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Sequence Of Multi-Step Higher Order Iteration Schemes For Nonlinear Scalar Equations

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Abstract. Two new algorithms of fourth and fifth order convergence have been introduced. We have used Modified decomposition technique to develop our algorithms. Convergence analysis of newly introduced algorithms have been discussed. To see the efficiency and performance of these algorithms, we have made comparison of these algorithms with some well known algorithms existing in literature by solving some nonlinear equations.

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1. Introduction

In this paper, we consider iterative methods to find a simple root α of a nonlinear equation

$$f(x) = 0 \tag{1.1}$$

where $f : I \subseteq R \rightarrow R$ for an open interval I, is a scalar function and it is sufficiently smooth in a neighborhood of α.To find the root of (1.1) is the oldest and basic problem in numerical analysis. Newton method is one of the oldest and most powerful formula to approximate the root of nonlinear equations. It has second order convergence. Many modifications in Newton method have been made to increase its convergence order using various techniques. Recently, many iterative methods with higher order convergecne have been established using different techniques like Taylor's series, Adomian decomposition, homotopy, modified homotopy, decomposition, modified decomposition, interpolation, quadrature rules and their many modifications. see $[1, 2, 3, 4, 5, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17]$

and references therein. Chun $[3]$ has introduced some multi-step iterative methods using Adomian decomposition. These methods require higher order derivatives. Later on, Noor $[4, 5]$ has established some multi-step iterative methods that do not require higher order derivatives using a differnet technique.

In this paper, some numerical methods based on decomposition technique are proposed for solving algebraic nonlinear equations. Daftardar Gejji and Jafari $[8]$ decomposition technique has been used to develop our new algorithms. In section 3, we have discussed the convergece analysis of our newly introduced algorithms. In the last section, numerical results are given to make comparison of these algorithm with some classical methods.

2. Iterative methods

Consider the nonlinear equation

$$f(x) = 0; \quad x \in \mathbb{R} \tag{2.1}$$

Assume α is a simple root of (1) and γ is an initial guess sufficiently close to α. We can write (2.1) in coupled system as follows;

$$f(\gamma) + (x-\gamma)(\frac{2f'^2(\gamma) - f(\gamma)f''(\gamma)}{2f'(\gamma)}) + g(x) = 0, \tag{2.2}$$

$$g(x) = f(x) - f(\gamma) - (x-\gamma)f'(\frac{2f'^2(\gamma) - f(\gamma)f''(\gamma)}{2f'(\gamma)}). \tag{2.3}$$

From Eq.(2.2), we have

$$x = \gamma - \frac{2f(\gamma)f'(\gamma)}{2f'^2(\gamma) - f(\gamma)f''(\gamma)} - \frac{2f'(\gamma)g(x)}{2f'^2(\gamma) - f(\gamma)f''(\gamma)}$$

Let

$$x = c + N(x), \tag{2.4}$$

where

$$c = \gamma - \frac{2f(\gamma)f'(\gamma)}{2f'^2(\gamma) - f(\gamma)f''(\gamma)}, \tag{2.5}$$

and

$$N(x) = -\frac{2f'(\gamma)g(x)}{2f'^2(\gamma) - f(\gamma)f''(\gamma)}. \tag{2.6}$$

Now, we construct a sequence of higher-order iteration schemes by applying technique introduced by Daftardar Gejji and Jafari $[8]$. This technique consists a solution of Equation (2.3) that can be written in the form of infinite series:

$$x = \sum_{i=0}^{\infty} x_i \tag{2.7}$$

and using Gejji and Jafari $[8]$ technique, the nonlinear operator N can be decomposed as;

$$N(\sum_{i=0}^{\infty} x_i) = N(x_0) - \sum_{i=1}^{\infty}\{N(\sum_{j=0}^{i} x_j) - N(\sum_{j=0}^{i-1} x_j)\} \tag{2.8}$$

