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Evaluation of Dialyses Patients by use of Stochastic Data Envelopment Analysis

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Abstract. Kidney Dialysis is considered as one of the world's major health problems. Level of occurrence of this illness is high and every year increases by $\%8$. Kidney Dialysis refers to temporary or permanent kidney damage which causes lack of proper functioning of kidneys. One of the dangerous problems for Dialysis patients is blood pressure. On the other hand life expectancy of these patients is matter of concern. Therefore finding a mathematical model which can link these two factors is of great importance. In this research it has been assumed that input outputs of the under evaluation units adhere to Rayleigh distribution. By considering suitable models for data envelopment analysis such as FDH, new methods for determining random value of under evaluation units are presented. Since one of the main Rayleigh distribution functions is about life expectancy, therefore the model is expanded to cover the dialysis patient's life expectancy.

AMS (MOS) Subject Classification Codes: 90;90C15;60 Exx
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1. INTRODUCTION

Kidney dialysis is one of the major health problems throughout the world which can cause irreversible progressive damage to kidneys and consequently to the person's general health. These include deteriorations in body liquid and/or electrolyte balance. This disease is responsible for over 60,000 deaths word wide every year.

Kidney function is filtering the blood and eliminating undesired substances out of it. Dialysis carries out many of the natural kidney functions. One of occurring complications during dialysis is low blood pressure. This condition carries many dangerous outcomes. Under low pressure conditions blood clothes easily and the dialysis operation is not carried out properly. This can endanger the patients life.

A family of statistical distribution analysis regarding life expectancy have been developed within the last century. Data Envelopment Analysis by use of CCR model in 1978 and then BCC model in 1984 for the evaluation of decision making units were introduced [2-1] . For evaluation of decision making units in different areas different DEA models using ranged, Fuzzy, certain, ... Data were introduced. In reality in many application areas the processed data is random and not certain. By considering that some of processed data in real applications would be random cooper et. al [3-4] suggested some models for these type of data, Li [11] , Huang and Li [6-7], Khodabakhshi [10], Khodabakhshi et. al [9] introduced DEA models for random data, Horrace and Schmidt [8] also examined the reliability of the random units.

The main problem with these models was that all input/output variables must have normal distribution. In reality however it may be otherwise and some of the input/output variables may not have a symmetrical distribution representation i.e. they are asymmetrical. Therefore evaluation of these data would lead to untrue conclusions and incorrect presentations. Section 2 refers to Rayliegh distribution which was formulated by Lord Rayliegh in 1919 [13]. This is a specific condition of weibull distribution which is used in cases of life span data analysis. In this article it has been assumed that the data have Rayliegh distribution. By considering that Kidney dialysis effects patients' health and life expectancy then the Rayliegh distribution which is very popular in life expectancy data modeling has been used.

The third section considers the suggested FDH random model. An application example regarding the suggested model is carried out in section 4and the conclusion is carried out in the fifth section.

2. PRELIMINARIES

In this section some of the concepts and theorems used in this article are introduced.

2.1. **Rayleigh distribution.** Rayleigh distribution is a probability distribution which was first introduced by Lord Rayleigh. This distribution is useful for evaluation of life span data analysis.

Definition 1: Random variable X with $\sigma > 0$ has Rayleigh distribution. It is presented by $X \sim Rayleigh(\sigma)$ such that probability density function and its cumulative distribution function are determined by [12]:

$$f(x;\sigma) = \begin{cases} \dfrac{x}{\sigma^2} e^{\left(\frac{-x^2}{2\sigma^2}\right)}; & x \geq o \\ \\ 0; & x < o \end{cases}$$

$$F(x;\sigma) = \begin{cases} 1 - e^{\left(\frac{-x^2}{2\sigma^2}\right)}; & x \geq o \\ \\ 0; & x < o \end{cases}$$

For Simplification and use in SFDH model that explained in subsequent future. If we put $\dfrac{1}{2\sigma^2} = \gamma$ then we have:

$$f(x;\gamma) = \begin{cases} 2\gamma x e^{-\gamma x^2}; & x \geq o \\ \\ 0; & x < o \end{cases}$$

$$F(x;\gamma) = \begin{cases} 1 - e^{-\gamma x^2}; & x \geq o \\ \\ 0; & x < o \end{cases}$$

2.2. **Properties of Rayleigh distribution.** now
Some of Properties Rayleigh distribution is:
1) The raw moments are given by:

$$\mu_k = \sigma^k 2^{k/2} \Gamma(1 + k/2);$$

Where $\Gamma(z)$ is the Gamma function.
2) The mean and variance of a Rayleigh random variable may be expressed as:

$$\mu(X) = \sigma \sqrt{\frac{\pi}{2}} \approx 1.253\sigma;$$

and

$$var(X) = \frac{4 - \pi}{2} \sigma^2 \approx 0.429\sigma^2;$$

3) The mode is σ and the maximum pdf is:

$$f_{max} = f(\sigma;\sigma) = \frac{1}{\sigma}exp - \frac{1}{2} \approx \frac{0.606}{\sigma};$$

