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The g-lift of Affine Connection in The Cotangent Bundle

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Abstract. The g-lifts of affine connection and curvature tensor are obtained via the musical isomorphism in the cotangent bundle.

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Key Words: g-lift, complete lift, connection, curvature tensor, musical isomorphism, tangent bundle, cotangent bundle.

1. Introduction

Musical isomorphism is provided between the tangent and cotangent bundles on the Riemannian Manifold. The exact origin of the term musical isomorphism is unknown. In 1971, the musical isomorphism was seen in the Berger and his collaborators' study and the term musical isomorphism was not encountered before this year [3]. So it is thought that the musical isomorphism was first studied by Berger and his collaborators. The musical isomorphism defined by g metric tensor was extended by Poor [7]. Making use of the complete lifts on TM tangent bundles, Cakan and her collaborators construct the complete lifts of some tensor fields with different type on T^*M cotangent bundles by means of a musical isomorphism [4]. The g-lifts of tensor fields are described via musical isomorphism newly by Cakan and Salimov [8]. Also the Riemannian manifolds and the tangent bundles studied a lot of authors [1, 2, 5, 6, 9, 10, 11, 12, 13] too. In the study that follows we solve two problems. Firstly we obtain the components of $g-$lift of affine connection ${}^G\overset{*}{\nabla}$ via the musical isomorphism in the cotangent bundle T^*M. Secondly we obtain components of $g-$lift of curvature tensor ${}^G\overset{*}{R}$ via the musical isomorphism in the cotangent bundle T^*M.

Let M be a pseudo-Riemannian manifold with n dimension. The tangent bundle on M is denoted by $TM = \cup_{x\in M} T_xM$.The local coordinates on TM are $(x^i, x^{\bar{i}}) = (x^i, y^i)$ where $\left(x^i\right)$ are local coordinates on M and $\left(y^i\right)$ are vector space coordinates according to the basis $\partial/\partial x^i$, i.e. $y_x = y^i \frac{\partial}{\partial x^i} \in T_xM$. The cotangent bundle on M is denoted by $T^*M = \cup_{x\in M} T^*_xM$. The local coordinates on T^*M are $(x^i, \widetilde{x}^{\bar{i}}) = (x^i, p_i)$ where $\left(x^i\right)$ are local coordinates on M and p_i are vector space coordinates according to the basis dx^i, i.e. $p_x = p_i dx^i \in T_x^* M$. We use the ranges of the index i being $\{1, ..., n\}$ and the index $\bar{i}$ being $\{n+1, ..., 2n\}$.

Let g be a pseudo Riemannian metric. $g^\sharp : T^*M \to TM$ is the musical isomorphism associated with g pseudo Riemannian metric with inverse given by $g^\flat : TM \to T^*M$.

The musical isomorphism $g^\sharp$ is described by

$$g^\sharp : \tilde{x}^M = (x^m, \tilde{x}^{\overline{m}}) = (x^m, p_m) \to x^J = (x^j, x^{\bar{j}}) = (\delta^j_m x^m, y^j = g^{jm} p_m)$$

where δ is the Kronecker delta.

In music notation, the sharp symbol $\sharp$ increases a note by a half step. Similar to this musical notation, the musical isomorphism $g^\sharp$ increases the indice of the vector space coordinate.

The musical isomorphism $g^\flat$ is described by

$$g^\flat : x^I = (x^i, x^{\bar{i}}) = (x^i, y^i) \to \tilde{x}^K = (x^k, \tilde{x}^{\bar{k}}) = (\delta^k_i x^i, p_k = g_{ki} y^i).$$

In another music notation, the flat symbol $\flat$ lowers a note by a half step. Similar to this musical notation, the musical isomorphism $g^\flat$ lowers the indice of the vector space coordinate.

The Jacobian matrices of $g^\flat$ is obtained by [4]

$$(g^\flat_*) = \left(\widetilde{A}^M_J\right) = \left(\frac{\partial \tilde{x}^M}{\partial x^J}\right) \begin{pmatrix} \delta^m_j & 0 \\ y^s \partial_j g_{ms} & g_{mj} \end{pmatrix} \tag{1. 1}$$

and the Jacobian matrices of $g^\sharp$ is obtained by

$$(g^\sharp_*) = \left(A^J_M\right) = \left(\frac{\partial x^J}{\partial \tilde{x}^M}\right) = \begin{pmatrix} \delta^j_m & 0 \\ p_s \partial_m g^{js} & g^{jm} \end{pmatrix}. \tag{1. 2}$$

Let ∇ be an affine connection on M and ${}^C\nabla$ be the complete lift of ∇. Then ${}^C\nabla$ is an affine connection on TM tangent bundle. We denote by Γ^k_{ij} the components of ∇ affine connection with respect to the local coordinates $\left(x^h\right)$ in M and ${}^C\Gamma^k_{ij}$ the components of ${}^C\nabla$ affine connection according to the induced coordinates $\left(x^h, y^h\right)$ in TM. The $g-$lifts are described newly by Cakan and Salimov. The $g-$lifts of some tensor fields are obtained by transferring complete lifts of some tensor field from TM tangent bundle to T^*M cotangent bundle via the musical isomorphism [8]. In this study we obtained the $g-$lift ${}^G\overset{*}{\nabla}$ of ∇ affine connection to the T^*M cotangent bundle via the musical isomorphism.

