Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol.47(2)(2015) pp. 35-47

Fully Developed Liquid Layer Flow Over a Convex Corner Considering Surface Tension Effects Using Numerical Methods

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Received: 02 April, 2015 / Accepted: 22 July, 2015 / Published online: 17 September, 2015

Abstract. . In this paper, liquid layer flow considering surface tension effect, encountering a convex corner has been discussed. The flow profile far upstream is fully developed. Half-Poiseuille gives exact solution far upstream. Due to small disturbance of $0(\delta)$, matched asymptotic technique has been opted to get the linearized solutions far downstream. The obtained equations have been solved numerically using Chebyshev Collocation method in collaboration with finite difference scheme. These results have been verified via computational work. The aforementioned method is beneficial, as we have successfully plotted graphs for the cases $s = 0.1$ and 0.2. Eventually, we have compared obtained results with the Gajjar's results [3]

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Half-poiseuille flow, Matched Asymptotic technique, Chebyshev Collocation method.

1. Introduction and main results

In this paper, we will discuss the liquid layer flow over a convex corner with the effect of surface tension. Free surface flows are related to various practical problems for instance; river flows, flow through part filled pipes etc. Previously a lot of work has been carried out by J. S Gajjar [3] about the behavior of liquid layer flow over a convex corner facing obstacles in free surface flow without taking into account surface tension effects. Gajjar has calculated analytic and numerical solutions for this problem, but for small corner angles by using matched asymptotic technique. He got the central idea of his work from the triple deck type structure to solve the problem which was first formulated by Stewartson [14]. Ludwig Prandtl (1875-1953) , German mathematician for the first time presented his paper in 1904 on boundary layer concept. Triple deck theory is an extension of the classical boundary layer theory proposed by Prandtl [10]. Prandtl, and his many students proceeded his work to formulate the essential principles of airfoil and propeller operation [11] . Recently, attention has been focused on the behavior of liquid layer flows under the effect of an electric field, as discussed in Tseluiko and Papageorgiou and also in Sadia farid and Gajjar's paper [18] [2]. Also, the work has been carried out on the behavior of boundary layers in external and internal flows by Stewartson [14] [15] and Smith [13].Moreover, transonic bi-convex corner flows have been discussed by Kung Ming-Chung, Po-Hsiung. Chang, Keh-Chin Chang, Frank K. Lu [1]. The asymptotic technique converts the corresponding equations into boundary layer equations along with no slip and matching conditions and also with interaction law i.e., $p = -sA - (1 - 1/C)A''$, including surface tension effects. Here, "s" represents the scaled measure of the angle of inclination of initial plane and "C" is capillary number. An extensive amount of work on asymptotic theory has also been done by Sychev [16]. With the passage of time, it became evident that his results were insufficient if corner angles are to be increased. In this particular situation, we need to smooth the corner angles so, that we could obtain fine results. We will solve this problem using numerical techniques such as, Chebyshev Collocation method and Finite difference method [9]. Primitive variables have been used instead of stream functions such as in [5] [8]. Use of primitive variables makes it possible to apply the adopted method for 3-D problems. The problem on undeveloped profile has been done by Gajjar [3].

1.1. **Development of the problem.** In this problem we will consider a 2D steady laminar flow of a liquid layer passing through a convex corner.. The Reynolds number is assumed to be large (i.e., $R >> 1$) and the flow profile is fully developed. The crucial part is to consider surface tension effects. To be precise, we focus on the flow of liquid layer down an inclined plane having angles $\beta^*, \beta^* + \alpha^*$ to the horizontal, upstream and downstream respectively.(See Figure 1). We note that far upstream the simple solution given by the half-Poiseuille form satisfies the Navier-Stokes equations exactly and that it is reasonable to expect that the flow sufficiently far downstream is also given by a similar half-Poiseuille profile. Global arguments e.g. mass flux considerations, can then be used to deduce useful relations between the upstream and downstream velocities and depths. However, we are primarily interested in how the boundary layer and the free surface first anticipate the presence of an obstacle such as a corner when the oncoming profile is fully developed, and our interest therefore will be centered more on the linearized solution of the problem. Before discussions of scalings it is necessary to throw some light on few important assumptions. If we say that distortion of bed is locally small of $0(\delta)$ say $(\delta << 1)$, then obviously the fluid is displaced by the same amount of $0(\delta)$, since the oncoming profile is fully developed and the viscous effects extend up to the free surface. It means that the displacement of the free surface is of the same order as that of the bed, i.e. that the basic flow has not been modified by an $0(1)$ amount due to the presence of an infinitesimally small obstacle, and this assumption is crucial to the analysis and scalings given below. We consider how fluid tackles the corner. It is useful to note the upstream behavior in channel flow theory [12], there a displacement of the core induces a pressure gradient across the channel which in turn drives the viscous wall layers and provides a self-sustaining mechanism for the upstream response. Here the fully developed profile can be considered as half a channel profile and essentially the same ideas as those in channel flow theory are applied further in this section