4) The skewness is given by:

$$\gamma_1 = \frac{2\sqrt{\pi}(\pi-3)}{(4-\pi)^{3/2}} \approx 0.631;$$

5) The excess kurtosis is given by:

$$\gamma_2 = -\frac{6\pi^2 - 24\pi + 16}{(4-\pi)^2} \approx -0.245;$$

6) The characteristic function is given by:

$$\varphi(t) = 1 - \sigma t e^{-\sigma^2 t^2/2}\sqrt{\frac{\pi}{2}}\Big(erfi\Big(\frac{\sigma t}{\sqrt{2}}\Big) - i\Big);$$

Where erfi(z) is the complex error function.
7) The moment generating function is given by

$$M(t) = 1 + \sigma t e^{\sigma^2 t^2/2}\sqrt{\frac{\pi}{2}}\Big(erfi\Big(\frac{\sigma t}{\sqrt{2}}\Big) + 1\Big);$$

Where erfi(z) is the error function.
8) The information entropy is given by

$$H = 1 + In\Big(\frac{\sigma}{\sqrt{2}}\Big) + \frac{\gamma}{2}$$

where σ is the EulerMascheroni constant.
With regard to the value of the parameter, Rayleigh distribution can have different forms. In the following figure the probability density function for parameter variations (σ) have been drawn:

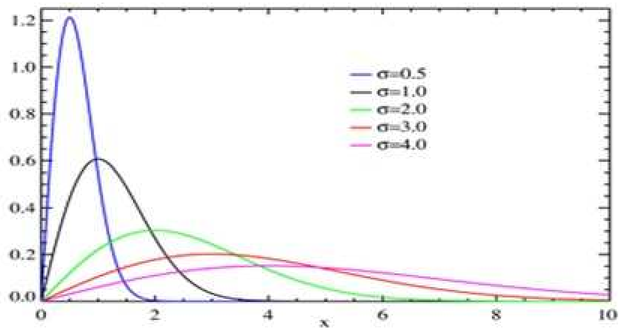

FIGURE 1. Probability density function of rayleigh distribution

The cumulative distribution function for the parameter variations (σ) of the Rayleigh distribution is as follows:

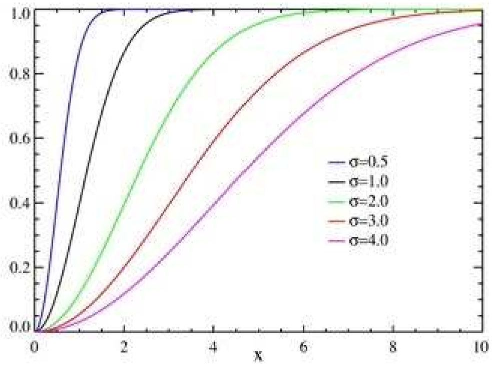

FIGURE 2. Cumulative distribution function of rayleigh distribution

2.3. **Output Orientated Free Disposal Hull (FDH).** Free disposal hull (FDH) models were introduced for the first time by Deprin, Tulkens and Simar [5]. In CCR and BCC models usually a linear combination of efficient units represents inefficient units, but since in many real world situations linear models do not represent a combination of several units thus FDH models were formulated. In these only one efficient unit represents a model for inefficient units. The difference between CCR and BCC data envelopment analysis model with FDH model is that the FDH technology does not limit itself to concave technology.
Suppose we have a set of n peer $DMU_s, \{DMU_j : j = 1, 2, \ldots, n\}$, which produce multiple outputs $Y_{rj}, (r = 1, 2, \ldots, s)$,by utilizing multiple inputs $X_{ij}, (i = 1, 2, \ldots, m)$. When a DMU_o is under evaluation by the FDH model, we have [14]:

$$\begin{aligned}
&\varphi_o^* = \max \varphi_o \\
s.t \quad &\sum_{j=1}^{n} \lambda_j X_{ij} \leq X_{io}; \quad i = 1, \ldots, m. \\
&\sum_{j=1}^{n} \lambda_j Y_{rj} \geq \varphi_o Y_{ro}; \quad r = 1, \ldots, s. \qquad (2.\ 1) \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \in \{0, 1\}; \quad j = 1, \ldots, n.
\end{aligned}$$

Where λ is a binary variable and φ_o is a free and continues variable.
Model (1) is always true because $\lambda_o = 1, \varphi_o = 1$ and $\lambda_j = 0$ for (each $j \neq 0$)) there is a true answer. For the model's deterministic solution (1) the below algorithm can be presented [14].

Algorithm 1 Preparation stage: the steps below must be taken first.
Step 0: input values, X_{ij} and output values, Y_{rj} are defined.
Repeater step: for each unit under evaluation such as DMU_o where $o \in \{1, 2, \ldots, n\}$ the steps bellow must be repeated.

Step 1: We eliminate all observations of j which are true for each r in the inequality $Y_{rj} < Y_{ro}$. Then we call the remaining observed values T_o .

Step 2: We eliminate all $j \in T_o$ which are true for each i in the inequality $X_{ij} > X_{io}$. Then we call all remaining observed values $\tilde{T_o}$.