Let R be curvature tensor of ∇ affine connection with R^h_{kji} components on M manifold. ${}^G\overset{*}{R}$ the $g-$lift of R curvature tensor is obtained to the T^*M cotangent bundle via the musical isomorphism.

2. The g-lift of Affine connection

Let M be a manifold with ∇ affine connection. There exists a unique ${}^C\nabla$ affine connection in TM which satisfies

$${}^C\nabla_{{}^CY}\, {}^CZ = {}^C\left(\nabla_Y Z\right)$$

for any $Y, Z \in \Im^1_0(M)$ [15].

Let Γ_{ji}^{h} be components of ∇ according to the local coordinates $\left(x^{h}\right)$ in M. Let ${}^{C}\Gamma_{JI}^{H}$ be components of the complete lift ${}^{C}\nabla$ according to the induced coordinates $\left(x^{h},y^{h}\right)$ in TM. The components of the ${}^{C}\nabla$ are given by

$$\begin{array}{llll} {}^{C}\Gamma_{ji}^{h}=\Gamma_{ji}^{h}, & {}^{C}\Gamma_{j\bar{i}}^{h}=0, & {}^{C}\Gamma_{\bar{j}i}^{h}=0, & {}^{C}\Gamma_{\bar{j}\bar{i}}^{h}=0 \\ {}^{C}\Gamma_{ji}^{\bar{h}}=y^{s}\partial_{s}\Gamma_{ji}^{h}, & {}^{C}\Gamma_{j\bar{i}}^{\bar{h}}=\Gamma_{ji}^{h}, & {}^{C}\Gamma_{\bar{j}i}^{\bar{h}}=\Gamma_{ji}^{h}, & {}^{C}\Gamma_{\bar{j}\bar{i}}^{\bar{h}}=0 \end{array} \tag{2. 3}$$

according to the induced coordinates $\left(x^{h},y^{h}\right)$ in TM [14]. And Let ${}^{C}\overset{*}{\Gamma}{}_{JI}^{H}$ be components of the complete lift ${}^{C}\overset{*}{\nabla}$ according to the induced coordinates $\left(x^{h},p_{h}\right)$ in $T^{*}M$. The components of the ${}^{C}\overset{*}{\nabla}$ are given by

$$\begin{array}{lll} {}^{C}\overset{*}{\Gamma}{}_{ji}^{h} & = & \Gamma_{ji}^{h}, \quad {}^{C}\overset{*}{\Gamma}{}_{j\bar{i}}^{h}=0, \quad {}^{C}\overset{*}{\Gamma}{}_{\bar{j}i}^{h}=0, \quad {}^{C}\overset{*}{\Gamma}{}_{\bar{j}\bar{i}}^{h}=0 \\ {}^{C}\overset{*}{\Gamma}{}_{ji}^{\bar{h}} & = & p_{s}\left(\partial_{h}\Gamma_{ji}^{s}-\partial_{j}\Gamma_{ih}^{s}-\partial_{i}\Gamma_{jh}^{s}+2\Gamma_{ht}^{s}\Gamma_{ji}^{t}\right) \\ {}^{C}\overset{*}{\Gamma}{}_{j\bar{i}}^{\bar{h}} & = & -\Gamma_{jh}^{i}, \quad {}^{C}\overset{*}{\Gamma}{}_{\bar{j}i}^{\bar{h}}=-\Gamma_{hi}^{j}, \quad {}^{C}\overset{*}{\Gamma}{}_{\bar{j}\bar{i}}^{\bar{h}}=0 \end{array} \tag{2. 4}$$

according to the induced coordinates $\left(x^{h},p_{h}\right)$ in $T^{*}M$ [15].

Theorem 2.1. *Let M be a $n-$dimensional pseudo Riemannian manifold with pseudo Riemannian metric g. Let ${}^{C}\nabla$ and ${}^{C}\overset{*}{\nabla}$ be complete lifts of ∇ affine connection to TM and $T^{*}M$, respectively. Then the differential of ${}^{C}\nabla$ by $g^{\flat}$, i.e. a $g-$lift ${}^{G}\overset{*}{\nabla}$ in the cotangent bundle $T^{*}M$, coincides with the complete lift ${}^{C}\overset{*}{\nabla}$ in the cotangent bundle $T^{*}M$ if ∇ is a Riemannian connection which is a metric connection with vanishing torsion.*

Proof. Using (1.1) and (1.2) from $\widetilde{A}_{JI}^{K}=\frac{\partial^{2}x^{K}}{\partial x^{J}\partial x^{I}}$ equation we obtain the components

$$\begin{array}{lll} \widetilde{A}_{ji}^{k} & = & 0, \quad \widetilde{A}_{j\bar{i}}^{k}=0, \quad \widetilde{A}_{\bar{j}i}^{k}=0, \quad \widetilde{A}_{\bar{j}\bar{i}}^{k}=0 \\ \widetilde{A}_{ji}^{\bar{k}} & = & \dfrac{p_{s}\partial^{2}g^{ks}}{\partial x^{j}\partial x^{i}}, \quad \widetilde{A}_{j\bar{i}}^{\bar{k}}=\partial_{j}g^{ki}, \quad \widetilde{A}_{\bar{j}i}^{\bar{k}}=\partial_{i}g^{kj}, \quad \widetilde{A}_{\bar{j}\bar{i}}^{\bar{k}}=0. \end{array} \tag{2. 5}$$