Thus from Equation (2.4), (2.7) and (2.8), we have

$$\sum_{i=0}^{\infty} x_i = x_0 + N(x_0) + \sum_{i=1}^{\infty}\{N(\sum_{j=0}^{i} x_j) - N(\sum_{j=0}^{i-1} x_j)\}$$

Thus, we have

$$\begin{aligned} x_0 &= c \\ x_1 &= N(x_0) \\ x_2 &= N(x_0 + x_1) - N(x_0) \\ &. \\ &. \\ &. \\ x_{n+1} &= N(x_0 + x_1 + ... + x_n) - N(x_0 + x_1 + ... + x_{n-1}),\ n = 1, 2, .. \end{aligned}$$

When

$$\begin{aligned} x &\approx x_0 \\ &= c \\ &= \gamma - \frac{2f(\gamma)f'(\gamma)}{2f'^2(\gamma) - f(\gamma)f''(\gamma)} \end{aligned}$$

From above relation, we can formulate the algorithm as follows;

Algorithm 2.1; For any initial guess x_0, we approximate x_{n+1}, by the iteration scheme;

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}$$

This is well known Halley's method and has third order convergence.

When

$$\begin{aligned} x &\approx x_0 + x_1 \\ &= \gamma - \frac{2f(\gamma)f'(\gamma)}{2f'^2(\gamma) - f(\gamma)f''(\gamma)} - \frac{2f'(\gamma)g(x_0)}{2f'^2(\gamma) - f(\gamma)f''(\gamma)} \end{aligned}$$

From Eq. (2.3), we see that

$$g(x_0) = f(x_0)$$

By substituting in above, we get

$$x = \gamma - \frac{2f(\gamma)f'(\gamma)}{2f'^2(\gamma) - f(\gamma)f''(\gamma)} - \frac{2f'(\gamma)f(x_0)}{2f'^2(\gamma) - f(\gamma)f''(\gamma)}$$

Above relation allows us to formulate the iteration scheme as follows;

Algorithm 2.2; For any initial guess x_0, we approximate x_{n+1}, by the iteration scheme;

Predictor Steps; $$y_n = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)},$$

Corrector Step; $$x_{n+1} = y_n - \frac{2f(y_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}.$$

When

$$\begin{aligned} x &\approx x_0 + x_1 + x_2 \\ &= x_0 + N(x_0 + x_1) \\ &= x_0 - \frac{2f'(\gamma)g(x_0 + x_1)}{2f'^2(\gamma) - f(\gamma)f''(\gamma)} \\ &= x_0 - \frac{2f'(\gamma)\{f(x_0 + x_1) + f(x_0)\}}{2f'^2(\gamma) - f(\gamma)f''(\gamma)} \end{aligned}$$

From above relation, we formulate iteration scheme as follows;

Algorithm 2.3; For any initial guess x_0, we approximate x_{n+1}, by the iteration scheme;

Predictor Steps;

$$\begin{aligned} y_n &= x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}, \\ z_n &= y_n - \frac{2f(y_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}, \end{aligned}$$

Corrector Step;

$$x_{n+1} = y_n - \frac{2f'(x_n)\{f(y_n) + f(z_n)\}}{2f'^2(x_n) - f(x_n)f''(x_n)}.$$

3. Convergence Analysis

This section deals the convergence analysis of algorithm 2.2 and 2.3 that have been introduced in this paper.

Theorem 3.1 Let I be an open interval and $f : I \subset \mathbb{R} \to \mathbb{R}$ be a function. Let $\alpha \in I$ be simple zero of $f(x) = 0$. If f is sufficiently differentiable and x_0 is sufficiently close to α then algorithm 2.2 has at least fourth order convergence.