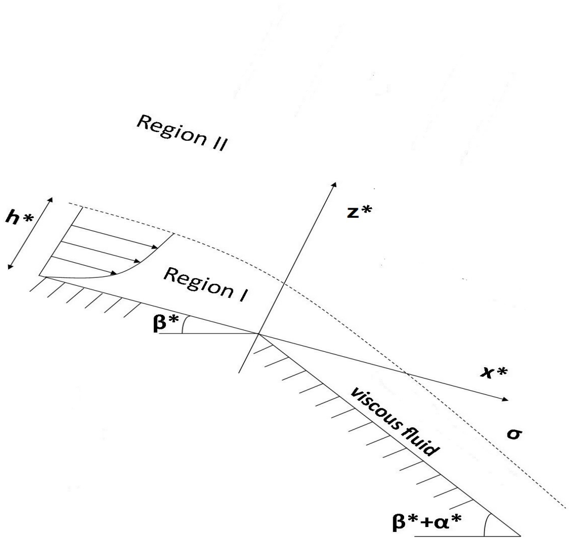

FIGURE 1. Problem Model.

below. The basic equations in Region I are incompressible Navier-Stokes equations.

$$uu_x + wu_z = -\frac{p_x}{\rho} + \nu(u_{xx} + u_{zz}) + g\sin\beta^*, \tag{1. 1}$$

$$uw_x + ww_z = -\frac{p_z}{\rho} + \nu(w_{xx} + w_{zz}) - g\cos\beta^*, \tag{1. 2}$$

$$u_x + w_z = 0. \tag{1. 3}$$

The boundary conditions are;

- No slip condition holds at the wall, $u|_{z=0} = 0, w|_{z=0} = 0$.
- The kinematic condition is $w = uh_x$.

At the interface $z = h(x)$ we must satisfy the kinematic condition and balance of normal and tangential stresses. The outward normal (n) and unit tangent (t) vectors at any point on $z = h(x)$ are;

$$n = \frac{(-h_x, 1)}{\sqrt{1 + h_x^2}} \tag{1. 4}$$

And,

$$t = \frac{(1, h_x)}{\sqrt{1 + h_x^2}} \tag{1. 5}$$

respectively. An exact solution of the problem exists and is given by [4] [5] [6],

$$U_B^* = \frac{g\sin\beta^*}{2\nu}(2h_0 z - z^2), \tag{1. 6}$$

$$W_B^* = 0, \tag{1. 7}$$

$$P_B^* = P_{atm} - \rho g(z - h_0)\cos\beta^*. \tag{1. 8}$$

U_B^* and W_B^* represents velocities in x and z direction respectively. Where P_B^* is the pressure and P_{atm} shows atmospheric pressure, "ν" is kinematic viscosity of the fluid, "ρ" represents the density and "g" is acceleration due to the gravity. We observe that velocity profile is parabolic in nature. We non-dimensionalize with respect to the length scale "h" using the following dimensionless quantities,

$$\acute{x} = \frac{x}{h_0}, \quad \acute{z} = \frac{z}{h_0}, \quad \acute{u} = \frac{u}{U_c}, \quad \acute{h} = \frac{h}{h_0}, \quad \acute{w} = \frac{w}{U_c}, \quad \acute{p} = \frac{p}{\rho U_c^2}, \tag{1. 9}$$

Here U_c represents characteristic velocity and $U_c = 2U_B^*(h_0)$. Substituting these into Navier-Stokes equations leads to:

$$uu_x + wu_z = -p_x + \frac{1}{R}(u_{xx} + u_{zz}) + \frac{2}{R}, \tag{1. 10}$$

$$uw_x + ww_z = -p_z + \frac{1}{R}(w_{xx} + w_{zz}) - \frac{2}{R}\cot\beta^*, \tag{1. 11}$$

Along with the boundary conditions of no slip, kinematic condition and the stress of normal and tangent balance that is

$$u = 0, \quad w = 0, \quad atz = 0 \tag{1. 12}$$

$$w = uh_x,$$

$$(1 - h_x^2)(u_z + w_x) + 4h_x w_z = 0,$$

$$\Big(\frac{1 + h_x^2}{1 - h_x^2}\Big)w_z + \frac{1}{2}R(\acute{p}_{atm} - p) = \frac{\sigma h_{xx}}{2C(1 + h_x^2)^{\frac{3}{2}}} \tag{1. 13}$$