Step 3: for each observation of $j \in \tilde{T_o}$ value of φ_o^j is determined from the below relation.

$$\varphi_o^j = \max\left\{\varphi_o \middle| \varphi_o \leq \frac{Y_{rj}}{Y_{ro}}\ ,\ Y_{ro} > 0,\ r = 1, \ldots, s\right\} = \min_{1 \leq r \leq s}\left\{\frac{Y_{rj}}{Y_{ro}}\ ,\ Y_{ro} > 0\right\}$$

Step 4: φ_o^* is calculated from the below relationship:

$$\varphi_0^* = \max_{j \in \tilde{T_o}}\{\varphi_0^j\} = \max_{j \in \tilde{T_o}}\ \min_{1 \leq r \leq s}\left\{\frac{Y_{rj}}{Y_{ro}}\ ,\ Y_{ro} > 0\right\}$$

Definition 2: if $\varphi_{FDH_o}^* = 1$ then unit under evaluation by DMU_o is efficient.

3. Output orientated SFDH model with Rayleigh distribution

In this section output orientated SFDH model with Rayleigh distribution is introduced. For every DMU_j assume $X_j = (X_{1j}, X_{2j}, \ldots, X_{mj})$ and $Y_j = (Y_{1j}, Y_{2j}, \ldots, Y_{sj})$ are random input output vectors respectively, such that $X_{ij} \sim Rayleigh(\alpha_{ij}), Y_{rj} \sim Rayleigh(\beta_{rj})$ and also inputs are independent of each other. Likewise outputs are also independent of each other. But inputs and outputs are not independent of each other. By using model (1) output orientated SFDH model is defined as below:

$$\begin{aligned} \varphi_o^* = \max_{1 \leq j \leq n}\ & \max \varphi_j^o \\ s.t.\quad P(X_{ij} \leq X_{io}) &\geq 1 - \alpha; \quad i = 1, \ldots, m. \\ P(Y_{rj} \geq \varphi_j^o Y_{ro}) &\geq 1 - \alpha; \quad r = 1, \ldots, s. \end{aligned} \tag{3. 2}$$

Output orientated FDH model can be presented in a compact format as: follow:

$$\begin{aligned} \varphi_o^* &= \max \varphi \\ s.t.\quad P\Big(\sum_{j=1}^{n} \lambda_j X_{ij} \leq X_{io}\Big) &\geq 1 - \alpha; \quad i = 1, \ldots, m. \\ P\Big(\sum_{j=1}^{n} \lambda_j Y_{rj} \geq \varphi_o Y_{ro}\Big) &\geq 1 - \alpha; \quad r = 1, \ldots, s. \\ \sum_{j=1}^{n} \lambda_j &= 1 \\ \lambda_j \in \{0, 1\}; &\qquad j = 1, \ldots, n. \end{aligned} \tag{3. 3}$$

3.1. **Deterministic output orientated SFDH model at presence of Rayleigh distribution.** For transfer of random model (2) to a determinate model first we state and prove the fundamental theorem below.

Theorem 1: If X and Y are 2 independent random variables, such that $X \sim Rayleigh(\alpha)$ and $Y \sim Rayleigh(\beta)$ then for each $c > 0$ random variable $Z = X - cY$ has the accumulative distribution function below at zero.

$$F_z(0) = \frac{\alpha c^2}{\beta + \alpha c^2}$$

Proof: See appendix.

By putting $Z = Y_{rj} - \varphi_o Y_{ro}$ in theorem (1), also considering the first restraint in model (2) we get:

$$P\big(Y_{rj} \ \geq \varphi_o Y_{ro}\big) \geq 1 - \alpha \Leftrightarrow P\big(Y_{rj} - \varphi_0 Y_{ro} \leq 0\big) \leq \alpha$$

$$\Leftrightarrow F_z(o) \leq \alpha \Leftrightarrow \frac{\beta_{ro}\varphi_o^2}{\beta_{ro}\varphi_o^2 + \beta_{rj}} \leq \alpha \Leftrightarrow \varphi_o^2 \leq \frac{\alpha}{1-\alpha}\frac{\beta_{rj}}{\beta_{ro}}$$

$$\overset{\varphi_o \geq 1}{\Longleftrightarrow} \varphi_o \leq \sqrt{\frac{\alpha}{1-\alpha}}\sqrt{\frac{\beta_{rj}}{\beta_{ro}}}$$

Similarly by putting $Z = X_{ij} - X_{io}$ in theorem (1), also considering the second restraint in model (2) we get:

$$P\Big(X_{ij} \leq X_{io}\Big) \geq 1 - \alpha \Leftrightarrow P\Big(X_{ij} - X_{io} \leq 0\Big) \geq 1 - \alpha \Leftrightarrow$$

$$F_z(0) \geq 1 - \alpha \Leftrightarrow \frac{\alpha_{io}}{\alpha_{io} + \alpha_{ij}} \geq 1 - \alpha \Leftrightarrow \frac{\alpha_{ij}}{\alpha_{io}} \leq \frac{\alpha}{1-\alpha}$$

Thus under these conditions deterministic model is fully defined.
Now for each $0 \in \{1, 2, , \ldots, n\}$ and for $0 < \alpha < 1$ similar to deterministic algorithm (1) (output orientated FDH) for the random model (1) G_o and $\tilde{G}_o$ can be presented as $G_o(\alpha)$ and $\tilde{G}_o(\alpha)$ respectively. These are defined as below:

$$G_o(\alpha) = \Big\{ j \Big| \frac{\alpha_{ij}}{\alpha_{io}} \leq \frac{\alpha}{1-\alpha}, \quad i = 1, \ldots, m \Big\} \tag{3. 4}$$