Proof. Expanding $f(x_n)$, $f'(x_n)$ and $f''(x_n)$ by Taylor's series about α, we have

$$f(x_n) = f^{'}(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + ...] \tag{3.1}$$

where $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f^{'}(\alpha)}$, and $e_n = x_n - \alpha$.

$$f^{'}(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + ...] \tag{3.2}$$

$$f''(x_n) = f'(\alpha)[2c_2 + 6c_3e_n + 12c_4e_n^2 + 20c_5e_n^3 + 30c_6e_n^4 + 42c_7e_n^5...] \tag{3.3}$$

$$y_n = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}$$

By substituting values and after simplifying, we have

$$\begin{aligned} y_n = \; & \alpha + (-c_3 + c_2^2)e_n^3 + (6c_2c_3 - 3c_4 - 3c_2^3)e_n^4 + (12c_2c_4 + 6c_3^2 - 6c_5 + \\ & 6c_2^4 - 18c_3c_2^2)e_n^5 + ...3.4 \end{aligned} \tag{3. 1}$$

Now, expansion of $f(y_n)$ by Taylor's series about α yeilds,

$$\begin{aligned} f(y_n) = \; & f'(\alpha)\{(-c_3 + c_2^2)e_n^3 + (6c_2c_3 - 3c_4 - 3c_2^3)e_n^4 + (12c_2c_4 + 6c_3^2 - 6c_5 + \\ & 6c_2^4 - 18c_3c_2^2)e_n^5 + ...\}3.5 \end{aligned} \tag{3. 2}$$

Corrector step of algorithm 2.2 is

$$x_{n+1} = y_n - \frac{2f(y_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}$$

By substituting and simplifying, we have

$$x_{n+1} = \alpha + (-c_2c_3 + c_2^3)e_n^4 + (-3c_2^4 + 6c_3c_2^2 - 3c_2c_4)e_n^5 + ... \quad (3.6)$$

□

Theorem 3.2 Let I be an open interval and $f : I \subset \mathbb{R} \to \mathbb{R}$ be a function. Let $\alpha \in I$ be simple zero of $f(x) = 0$. If f is sufficiently differentiable function and x_0 is sufficiently close to α then algorithm 2.3 has at least fifth order convergence and satisfies the error equation

$$x_{n+1} = \alpha + (-c_3c_2^2 + c_2^4)e_n^5 + O(e_n^6), \text{ and } e_n = x_n - \alpha$$

Proof. From Eq. (3.6), we have

$$z_n = \alpha + (-c_2c_3 + c_2^3)e_n^4 + (-3c_2^4 + 6c_3c_2^2 - 3c_2c_4)e_n^5 + ... \quad (3.7)$$

Expanding $f(z_n)$ by Taylor's series about α, we have

$$f(z_n) = f'(\alpha)[(-c_2c_3 + c_2^3)e_n^4 + (-3c_2^4 + 6c_3c_2^2 - 3c_2c_4)e_n^5 + ...] \quad (3.8)$$

Now

$$x_{n+1} = y_n - \frac{2f'(x_n)\{f(y_n) + f(z_n)\}}{2f'^2(x_n) - f(x_n)f''(x_n)}$$

By substituting values from Eq. (3.1), (3.2), (3.3), (3.4), (3.5), (3.7) and (3.8) and simplifying, we have

$$x_{n+1} = \alpha + (-c_3c_2^2 + c_2^4)e_n^5 + O(e_n^6)$$

Hence algorithm 2.3 has fifth order convergence. □

4. Applications

Some numerical examples have been discussed to examine the validity and efficieny of our newly introduced algorithms namely, algorithm 2.2 and algorithm 2.3. We have also provided the comparison of these algorithms with Newton method(NM), Abbasbandy method (AM), Halley's method (HM) and Chun method (CM2 and CM3) [3] . We use $\epsilon = 10^{-25}$.