Since,

$$U_c = \frac{gh_0^2}{2\nu}\sin\beta^*$$

So, the other dimensionless parameters are Reynolds number R (measuring the ratio of viscous forces to inertial forces), Capillary number C (measuring the ratio of viscous to capillary forces) and are given by

$$R = \frac{U_c h_0}{\nu} = \frac{gh_0^3 sin\beta^*}{\nu^2}$$

$$C = \frac{U_c \mu}{\sigma} = \frac{\rho g h_0^2 sin\beta^*}{\sigma}$$

Similarly, in non-dimensional form the upstream profile become,

$$U_B = z - \frac{1}{2}z^2, \tag{1. 14}$$

$$W_B = 0, \tag{1. 15}$$

$$P_B = \frac{(1 - z)}{Rtan\beta^*} \tag{1. 16}$$

Consider a perturbation of $0(\delta)$ to the oncoming flow. Gajjar postulates that the interaction here occurs because of two factors , (a) the displacement of the free surface which induces a pressure of $0(\delta/Rtan\beta^*)$ to counteract the change in the hydrostatic pressure and (b) the displacement of the free surface which sets up a transverse pressure gradient of $0(\delta)$, as in channel flow Smith [3] [12]. We consider the case where (a) and (b) are comparable. From the z momentum equation the pressure is $0(\frac{\delta}{L^2})$, where "L" is the unknown stream wise length scale and assumed to be large. Hence;

$$\lambda \sim \frac{\delta}{Rtan\beta^*} \sim \frac{\delta}{L^2}, \tag{1. 17}$$

The rest of the argument is same as given by Smith [12] and further details may be found there. Viscous-inertial and pressure balances imply;

$$\frac{\delta^2}{L} \sim \frac{\lambda}{L} \sim \frac{\delta}{R\delta^2}, \quad (1.\ 18)$$

These equations (1. 17) and (1. 18) give the crucial scalings:

$$\delta \sim R, L \sim R^{\frac{-2}{7}}, \lambda \sim R^{\frac{-4}{7}}, tan\beta^* \sim R^{\frac{-5}{7}}, \quad (1.\ 19)$$

With regard to (a) and (b) above, if $\beta^* << R^{\frac{-5}{7}}$ then the hydrostatic pressure is important and if $\beta^* >> R^{\frac{-5}{7}}$ then the transverse pressure is dominant. δ is of order "R". It is convenient to set $\epsilon = R^{\frac{-5}{7}}$, $tan\beta^* = R^{\frac{-5}{7}}\bar{s}$and $z = \alpha\epsilon^2 F(\epsilon x)$ where $\bar{F}(x)$ is the reduced wall shape, and $\bar{F}(x) \to x$ for $x >> 1$. Hence, far downstream the angle α^* is of $0(R^{\frac{-3}{7}})$.

1.2. **Solution Of the problem.** Let $\bar{X} = \epsilon x$ be the scaled stream-wise co-ordinate and $z = 1 + \epsilon^2\eta + ...$ be the free surface. Then the flow in layer I where $z\ 0(1)$ is

$$u \sim U_B + \epsilon^2 u_1 + o(\epsilon^4), \quad (1.\ 20)$$

$$w \sim \epsilon^3 w_1 + o(e^5), \quad (1.\ 21)$$

$$p \sim \epsilon^2 \bar{s}(1-y) + \epsilon^4 p_1 + o(\epsilon^6). \quad (1.\ 22)$$

Here $u_B = z - z^2/2$ is the basic flow.Substituting into dimensionless NS equations (1. 1)–(1. 3),) gives the simple displacement solution i.e.

$$u_1 = \bar{A}(\bar{X})\bar{U}_{Bz}, \quad (1.\ 23)$$

$$w = -\bar{A}'(\bar{X})U_B, , \quad (1.\ 24)$$

$$p_1 = \bar{P}(\bar{X}) + \bar{A}''(\bar{X})\int_0^z U_B^2 dt. \quad (1.\ 25)$$

Here "t" is taken as variable of integration.$\bar{A}(\bar{X})$, $\bar{P}(\bar{X})$ are unknown and both approaches to zero as $\bar{X} \to -\infty$. Implementing the conditions at the free surface $z = 1 + \epsilon^2\bar{\eta}$, gives

$$\bar{\eta}(\bar{X}) = \bar{A}(\bar{X}), \quad (1.\ 26)$$

$$\bar{P}(\bar{X}) = -\bar{s}\bar{\eta}(\bar{X}) + \frac{1}{C}\bar{A}\bar{X} - \bar{A}''(\gamma), \quad (1.\ 27)$$

$$\gamma = \int_0^1 U_B^2 dz, \quad (1.\ 28)$$

Now we need a viscous layer of thickness $O(\epsilon^2)$ to reduce the slip velocity in equation (1. 20 - 1. 22) to zero, and therefore $z = \epsilon^2\bar{Z}, \bar{Z} \sim 0(1)$

$$u \sim \epsilon^2\bar{U}_1 + ..., \quad (1.\ 29)$$

$$w \sim \epsilon^5\bar{w} + ..., \quad (1.\ 30)$$

$$p \sim \epsilon^2\bar{s} + \epsilon^4(-s\bar{z} + \bar{P}_1) + ..., . \quad (1.\ 31)$$