In the deterministic FDH model by placing $\tilde{G}_o$ for φ_o level of efficiency i.e. a number is substituted.Therefore under random conditions for φ_o level of efficiency i.e. number

$\sqrt{\dfrac{\alpha}{1-\alpha}}$ is placed in the first restraint in model (2)

$$P\Big(Y_{rj} \geq \varphi_o Y_{ro}\Big) \geq 1-\alpha \Leftrightarrow \varphi_o \leq \sqrt{\frac{\alpha}{1-\alpha}}\sqrt{\frac{\beta_{rj}}{\beta_{ro}}} \Leftrightarrow$$
$$\sqrt{\frac{\alpha}{1-\alpha}} \leq \sqrt{\frac{\alpha}{1-\alpha}}\sqrt{\frac{\beta_{rj}}{\beta_{ro}}} \Leftrightarrow \sqrt{\frac{\beta_{rj}}{\beta_{ro}}} \geq 1$$

Therefore $\tilde{G}_o(\alpha)$ is defined as:

$$\tilde{G}_o(\alpha) = \left\{ j \in G_o(\alpha) \middle| \sqrt{\frac{\beta_{rj}}{\beta_{ro}}} \geq 1, \quad r = 1, \ldots, s \right\} \tag{3. 5}$$

Finally by definitions of $G_o(\alpha)$ and $\tilde{G}_o(\alpha)$ and algorithm (1) output orientated deterministic FDH model can be presented as an efficient algorithm for solving output orientated random FDH model (2) as below.

Algorithm 2
Preliminary stage: first the fallowing steps must be carried out:
Step 0: value of input (α_{ij}) output (β_{rj}) parameters are determined.
Step 1: determine the value for α.
Repeater step: for each unit under evaluation such as DMU_o where $0 \in \ 1, 2, \ldots, n\}$the fallowing steps must be repeated.
Step 2: The sets $G_o(\alpha)$ and $\tilde{G}_o(\alpha)$ are determined in relationships (4) and (5).
Step 3: for each observation of $j \in \tilde{G}_o(\alpha)$ value of $\varphi_o^j(\alpha)$ is found from the relationship below.

$$\varphi_o^j(\alpha) = \max\left\{\varphi_o \middle| \varphi_o \leq \sqrt{\frac{\alpha}{1-\alpha}}\sqrt{\frac{\beta_{rj}}{\beta_{ro}}}, \quad r = 1, \ldots, s\right\} = \min_{1\leq r\leq s}\left\{\sqrt{\frac{\alpha}{1-\alpha}}\sqrt{\frac{\beta_{rj}}{\beta_{ro}}}\right\}$$

Step 4: $\varphi_o^*(\alpha)$ is calculated from:

$$\varphi_o^*(\alpha) = \max_{j\in\tilde{G}_o(\alpha)} \varphi_o^j(\alpha) = \max_{j\in\tilde{G}_o(\alpha)} \min_{1\leq j\leq s}\left\{\sqrt{\frac{\alpha}{1-\alpha}}\sqrt{\frac{\beta_{rj}}{\beta_{ro}}}\right\}$$

In this case $\varphi^*_{FDH_o}(\alpha)$ equals to:

$$\varphi^*_{FDH_o}(\alpha) = \varphi_o^*(\alpha)$$

Definition 3: If the unit under evaluation DMU_o at the error level α has no other reverence other than itself i.e. $\tilde{G}_o(\alpha) = \{O\}$ then $\varphi^*_{FDH_o}(\alpha) = \sqrt{\dfrac{\alpha}{1-\alpha}}$ then DMU_o is α-random efficient. In this definition DMU_o is called an extreme point on stochastic frontier.

Definition 4: If the unit under evaluation DMU_o has another error reference point than itself and $\varphi^*_{FDH_o}(\alpha) = \sqrt{\dfrac{\alpha}{1-\alpha}}$ then DMU_o is α-random efficient. In this definition

DMU_o is called an extreme point on stochastic frontier.

Definition 5: If the unit under evaluation DMU_o has another error reference than itself and $\varphi^*_{FDH_o}(\alpha) \neq \sqrt{\dfrac{\alpha}{1-\alpha}}$ then DMU_o is α-random inefficient.

Definition 6: If the unit under evaluation DMU_o at error levelat error level α is not overcome by any of the decision making units i.e. $(\tilde{G}_o(\alpha) = \emptyset)$ then DMU_o is α-random efficient. Its efficiency is expressed as $\varphi^*_{FDH_o}(\alpha) = \sqrt{\dfrac{\alpha}{1-\alpha}}$.

Definition 7: if $\tilde{G}_o(\alpha) \neq \emptyset$ and $o \notin \tilde{G}_o(\alpha)$ then the unit under evaluation DMU_o at error level falls on stochastic frontier which is inefficient. Under this condition DMU_o the unit under evaluation is α-random inefficient.

Notice: when $G_o(\alpha) = \emptyset$ then its corresponding restraints will be redundant (input restraints) and $\tilde{G}_o(\alpha)$ is calculated for all observations of $(1 \leq j \leq n)$ and the algorithm will be fallowed.