We use the following six nonlinear scalar equations to make numerical comparison of our newly introduced algorithms with above mentioned methods:

$$\begin{aligned} f_1(x) &= x^3 + x^2 - 2 = 0 \\ f_2(x) &= x^3 + 4x^2 - 10 = 0 \\ f_3(x) &= \sin^2 x - x^2 + 1 = 0 \\ f_4(x) &= \cos x - x - 1 = 0 \\ f_5(x) &= x^{10} - 1 = 0 \\ f_6(x) &= e^{x^2+7x-30} - 1 = 0 \end{aligned}$$

Comparison Table

Examples	Iterations	x_n	$f(x_n)$
$f_1,\ x_0 = -0.5$			
NM	13	1.0000000000000000000000000	$3.33e^{-41}$
AM	20	1.0000000000000000000000000	$2.00e^{-127}$
HM	8	1.0000000000000000000000000	$8.89e^{-46}$
$CM2$	28	1.0000000000000000000000000	$1.28e^{-40}$
$CM3$	34	1.0000000000000000000000000	$12.17e^{-61}$
Alg. 2.2	4	1.0000000000000000000000000	$2.54e^{-35}$
Alg. 2.3	3	1.0000000000000000000000000	$2.75e^{-54}$
$f_2,\ x_0 = 0.5$			
NM	8	1.3652300134140968457608068	$8.83e^{-57}$
$A.M$	8	1.3652300134140968457608068	$3.76e^{-126}$
HM	5	1.3652300134140968457608068	$1.55e^{-71}$
$CM2$	8	1.3652300134140968457608068	$1.48e^{-36}$
$CM3$	5	1.3652300134140968457608068	$1.97e^{-41}$
Alg. 2.2	3	1.3652300134140968457608068	$1.87e^{-35}$
Alg.2.3	3	1.3652300134140968457608068	$1.67e^{-81}$
$f_3,\ x_0 = 0.5$			
NM	9	1.4044916482153412260350868	$2.98e^{-44}$
AM	14	1.4044916482153412260350868	$1.00e^{-127}$
HM	6	1.4044916482153412260350868	$6.51e^{-73}$
$CM2$	8	1.4044916482153412260350868	$1.48e^{-36}$
$CM3$	15	1.4044916482153412260350868	$6.46e^{-84}$
Alg. 2.2	4	1.4044916482153412260350868	$5.81e^{-42}$
Alg.2.3	4	1.4044916482153412260350868	$1.00e^{-127}$
$f_4,\ x_0 = 0$			
NM	8	0.7390851332151606416553120	$1.51e^{-41}$
AM	5	0.7390851332151606416553120	$2.96e^{-92}$
HM	4	0.7390851332151606416553120	$1.28e^{-43}$
$CM2$	4	0.7390851332151606416553120	$1.38e^{-73}$
$CM3$	4	0.7390851332151606416553120	$1.53e^{-126}$
Alg. 2.2	3	0.7390851332151606416553120	$6.32e^{-53}$
Alg. 2.3	3	0.7390851332151606416553120	$5.10e^{-91}$

Examples	IT	x_n	$f(x_n)$
$f_5,\ x_0 = -0.5$			
NM	44	$-1.000000000000000000000000000$	$4.02e^{-32}$
AM	88	$-1.000000000000000000000000000$	$3.00e^{-127}$
HM	7	$-1.000000000000000000000000000$	$9.78e^{-41}$
$CM2$		$Diverge$	
$CM3$		$Diverge$	
Alg. 2.2	5	$-1.000000000000000000000000000$	$7.56e^{-82}$
Alg. 2.3	4	$-1.000000000000000000000000000$	$3.58e^{-39}$
$f_6,\ x_0 = 1$			
NM		$Diverge$	
AM		$Diverge$	
HM	15	$3.000000000000000000000000000$	$1.41e^{-66}$
$CM2$		$Diverge$	
$CM3$		$Diverge$	
Alg. 2.2	9	$3.000000000000000000000000000$	$6.31e^{-82}$
Alg. 2.3	6	$3.000000000000000000000000000$	$3.62e^{-44}$

5. Conclusion

We have introduced two new algorithms of fourth and fifth order convergence by using modified decomposition technique to coupled system. We have discussed convergence analysis of these newly introduced algorithms. Computational comparison of these algorithms have been made with some well known existing methods. From numerical table, we see that our new methods converge fastly to the true solution of equations and are comparable with some classical methods.

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