So,$\bar{U}_1$, $\bar{W}_1$, $\bar{P}_1$ satisfy the boundary layer equations:

$$\bar{U}_{1\bar{X}} + \bar{W}_{1\bar{Z}} = 0, \quad (1.\ 32)$$

$$\bar{U}_1\bar{U}_{1\bar{X}} + \bar{W}_1\bar{U}_{1\bar{Z}} = -\bar{P}_{1\bar{X}} + \bar{U}_{\bar{Z}\bar{Z}}, \quad (1.\ 33)$$

$$\bar{P}_{1\bar{Z}} = 0. \quad (1.\ 34)$$

along with the boundary conditions

$$\bar{U}_1 = \bar{W}_1 = 0 \qquad at \qquad \bar{Z} = -\alpha \bar{F}(\bar{X}), \tag{1. 35}$$

$$\bar{U}_1 \to \bar{Z} + \bar{A}(\bar{X}) \quad as \quad \bar{Z} \to -\infty, \tag{1. 36}$$

$$\bar{U}_1 \to \bar{Y} \quad as \quad \bar{X} \to \infty, \tag{1. 37}$$

$$\bar{P}_1 = \bar{P}(\bar{X}). \tag{1. 38}$$

To match with I, and for no slip at the wall. We can set the factor γ equal to unity in equation (1. 27) with the normalization [3] [4]:

$$\begin{aligned}[U, W, X, Y, A, \eta, P, s, F(X)] =& [\gamma^{\frac{-1}{7}} \bar{U}_1, \gamma^{\frac{1}{7}} \bar{W}_1, \gamma^{\frac{-3}{7}} \bar{X}, \gamma^{\frac{-1}{7}} \bar{Y}, \\ & \gamma^{\frac{-1}{7}} \bar{A}, \gamma^{\frac{-1}{7}} \bar{\eta}, \gamma^{\frac{-2}{7}} \bar{P}_1, \gamma^{\frac{-1}{7}} \bar{s}, \gamma^{\frac{-1}{7}} \bar{F}(\bar{X})]\end{aligned} \tag{1. 39}$$

After applying Prandtl transformation $Y = Z + \alpha F(X)$,$V = W + \alpha F'(U)$ to the set of equations (1. 35) $-$ (1. 38), we obtain

$$U_X + V_Y = 0, \tag{1. 40}$$

$$UU_X + VU_Y = -P_X + U_{YY} \quad , P = P(X) \tag{1. 41}$$

$$U = V = 0, \quad on \quad Y = 0, \tag{1. 42}$$

$$U \to Y + A - \alpha F(X) \quad as \quad Y \to \infty, \tag{1. 43}$$

$$U \to Y \quad as \quad X \to -\infty, \tag{1. 44}$$

$$P = -sA - (1 - \frac{1}{C})A" \quad as \quad , \eta = -A. \tag{1. 45}$$

Our aim is to obtain the linearized solutions of the system (1. 40) to (1. 45) numerically. In this paper, we have obtained the results for pressure (P), displacement (A) and skin friction $(\tau = (U_y)_{y=0}$.

2. Numerical Method

The equations under consideration have been briefed earlier. The method adopted in this paper is shaped in the form of primitive variables in which finite difference method is applied in X-direction and Chebyshev collocation method in Y-direction. In Chebyshev collocation method Y domain is mapped into Chebyshev domain $z \in [-1, 1]$.Using standard differentiation matrix we deal with derivatives of Y in discrete form. Then to tackle with continuity equation it will be converted first into discrete form via inverse standard differentiation matrix B and then the obtained result is substituted into momentum equation. This method is based on iterative method known as Newton linearization. Sum of an initial guess and small correction is provided to each unknown parameter. Products of correction can be neglected. And the iteration continues in the same fashion until we get small correction. It depicts that the initial guess has converged to final equation of the solution [5] [9].

The X domain is discretized into a uniform grid defined over $[x_{min}, x_{max}]$ given as,

$$X_f = X_{min} + (f - 1)\Delta X \quad , f = 1, 2, 3, ..., n \tag{2. 46}$$

And to apply Chebyshev method Y domain is first mapped into $-1 < Z < 1$ via,

$$Y = \frac{1}{2} Y_{max}(Z + 1) \tag{2. 47}$$

After discretization we get,

$$Y = \frac{1}{2} Y_{max}(z_r + 1) \tag{2. 48}$$

Where z_r is set of Chebychev points given as,

$$z_r = -\cos(\frac{r\pi}{M}), \qquad r = 0, 1, ., M \tag{2. 49}$$

By the construction of mesh, we define

$$U(X_f, Y(z_r)) = U_{fr}, V(X_f, Y(z_r)) = V_{fr}, P(X_f) = P_f, A(X_f) = A_f, F(X_f) = F_f \tag{2. 50}$$

For the numerical calculations we have considered the wall shaped function as $F(X) = \frac{\alpha}{2}(X + (\sqrt{x^2 + r^2})$, with "$r$" as smoothing convex corner co-efficient.We will use the following matrix operation.