4. Application

In this section an application problem regarding life expectancy has been used to present functioning of an output orientated SFDH model in determining the efficacy of systems on life expectancy by use of Rayleigh distribution. Considering importance of life span and lowering of blood pressure output orientated model was used.
Please take note: to make certain that the results of solving the random models be sound and reliable then according to uncertainty principles a value for must be selected such that to be near $1/2$, so the results obtained from random models and deterministic models would be the same.

4.1. A Practical Application Regarding Life Expectancy.

In this example an output orientated SFDH model was exemplified by considerations of measuring random efficiency of decision making units. Thirty dialysis patients were presented by $DMU_j (j = 1, 2, \ldots, n)$. Each DMU_j consists of two input elements which are: The average weekly time under dialysis and the time period on the waiting list for Kidney transplant, measured annually represented as $X_{ij}, i = 1, 2$ These adhere to goodness of fit test according to Rayleigh distribution $X_{ij} \sim Rayleigh(\gamma_{ij})$ (for inputs). Patients' life span and low blood pressure are each a random variable in $Y_{rj}, r = 1, 2$ which according to goodness of fit test have Rayleigh distribution $Y_{rj} \sim Rayleigh(\beta_{rj})$ (for outputs).
Input output parameters are presented in table (1). By employing Algorithm (2) random efficiency for each of the 30 patients was determined.

TABLE 1. inputs and outputs

DMU_j	Average weekly hours of dialysis (first input)	Time period before Kidney transplant in years (second input)	Life span expectancy In years (first output)	Blood pressure (second output)
DMU_{01}	$X_{11} \sim Rayleigh(6)$	$X_{21} \sim Rayleigh(2)$	$Y_{11} \sim Rayleigh(15)$	$Y_{21} \sim Rayleigh(9)$
DMU_{02}	$X_{12} \sim Rayleigh(12)$	$X_{22} \sim Rayleigh(5)$	$Y_{12} \sim Rayleigh(11)$	$Y_{22} \sim Rayleigh(8)$
DMU_{03}	$X_{13} \sim Rayleigh(15)$	$X_{23} \sim Rayleigh(1)$	$Y_{13} \sim Rayleigh(2)$	$Y_{23} \sim Rayleigh(4)$
DMU_{04}	$X_{14} \sim Rayleigh(8)$	$X_{24} \sim Rayleigh(3)$	$Y_{14} \sim Rayleigh(20)$	$Y_{24} \sim Rayleigh(9)$
DMU_{05}	$X_{15} \sim Rayleigh(16)$	$X_{25} \sim Rayleigh(0.5)$	$Y_{15} \sim Rayleigh(11)$	$Y_{25} \sim Rayleigh(3)$
DMU_{06}	$X_{16} \sim Rayleigh(20)$	$X_{26} \sim Rayleigh(0.67)$	$Y_{16} \sim Rayleigh(0.02)$	$Y_{26} \sim Rayleigh(2)$
DMU_{07}	$X_{17} \sim Rayleigh(9)$	$X_{27} \sim Rayleigh(20)$	$Y_{17} \sim Rayleigh(7)$	$Y_{27} \sim Rayleigh(9)$
DMU_{08}	$X_{18} \sim Rayleigh(8)$	$X_{28} \sim Rayleigh(2)$	$Y_{18} \sim Rayleigh(9)$	$Y_{28} \sim Rayleigh(9)$
DMU_{09}	$X_{19} \sim Rayleigh(12)$	$X_{29} \sim Rayleigh(6)$	$Y_{19} \sim Rayleigh(15)$	$Y_{29} \sim Rayleigh(9)$
DMU_{10}	$X_{110} \sim Rayleigh(6)$	$X_{210} \sim Rayleigh(3)$	$Y_{110} \sim Rayleigh(18)$	$Y_{210} \sim Rayleigh(7)$
DMU_{11}	$X_{111} \sim Rayleigh(15)$	$X_{211} \sim Rayleigh(2)$	$Y_{111} \sim Rayleigh(13)$	$Y_{211} \sim Rayleigh(3)$
DMU_{12}	$X_{112} \sim Rayleigh(15)$	$X_{212} \sim Rayleigh(2.5)$	$Y_{112} \sim Rayleigh(12)$	$Y_{212} \sim Rayleigh(2)$
DMU_{13}	$X_{113} \sim Rayleigh(20)$	$X_{213} \sim Rayleigh(7)$	$Y_{113} \sim Rayleigh(0.02)$	$Y_{213} \sim Rayleigh(2)$
DMU_{14}	$X_{114} \sim Rayleigh(16)$	$X_{214} \sim Rayleigh(9)$	$Y_{114} \sim Rayleigh(3.5)$	$Y_{214} \sim Rayleigh(4)$

The data in table (1) is evaluated according to algorithm (2) and processed for each level of $\alpha \in \{0.1, 0.3, 0.5, 0.525\}$and by use of EXECL represented in table (2).

