015804)	0.097895 (0.026096)	0.139146 (0.054761)	0.985056 (0.045674)
80	0.091929 (0.022513)	0.117802 (0.018509)	0.114096 (0.008126)	0.096090 (0.013177)	0.131298 (0.029974)	0.980970 (0.022947)

Numerical results of the simulation study presented in Tables $1 - 18$ reveal interesting properties of the proposed Bayes estimators. The estimated values of the parameters converge to the true values, and amounts of posterior risks tend to decrease for lager choice of sample size. Another interesting point concerning the posterior risks of the estimates of (λ_1, λ_2) is that increasing (decreasing) the proportion of the component in mixture reduces (increases) the amount of the posterior risk for the estimates of λ_1.In addition, when $p_1 = 0.45$ and values of λ_i are relatively smaller i.e. for $(\lambda_1, \lambda_2) = (0.1, 0.12)$ and $(1, 1.2)$, the Bayes estimates assuming Maxwell prior are more precise than the rest of the informative priors, as the averaged posterior risks of the mixture components are smaller as compared to those under other priors. On the other hand, for quite larger values of parameters, i.e. for $(\lambda_1, \lambda_2) = (10, 12)$, the estimates under Maxwell prior are again observed to be more efficient than those under square root gamma and half normal priors, but degree of over/under estimation is higher under Maxwell prior. Moreover, in case where we take the significantly different values of the parameters, i.e. for $(\lambda_1, \lambda_2) = (0.1, 12)$,the estimates under square root gamma (with few exceptions) perform better than those under Maxwell and half normal priors. When $p_1 = 0.6$, the estimates under Maxwell prior using both loss functions are found to be the most efficient for relatively closer values of the parameters with few exceptions. And when the values of the parameters representing both components are assumed to be quite different, the square root gamma prior performs better than other priors. So, we can conclude that for the relatively closer choice of the parametric values, the estimates under

Maxwell prior are the best under both loss functions with few exceptions. Similarly, for highly variant choice of the parametric values for the two components of the mixture, the performance of square root gamma prior seems to better than other priors.

The Bayes estimates of the lifetime parameters are over/under-estimated but the size of over/under-estimation is greater under squared error loss function. On the other hand, estimates of the mixing proportion parameter have mixed behavior sometimes over-estimated and sometimes under-estimated but the Bayes estimates under half normal prior are much closer to the true parametric value.

In comparison of loss functions it has been assessed that the magnitudes of posterior risks under squared error loss function are smaller than those under k-loss function for smaller choice of true parametric values. However, for larger values of the true parametric values, the k-loss function produces the better results. It should also be mentioned here that the squared error loss function produces better convergence than k-loss function. It may also be mentioned here that because of space restriction, results for all the variations in the parameters are not shown here. Only selected results are included.

6 Real Data Analysis

In this section, we have analyzed real data sets to illustrate the methodology discussed in the previous sections. In order to show the usefulness of the proposed mixture model, we applied the findings of the paper to the survival times (in years) of a group of patients given chemotherapy treatment; the data has been reported by Bekker et. al. [21]. We have used the Kolmogorov-Smirnov and chi square tests to see whether the data follow the Rayleigh distribution. These tests say that the data follow the Rayleigh distribution at 5% level of significance with p-values 0.2170 and 0.2681 respectively. The data consisting of 46 survival times (in years) for 46 patients are:

Table 19: Survival times (in years) of patients given chemotherapy treatment

0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529,
0.534, 0.540, 0.570, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589,
2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033

Now we consider the case when the data are doubly type II censored. Data are randomly grouped into two sets using probabilistic mixing for $p_1 = 0.60$.