$$\underline{\phi}^{'} = B_M \underline{\phi} \tag{2. 51}$$

Where $\underline{\phi} = (\phi_1, \phi_2, ..., \phi_M)^T, \underline{\phi}^{'} = (\phi_1^{'}, \phi_2^{'}, ..., \phi_M^{'})^T$ and B_M is $(M+1) \times (M+1)$ matrix [17]. To explain the construction of B_M, the cases $M = 1$ and $M = 2$ are now evaluated.For $M = 1$, the chebychev points are given by $z_0 = -1$ and $z_1 = 1$.The polynomial $p(z)$ is formed using Lagrangian interpolation and is given by,

$$p(z) = \frac{1}{2}(1-z)\phi_0 + \frac{1}{2}(1+z)\phi_1 \tag{2. 52}$$

Differentiating this gives,

$$p^{'}(z) = \frac{1}{2}\phi_0 + \frac{1}{2}\phi_1 \tag{2. 53}$$

Now it is obvious that above expression can be put into he form of equation (2. 51) with the differentiation matrix taking the form

$$M_1 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Same is the case with M=2. Now arranging from 0 to M,

$$(B_M)_{00} = -\frac{(2M^2+1)}{6}, \qquad (B_M)_{MM} = \frac{(2M^2+1)}{6}, \tag{2. 54}$$

$$(B_M)_{rr} = \frac{(-z_r)}{(2(1-z_r^2))}, \qquad r = 1, 2, ..., M-1 \tag{2. 55}$$

$$(B_M)_{ir} = \frac{c_i(-1)^{(i+r)}}{c_i(z_i - z_r)}, \qquad i \neq r, i, r = 0, 1, ..., M, \tag{2. 56}$$

$$C_i = \begin{cases} 2\,, & i = 0, M \\ 1\,, & \text{otherwise} \end{cases} \tag{2. 57}$$

U is used as variable and n^{th} Y derivative is calculated in discretized manner via,

$$(\frac{\partial^n U}{\partial Y^n}) = (\frac{2}{Y_{max}^n}) \sum_{(k=0)}^{M} (B_M^n)_{fk} U_{kr} \tag{2. 58}$$

This is going to be used such as $(B^n U)_{fr}$.The Continuity equation can be reorganized as, $V = -\int_0^Y U_X dY$ and inverse of B_M is here

$$I_M = (\frac{Y_{max}}{2}) B_M^{-1} \tag{2. 59}$$

The continuity equation can be rewritten as

$$\underline{V_f} = -\frac{1}{2\Delta X} I_M(\underline{U_{f-2}} - 4\underline{U_{f-1}} + 3\underline{U_f}), \tag{2. 60}$$

Using second order backward difference scheme, momentum equation can be expressed in the following form

$$\frac{1}{2\Delta X}U_{fr}(U_{f-2r} - 4U_{f-1r} + 3U_{fr}) + V_{fr}(BU)_{fr} = -\frac{1}{2\Delta X}(P_{f-2} - 4P_{f-1} + 3P_f) + (B^2U)_{fr}, \quad 3 \leq f \leq n-1 \quad (2.\ 61)$$

Newton linearization is the foundation of this method. So, we will linearize the above momentum equation by using following equations each of which is the sum of initial guess and small correction.

$$U_{fr} = u_{fr} + J_{fr}, \qquad V_{fr} = v_{fr} + K_{fr}, \quad (2.\ 62)$$

$$P_f = p_f + \bar{p_f}, \qquad A_f = a_f + \bar{a_f}. \quad (2.\ 63)$$

Substitute above equations (2. 62)-(2. 63) into (2. 61) and neglecting quadratically small terms, we acquired,

$$\frac{1}{2\Delta X}u_{fr}(J_{f-2r} - 4J_{f-1r} + 3J_{fr}) + \frac{1}{2\Delta X}J_{fr}(u_{f-2r} - 4u_{f-1r} + 3u_{fr}) + V_{fr}(BJ)_{fr} + K_{fr}(Bu)_{fr} + \frac{1}{2\Delta X}(\bar{p}_{f-2} - 4\bar{p}_{f-1} + 3\bar{p_f}) - (B^2J)_{fr}) = R_fr \quad ,3 \leq f \leq n-1 \quad (2.\ 64)$$

$$R_{fr} = -\frac{1}{2\Delta X}u_{fr}(u_{f-2r} - 4u_{f-1r} + 3u_{fr}) - v_{fr} - \frac{1}{2\Delta X}(p_{f-2} - 4p_{f-1} + 3p_f) + (B^2U)_{fr}, \quad (2.\ 65)$$

Similarly

$$\underline{v}_f = -\frac{1}{2\Delta X}I_M(\underline{u}_{f-2} - \underline{4u}_{f-1} + 3\underline{u}_f), \quad (2.\ 66)$$

$$\underline{K}_f = -\frac{1}{2\Delta X}I_M(\underline{J}_{f-2} - \underline{4J}_{f-1} + 3\underline{J}_f). \quad (2.\ 67)$$