DMU_j	Average weekly hours of dialysis (first input)	Time period before Kidney transplant in years (second input)	Life span expectancy In years (first output)	Blood pressure (second output)
DMU_{15}	$X_{115} \sim Rayleigh(12)$	$X_{215} \sim Rayleigh(11)$	$Y_{115} \sim Rayleigh(2)$	$Y_{215} \sim Rayleigh(3)$
DMU_{16}	$X_{116} \sim Rayleigh(9)$	$X_{216} \sim Rayleigh(5)$	$Y_{116} \sim Rayleigh(19)$	$Y_{216} \sim Rayleigh(8)$
DMU_{17}	$X_{117} \sim Rayleigh(16)$	$X_{217} \sim Rayleigh(2)$	$Y_{117} \sim Rayleigh(8.5)$	$Y_{217} \sim Rayleigh(5)$
DMU_{18}	$X_{118} \sim Rayleigh(9)$	$X_{218} \sim Rayleigh(13)$	$Y_{118} \sim Rayleigh(11)$	$Y_{218} \sim Rayleigh(7)$
DMU_{19}	$X_{119} \sim Rayleigh(6)$	$X_{219} \sim Rayleigh(4)$	$Y_{119} \sim Rayleigh(16)$	$Y_{219} \sim Rayleigh(9)$
DMU_{20}	$X_{120} \sim Rayleigh(15)$	$X_{220} \sim Rayleigh(15)$	$Y_{120} \sim Rayleigh(14)$	$Y_{220} \sim Rayleigh(4)$
DMU_{21}	$X_{121} \sim Rayleigh(16)$	$X_{221} \sim Rayleigh(14)$	$Y_{121} \sim Rayleigh(9.5)$	$Y_{221} \sim Rayleigh(3)$
DMU_{22}	$X_{122} \sim Rayleigh(12)$	$X_{222} \sim Rayleigh(10)$	$Y_{122} \sim Rayleigh(14)$	$Y_{222} \sim Rayleigh(4)$
DMU_{23}	$X_{123} \sim Rayleigh(20)$	$X_{223} \sim Rayleigh(3.5)$	$Y_{123} \sim Rayleigh(10)$	$Y_{223} \sim Rayleigh(2)$
DMU_{24}	$X_{124} \sim Rayleigh(16)$	$X_{224} \sim Rayleigh(1.5)$	$Y_{124} \sim Rayleigh(8)$	$Y_{224} \sim Rayleigh(6)$
DMU_{25}	$X_{125} \sim Rayleigh(8)$	$X_{225} \sim Rayleigh(6)$	$Y_{125} \sim Rayleigh(12.5)$	$Y_{225} \sim Rayleigh(8)$
DMU_{26}	$X_{126} \sim Rayleigh(15)$	$X_{226} \sim Rayleigh(9.5)$	$Y_{126} \sim Rayleigh(2)$	$Y_{226} \sim Rayleigh(3)$
DMU_{27}	$X_{127} \sim Rayleigh(16)$	$X_{227} \sim Rayleigh(3)$	$Y_{127} \sim Rayleigh(2)$	$Y_{227} \sim Rayleigh(2)$
DMU_{28}	$X_{128} \sim Rayleigh(16)$	$X_{228} \sim Rayleigh(5.5)$	$Y_{128} \sim Rayleigh(11)$	$Y_{228} \sim Rayleigh(4)$
DMU_{29}	$X_{129} \sim Rayleigh(16)$	$X_{229} \sim Rayleigh(4)$	$Y_{129} \sim Rayleigh(12)$	$Y_{229} \sim Rayleigh(3)$
DMU_{30}	$X_{130} \sim Rayleigh(20)$	$X_{230} \sim Rayleigh(20)$	$Y_{130} \sim Rayleigh(1)$	$Y_{230} \sim Rayleigh(2)$

The results presented in table (2) indicate that at $\alpha = 0.05$ level, %20 of the patients i.e. patients number 1,4,7,8,9,19 and at $\alpha = 0.1$ level, %20 of patients number 1,4,7,8,9 and 19 and at $\alpha = 0.3$ level, %27 of number 1,4,7,8,9,17,19 and 24, and at $\alpha = 0.5$ level, patients number 1,3,4,5,7,8,9,10,19 and 24 were %33 of patients, and at $\alpha = 0.525$ level patients number 1,3,4,5,7,8,9,10,19 and 24 i.e. %33 of patients showed random efficiency.
Other patients showed random inefficiency. In addition patients number 1,4,7,8,9 and19 were randomly efficient at all levels. These represent %20 of the patients (according to definitions 3 to 7). It could be concluded that patients number 1, 4,7,8,9 and 19 was efficient i.e. the kidney transplant operation was carried out at a suitable time.
Therefore by considering hospital clinical conditions such as dialysis equipment and quality of dialysis, investigating medical team performance including diagnosis by the physicians, nursing staff, considerations of diet, reduction of liquid taking and operation timing which all contributed to lowering patient blood pressure and lengthening their life expectancy showed the validity of physicians decision for time of operation.
For inefficient units decisions made as well as quality of operation and care plus diet considerations were weak. It may be considered that at $\alpha = 0.5$ level the obtained efficiency results from random SFDH and the results from output orientated deterministic FDH models were equal.