Table 20: Doubly censored mixture real life data regarding survival times of patients given chemotherapy treatment

Population-I	Population-II
0.197, 0.534, 0.115, 0.296, 0.121, 0.466, 0.529, 1.447, 0.863, 0.132, 0.395, 0.696, 2.825, 3.658, 3.978, 3.743, 2.343, 2.178, 0.540, 4.003,1.553, 1.485, 2.83, 2.416	0.260, 1.099, 0.501, 0.458, 0.641, 0.334, 0.570, 0.164, 0.203, 0.282, 0.047, 1.271, 1.589, 1.326, 0.841, 2.444

The following characteristics are extracted from censored data for the analysis of mixture model:
$p_1 = 0.6$

$n = 40$, $r = 5$, $r_1 = 2$, $r_2 = 3$, $n - r = 9$, $s = 36$, $s_1 = 22$, $s_2 = 14$, $n_1 = 24$, $n_2 = 16, x_{r_1} = 0.121$, $x_{s_1} = 3.978$, $x_{r_2} = 0.203$, $x_{s_2} = 2.444$, ${}^{s_1}_{i=r_1} x^2_{1(i)} = 84.6037$ and ${}^{s_2}_{i=r_2} x^2_{2(i)} = 15.2833$.

The similar methodology has been employed when $p_1 = 0.45$.

$n = 40$, $r = 5$, $r_1 = 2$, $r_2 = 3$, $n - r = 9$, $s = 36$, $s_1 = 16$, $s_2 = 20$, $n_1 = 18$, $n_2 = 22, x_{r_1} = 0.121$, $x_{s_1} = 3.658$, $x_{r_2} = 0.164$, $x_{s_2} = 3.978$, ${}^{s_1}_{i=r_1} x^2_{1(i)} = 48.704$ and ${}^{s_2}_{i=r_2} x^2_{2(i)} = 37.1999$.

Bayes estimates are obtained assuming informative priors under squared error loss function, and k-loss function.

Table 21: B.Es and P.Rs under squared error loss function, and k-loss function for real data set.

Priors	**squared error loss function**			**k-loss function**		
$p_1 = 0.6$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
Square root prior	0.392190	0.980312	0.665364	0.389475	0.843369	0.660755
	(0.001681)	(0.016213)	(0.005625)	(0.022397)	(0.035861)	(0.027171)
Maxwell Prior	0.39643	0.996689	0.664565	0.394281	0.988384	0.660083
	(0.001680)	(0.016199)	(0.005436)	(0.02186)	(0.033752)	(0.024248)
Half Normal Prior	0.387845	0.962886	0.678225	0.385647	0.954269	0.673898
	(0.001683)	(0.016202)	(0.005553)	(0.022862)	(0.036283)	(0.025765)
$p_1 = 0.45$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
Square root prior	0.405184	0.74061	0.502853	0.401992	0.734712	0.49585
	(0.002667)	(0.008064)	(0.006596)	(0.031881)	(0.032240)	(0.056882)
Maxwell Prior	0.39419	0.749978	0.507659	0.391150	0.74476	0.500983
	(0.002400)	(0.007449)	(0.006380)	(0.031212)	(0.028121)	(0.053662)
Half Normal Prior	0.382825	0.731263	0.52014	0.379689	0.725883	0.513596
	(0.002599)	(0.007484)	(0.006417)	(0.033171)	(0.029756)	(0.051374)

The findings from the analysis are in close accordance with those of simulation study, suggesting the preference of Maxwell prior along with squared error loss function.

7 Conclusions

In this article, the Bayesian inference of the mixture of Rayleigh model under doubly type II censoring has been considered assuming informative priors. The simulation study has displayed some interesting properties of the Bayes estimates. It is noted in each case the posterior risks of estimates of lifetime parameters are reduced as the sample size increases. The results indicated that for the relatively closer choice of the parametric values, the estimates under Maxwell prior are the best for almost all cases. On the other hand, for significantly different choice of the parametric values for the two components of the mixture, the performance of square root gamma prior seems to better than other priors. The performance of the squared error loss function is better than k-loss function for smaller choice of true parametric values. However, for larger values of the true parametric values, the k-loss function produces the better results. It should also be mentioned here that the squared error loss function produces better convergence than k-loss function for almost all the cases. The real life example further strengthened the findings from the simulation study. The study can further be extended by considering some other censoring techniques, and using some more flexible probability distribution.

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