Combining relevant matrix equations, we get the matrix equation,

$$J_f\underline{\phi}_{f-2} + L_f\underline{\phi}_{f-1} + C_f\underline{\phi}_f + E_f\underline{\phi}_{f+1} = \underline{\bar{R}_f} \quad (2.\ 68)$$

The no slip condition at $Z = 0$ is $U(X, 0) = o, U_{f,0} = 0, u_{f0} + J_{f0} = 0, J_{f0} = -u_{f0}$. The condition at $Z = \infty$ for $(M+1)^{th}$ row is given $J_{fM} - \bar{a_f} = Z_M + a_f - u_{fM} - \alpha F_f$ and for $(M + 2)^{nd}$ row $(BJ)_{fM} \to 1 - (BU)_{fM}$ respectively.Ultimately pressure displacement law is implemented into the method through row $(M + 3)^{rd}$ of (2. 68) is

$$P_f = -sA_f - (1 - \frac{1}{C})A_f^{''} \quad (2.\ 69)$$

$$\bar{p}_f + s\bar{a}_f + (\frac{C-1}{C\Delta X^2})[\bar{a}_{f+1} - 2\bar{a}_f - \bar{a}_{f-1}] = -p_f - sa_f - (\frac{C-1}{C\Delta X^2})[a_{f+1} - 2a_f - a_{f-1}] \quad (2.\ 70)$$

This method is complete for $3 \leq f \leq n-1$. we will repeat the same process of discretization and linearization for $f = 2$, but this time we are going to use second order difference scheme instead of three point backward difference method. So, after doing linearization we get,

$$\frac{1}{2\Delta X}(u_{2r} + u_{1r})(G_{2r} - G_{1r}) + \frac{1}{2\Delta X}(J_{2r} - G_{1r})(u_{2r} - u_{1r}) + v_{\frac{3}{2}r}((BJ)_{2r} + (BJ)_{1r}) + H_{\frac{3}{2}r}((Bu)_{2r} + (Bu)_{1r}) + (\frac{\bar{P}_2 - \bar{P}_1}{\Delta X}) - \frac{1}{2}((B^2J)_{2r} + (B^2J_{1r})) = R_{2r}. \quad (2.\ 71)$$

Where, $R_{2r} = -\frac{1}{2}(u_{2r} + u_{1r})\frac{1}{\Delta X}(u_{2r} - u_{1r}) - v_{\frac{3}{2}}\frac{1}{2}((Bu)_{2r} + (Bu)_{1r})...\frac{p_2-p_1}{\Delta X} + \frac{1}{2}((B^2u_{2r}) + (B^2u_{1r}))$

This can be written in the form of (2. 68) and then we get matrices for $J_f, B_f, C_f, E_f and \underline{\bar{R}}_f$ same as above. For $f = n$, we linearize this problem on the same lines as before, but again, we will imply different discretization scheme. Since, additional node is required outside the domain so second order backward differencing scheme is used for A'' term. So, after linearization the obtained pressure displacement expression turns to be,

$$\bar{p_f} + s\bar{a}_f + (\frac{C-1}{C\Delta X^2})[\bar{a}_{f-3} - 4\bar{a}_{f-2} - 5\bar{a}_{f-1} - 2\bar{a}_f] = -p_f - sa_f - (\frac{C-1}{C\Delta X^2}) [a_{f-3} - 4a_{f-2} + 5a_{f-1} - 2a_f] \quad (2.\ 72)$$

Since, it carries an extra matrix to incorporate the term at the $(f-3)$ station. So for $f = n$ the matrix system takes the form

$$K_f\underline{\phi_{f-3}} + J_f\underline{\phi_{f-2}} + L_f\underline{\phi_{f-1}} + C_f\underline{\phi_f} = \underline{\bar{R_f}} \quad (2.\ 73)$$

We have calculated sparse matrices for the system in (2. 68) for all $f = n$. The completed reduced form of matrix is

$$\begin{pmatrix} C_1 & E_1 & 0_{M+3} & 0_{M+3} & 0_{M+3} & 0_{M+3} & \cdots & 0_{M+3} \\ L_2 & C_2 & E_2 & 0_{M+3} & 0_{M+3} & 0_{M+3} & \cdots & 0_{M+3} \\ J_3 & L_3 & C_3 & E_3 & 0_{M+3} & 0_{M+3} & \cdots & 0_{M+3} \\ 0_{M+3} & J_4 & L_4 & C_4 & E_4 & 0_{M+3} & \cdots & 0_{M+3} \\ 0_{M+3} & 0_{M+3} & J_5 & L_5 & C_5 & E_5 & \cdots & 0_{M+3} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0_{M+3} & 0_{M+3} & \cdots & 0_{M+3} & K_n & J_n & L_n & C_n \end{pmatrix} \begin{pmatrix} \underline{\psi_1} \\ \underline{\psi_2} \\ \underline{\psi_3} \\ \underline{\psi_4} \\ \underline{\psi_5} \\ \vdots \\ \vdots \\ \underline{\psi_n} \end{pmatrix} = \begin{pmatrix} \underline{\bar{R_1}} \\ \underline{\bar{R_2}} \\ \underline{\bar{R_3}} \\ \underline{\bar{R_4}} \\ \underline{\bar{R_5}} \\ \vdots \\ \vdots \\ \underline{\bar{R_n}} \end{pmatrix}$$

Where $0_{M+3} \times 0_{M+3}$ matrix of Zeroes.