Appendix: Proof of Theorem 1
According to definition 1 we have

$$F_z(z) = P(Z \leq z) = P(X - cY \leq z) = P(X \leq cY + z)$$

if $cY + z \leq 0$ then $F_z(z) = 0$ because random variable X is supported within $(0, \infty)$

TABLE 2. shows the results of measuring random efficiency of the dialysis patients

DMU_j	$\alpha = 0.05$ Efficiency	$\alpha = 0.1$ Efficiency	$\alpha = 0.3$ Efficiency	$\alpha = 0.5$ Efficiency	$\alpha = 0.525$ Efficiency
DMU_{01}	0.2777	0.3333	0.9512	1.0000	1.0513
DMU_{02}	0.24555	0.3536	1.0089	1.0607	1.1151
DMU_{03}	0.29999	0.5000	1.4268	1.0000	1.0513
DMU_{04}	0.45555	0.3333	0.9512	1.0000	1.0513
DMU_{05}	0.56666	0.4495	1.2826	1.0000	1.0513
DMU_{06}	0.56666	0.7071	1.1650	1.2247	1.2247
DMU_{07}	0.24555	0.3333	0.9512	1.0000	1.0513
DMU_{08}	0.33333	0.3333	0.9512	1.0000	1.0513
DMU_{09}	0.24555	0.3333	0.9512	1.0000	1.0513
DMU_{10}	0.25666	0.3514	1.0026	1.0000	1.0513

DMU_j	$\alpha = 0.05$ Efficiency	$\alpha = 0.1$ Efficiency	$\alpha = 0.3$ Efficiency	$\alpha = 0.5$ Efficiency	$\alpha = 0.525$ Efficiency
DMU_{11}	0.24444	0.4134	1.1798	1.0742	1.1293
DMU_{12}	0.27777	0.4303	1.0635	1.1180	1.1754
DMU_{13}	0.276666	0.7071	2.0178	2.1213	2.2302
DMU_{14}	0.25555	0.5000	1.4268	1.5000	1.5770
DMU_{15}	0 .23333	0.5634	1.6078	1.6903	1.7770
DMU_{16}	0.245555	0.3420	0.9759	1.0260	1.0786
DMU_{17}	0.234444	0.4472	0.9512	1.3284	1.3966
DMU_{18}	0.23555	0.3780	1.0785	1.1339	1.1921
DMU_{19}	0.24555	0.3333	0.9512	1.0000	1.0513
DMU_{20}	0.2444	0.3984	1.1369	1.1952	1.2566

range. Then if $cY + z > 0$ then we get:

$$F_z(z) = \int_{-\frac{z}{c}}^{\infty} F_X(cY + z) f_y(y) dy$$

Thus considering random variable Y is supported within $(0, \infty)$ range:

$$F_z(z) = \begin{cases} \int_{-\frac{z}{c}}^{\infty} F_X(cY + z) f_Y(y)\, dy; & -\dfrac{z}{c} > 0 \\ \\ \int_0^{\infty} F_X(cY + z) f_Y y\, dy; & -\dfrac{z}{c} \leq 0 \end{cases}$$

DMU_j	$\alpha = 0.05$ Efficiency	$\alpha = 0.1$ Efficiency	$\alpha = 0.3$ Efficiency	$\alpha = 0.5$ Efficiency	$\alpha = 0.525$ Efficiency
DMU_{21}	0.277777	0.4837	1.3801	1.4510	1.5254
DMU_{22}	0.27888	0.3984	1.1369	1.1952	1.2566
DMU_{23}	0.25555	0.4714	1.3452	1.4142	1.4868
DMU_{24}	0.26666	0.4082	0.9512	1.0000	1.0513
DMU_{25}	0.27666	0.3536	1.0089	1.0607	1.1151
DMU_{26}	0.288888	0.5040	1.4381	1.5119	1.5894
DMU_{27}	0.45666	0.5634	1.3924	1.6903	1.7770
DMU_{28}	0.3454	0.4495	1.2826	1.3484	1.4176
DMU_{29}	0.476	0.4303	1.2280	1.2910	1.3572
DMU_{30}	0.4453	0.7071	2.0178	2.1213	2.2302

Now if $-\dfrac{z}{c} \leq 0$ which is equivalent to $z \geq 0$;

$$\begin{aligned} F_z(Z) &= \int_0^\infty F_X(cY+z) f_Y(y)\, dy \\ &= \int_0^\infty \left(1 - e^{-(cy+z)^2\alpha}\right) 2\beta y e^{-\beta y^2}\, dy \\ &= \int_0^\infty 2\beta y e^{-\beta y^2}\, dy - \int_0^\infty 2\beta y e^{-\beta y^2} e^{-(cy+z)^2\alpha}\, dy \\ &= \int_0^\infty 2\beta y e^{-\beta y^2}\, dy - \int_o^\infty 2\beta y e^{-\beta y^2 - (cy+z)^2\alpha}\, dy \end{aligned}$$

Since solving the above integration is difficult, the fallowing method is carried out. Also note that only value $F_Z(0) = P(Z \leq 0)$ is needed. Therefore by equating Z to zero or $Z = 0$ the required solution is found.