2.1. **Results and Dscussions.** Linearized solutions of numerical method have been obtained by using Matlab solver. We have taken $s = 0.1, 0.2$and $C = 5$ in computational work while considering grid size as 60 by 1001. So, graphs for linearized solutions of pressure (p), displacement (a) and skin friction (τ) have been plotted. An example of this can be seen as in [7].

vspace*4pc

2.2. **Comparison.** To check the validity, the results obtained are compared with Gajjar [3].When $C >> 1$, the surface tension effects are very low and can be neglected. This is also supported by the graphs (8- 9)in which our results are compared with Gajjar [3] results and they show good agreement.

After comparing our results with the problem of Gajjar, it is clear from the obtained graphs, that if we take $C >> 1$ then obtained graphs show good agreement with the plotted graphs of Gajjar [3]. So, if C is very large then we can ignore surface tension effects. Similarly, if we consider $S = 0$, along with $C >> 1$, then also it matches with the smith's results [8]. Also, rich set of new problems (e.g. stability, extensions to undeveloped flow...) is being investigated.

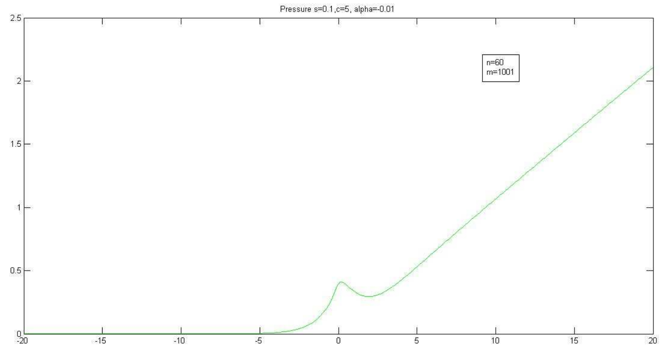

FIGURE 2. Plot of Pressure $\frac{p}{\alpha}$ for $C = 5$ and $s = 0.1$, $-\alpha = 0.01$.

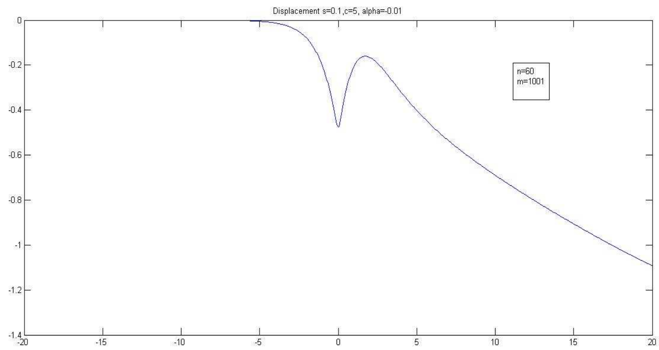

FIGURE 3. Plot of Displacement $\frac{A}{\alpha}$ for $C = 5$ and $s = 0.1$, $-\alpha = 0.01$.

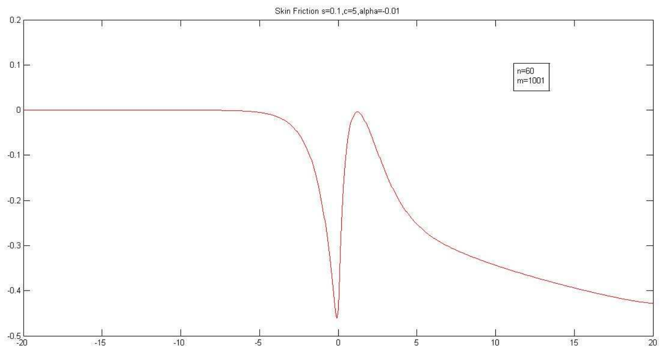

FIGURE 4. Plot of Skin Friction $\frac{(\tau-1)}{\alpha}$ for $C = 5$, $s = 0.1$ and $-\alpha = 0.01$.