$$P(X - cY \leq 0) = \int_0^\infty P(X - cY \leq 0 \Big| Y = y) f_Y(y)\, dy$$

$$= \int_0^\infty F_x(cy) f_Y(y)\, dy = \int_0^\infty \left(1 - e^{-\alpha(cy)^2}\right) 2\beta y e^{-\beta y^2}\, dy$$

$$= \int_0^\infty 2\beta y e^{-\beta y^2}\, dy - \beta \int_0^\infty 2y e^{-\beta y^2} e^{-\alpha(cy)^2}\, dy$$

$$\begin{aligned} &= \int_0^\infty 2\beta y e^{-\beta y^2}\, dy - \beta \int_0^\infty 2y e^{-y^2(\beta+\alpha c^2)}\, dy \\ &= -e^{-\beta y^2}\Big]_0^\infty + \left(\frac{\beta}{\beta + \alpha c^2}\right) e^{-y^2(\beta+\alpha c^2)}\ \Bigg]_0^\infty \end{aligned}$$

$$= \lim_{y\to\infty}\Big(-e^{-\beta y^2}\Big) + e^{-\beta(o)^2} + \lim_{y\to\infty}\left(\left(\frac{\beta}{\beta+\alpha c^2}\right)e^{-y^2(\beta+\alpha c^2)}\right)$$

$$-\left(\frac{\beta}{\beta+\alpha c^2}\right)e^{-(o)^2(\beta+\alpha c^2)} = 0+1+0-\left(\frac{\beta}{\beta+\alpha c^2}\right)$$

$$= 1-\left(\frac{\beta}{\beta+\alpha c^2}\right) = \frac{\alpha c^2}{\beta+\alpha c^2} =$$

$$1-\left(\frac{\beta}{\beta+\alpha c^2}\right) = \frac{\alpha c^2}{\beta+\alpha c^2}$$

Which proves the theorem.

5. Conclusion

For purposes of evaluation of decision making units many DEA models have been introduced. These models mainly have been developed for the purpose of using ranged, Fuzzy, definite, certain, ... data models. Under real conditions however data generated is usually uncertain and also not under control of decision making units, i.e. they are random. Under these conditions new methods 12 were required. More recent models consider random input output data having normal distribution. There are no suitable models with regards to determining efficiency of units in presence of statistical distributions as yet.
In this article random nature of data has been considered. By use input outputs orientated SFDH models with input outputs adhering to Rayleigh distribution efficiency of DMUs have been evaluated.
In the presented method by considering error level α possibility of occurrence of unforeseen conditions is derived. This error level must be analysed initially and decided upon. Any variation in this level of error will be reflected in the produced results. Therefore selection of this error is of great importance. It is advisable to select the error level as a number α to be near 0.5. Since one of the variables which fallow Rayleigh distribution is the random life expectancy which fits the condition of the dialysis patients therefore it was applied for 30 dialysis patients and its obtained results are presented.

References

[1] R. D. Banker, A. Charnes and W. W Cooper, *Some models for estimating technical and scale inefficiencies in data envelopment analysis*, Management Science **30**, (1984) 1078 -1092 .

[2] A. Charnes, W. W. Cooper and E. Rhodes, *Measuring the efficiency of decision making units*, European Journal of Operational Research **30**, (1978) 1078-1092.

[3] W. W. Cooper, Z. M. Huang, V. Lelas, S. X. Li and O. B. Olesen, *Chance constrained programming formulations for stochastic characterizations of efficiency and dominance in DEA,* , Journal of Productivity Analysis **9**, (1998) 53-79.

[4] W. W. Cooper, H. Deng, Z. M. Huang and S. X. Li, *Chance constrained programming approaches to technical efficiencies and inefficiencies in stochastic data envelopment analysis*,Journal of the Operational Research Society **53**, No. 12 (2002) 1347-13.

[5] D. Deprins, L. Simar and H. Tulkens, *Measuring labor inefficiency in post offices.In: M. Marchand, P. Pestieau, H. Tulkens (editors)*, The Performance of Public Enterprises: Concepts and Measurements, Amsterdam, North-Holland (1984) 243-267.

[6] Z. M. Huang and S. X. Li, *Dominance stochastic models in data envelopment analysis*, European Journal of Operational Research **95**, NO. 2 (1996) 390-403.

[7] Z. Huang and S. X. Li, *Stochastic DEA models with different types of inputoutput disturbances*, Journal of Productivity Analysis **15**, (2001) 95-113.

[8] W. C. Horrace and P. Schmidt, *Confidence Statements for Efficiency Estimates from Stochastic Frontier Models*, Journal of Productivity Analysis. **7**, (1996) 257-282.

[9] M. Khodabakhshi and M. Asgharian, *An input relaxation measure of efficiency in stochastic data envelopment analysis*, Applied Mathematical Modelling **33**, (2008) 2010-2023.

[10] M. Khodabakhshi, *Estimating most productive scale size with stochastic data in data envelopment analysis*, Economic Modelling **26**, No. 5 (2009) 968-973.

[11] S. X. Li, *Stochastic models and variable returns to scales in data envelopment analysis*, European Journal of Operational Research **104**, (1998) 532-548.

[12] Papoulis, Athanasios and S. Pillai, *Probability, Random Variables and Stochastic Processes*,ISBN 0073660116, ISBN 9780073660110. (2001)

[13] L. Rayleigh, *On the problem of random vibrations and of random flights in one, two, and three dimensions*, Phil. Mag , **37**, (1919) 321-347.

[14] H. Tulkens, *On FDH efficiency analysis: some methodological issues and applications to retail banking, courts, and urban transit*, In Productivity Issues in Services at the Micro Level , (1993) 179-206.