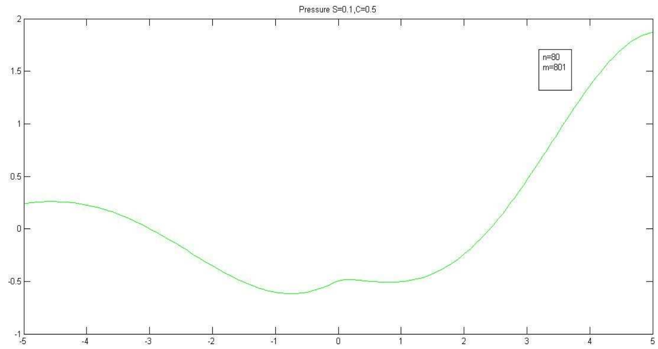

FIGURE 5. Plot of Pressure $\frac{p}{\alpha}$ for $C = 0.5$, $s = 0.1$, $-\alpha = 0.01$.

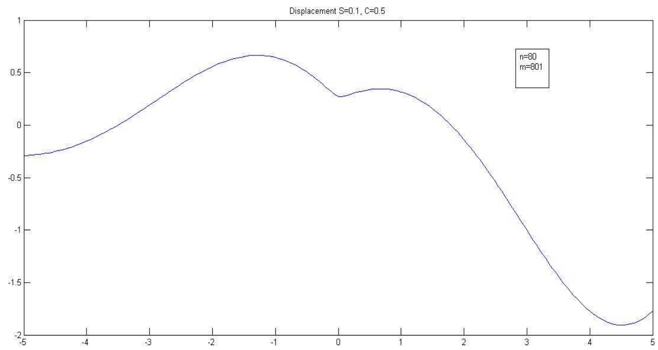

FIGURE 6. Plot of Displacement $\frac{A}{\alpha}$ for $C = 0.5$, $s = 0.1$ and $-\alpha = 0.01$.

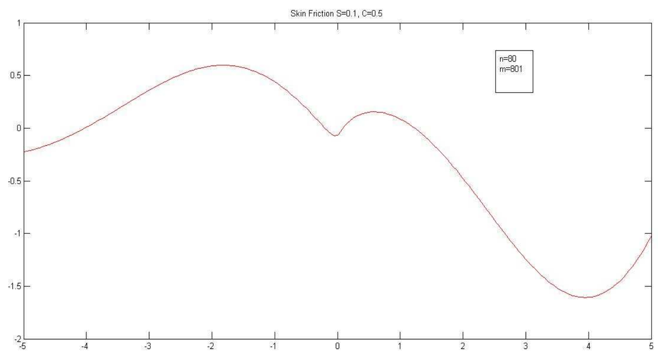

FIGURE 7. Plot of Skin Friction $\frac{(\tau-1)}{\alpha}$ for $C = 0.5$, $s = 0.1$ and $-\alpha = 0.01$.

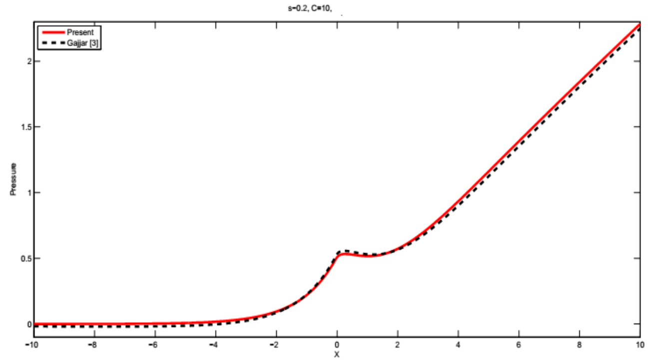

FIGURE 8. Plot of Pressure $C = 10$, $s = 0.2$ and $-\alpha = 0.01$.

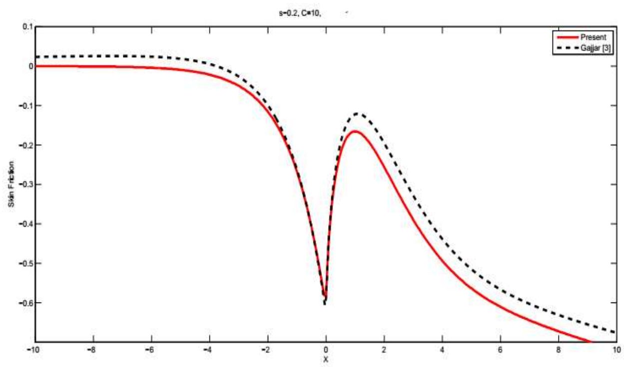

FIGURE 9. Plot of Skin Friction $C = 10$, $s = 0.2$ and $-\alpha = 0.01$.

3. CONCLUSION

In this paper we have studied the behavior of liquid layer flow when it passes through a convex corner and surface tension effects are not negligible. Inclusion of surface tension effects in the flow of liquid layer leads to the interaction law

$$P = -sA - (1 - \frac{1}{C})A^{''}$$

The linearized solutions are obtained for the problem and it is observed that the high surface tension effects cannot be ignored because it affects the behavior of flow. It can be seen from the graphs of pressure, displacement and skin friction for $C << 1$ (surface tension effects are high) that they are different from the results for $C >> 1$ (surface tension effects are low).

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