Using (1.1), (1.2), (2.3) and (2.5) the components of ${}^{G}\overset{*}{\nabla}$ are obtained from equation

$$g_{*}^{\flat}\,{}^{C}\Gamma=\left({}^{G}\overset{*}{\Gamma}{}_{JI}^{H}\right)=\widetilde{A}_{J}^{M}\widetilde{A}_{I}^{S}A_{K}^{H}\,{}^{C}\Gamma_{MS}^{K}+A_{K}^{H}\widetilde{A}_{JI}^{K}$$

where $H,I,...=1,...,2n$:

$$\begin{aligned} &{}^{G}\overset{*}{\Gamma}{}_{ji}^{h}=\widetilde{A}_{j}^{m}\widetilde{A}_{i}^{s}A_{k}^{h}\,{}^{C}\Gamma_{ms}^{k}+\widetilde{A}_{j}^{m}\widetilde{A}_{i}^{s}A_{\bar{k}}^{h}\,{}^{C}\Gamma_{ms}^{\bar{k}}+\widetilde{A}_{j}^{\bar{m}}\widetilde{A}_{i}^{s}A_{k}^{h}\,{}^{C}\Gamma_{\bar{m}s}^{k}+\widetilde{A}_{j}^{m}\widetilde{A}_{i}^{\bar{s}}A_{k}^{h}\,{}^{C}\Gamma_{m\bar{s}}^{k} \\ &+\widetilde{A}_{j}^{\bar{m}}\widetilde{A}_{i}^{\bar{s}}A_{k}^{h}\,{}^{C}\Gamma_{\bar{m}\bar{s}}^{k}+\widetilde{A}_{j}^{\bar{m}}\widetilde{A}_{i}^{s}A_{\bar{k}}^{h}\,{}^{C}\Gamma_{\bar{m}s}^{\bar{k}}+\widetilde{A}_{j}^{m}\widetilde{A}_{i}^{\bar{s}}A_{\bar{k}}^{h}\,{}^{C}\Gamma_{m\bar{s}}^{\bar{k}}+\widetilde{A}_{j}^{\bar{m}}\widetilde{A}_{i}^{\bar{s}}A_{\bar{k}}^{h}\,{}^{C}\Gamma_{\bar{m}\bar{s}}^{\bar{k}} \\ &+A_{k}^{h}\widetilde{A}_{ji}^{k}+A_{\bar{k}}^{h}\widetilde{A}_{ji}^{\bar{k}} \\ &=\delta_{j}^{m}\delta_{i}^{s}\delta_{k}^{h}\,\Gamma_{ms}^{k}=\Gamma_{ji}^{h} \end{aligned}$$

$$
\begin{aligned}
{}^{G}\overset{*}{\Gamma}{}^{h}_{j\bar{i}} &= \widetilde{A}^{m}_{j}\widetilde{A}^{s}_{\bar{i}}A^{h}_{k}\;{}^{C}\Gamma^{k}_{ms} + \widetilde{A}^{m}_{j}\widetilde{A}^{s}_{\bar{i}}A^{h}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{ms} + \widetilde{A}^{\bar{m}}_{j}\widetilde{A}^{s}_{\bar{i}}A^{h}_{k}\;{}^{C}\Gamma^{k}_{\bar{m}s} + \widetilde{A}^{m}_{j}\widetilde{A}^{\bar{s}}_{\bar{i}}A^{h}_{k}\;{}^{C}\Gamma^{k}_{m\bar{s}} \\
&+\widetilde{A}^{\bar{m}}_{j}\widetilde{A}^{\bar{s}}_{\bar{i}}A^{h}_{k}\;{}^{C}\Gamma^{k}_{\overline{ms}} + \widetilde{A}^{\bar{m}}_{j}\widetilde{A}^{s}_{\bar{i}}A^{h}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{\bar{m}s} + \widetilde{A}^{m}_{j}\widetilde{A}^{\bar{s}}_{\bar{i}}A^{h}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{m\bar{s}} + \widetilde{A}^{\bar{m}}_{j}\widetilde{A}^{\bar{s}}_{\bar{i}}A^{h}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{\overline{ms}} \\
&+A^{h}_{k}\widetilde{A}^{k}_{j\bar{i}} + A^{h}_{\bar{k}}\widetilde{A}^{\bar{k}}_{j\bar{i}} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
{}^{G}\overset{*}{\Gamma}{}^{h}_{\bar{j}i} &= \widetilde{A}^{m}_{\bar{j}}\widetilde{A}^{s}_{i}A^{h}_{k}\;{}^{C}\Gamma^{k}_{ms} + \widetilde{A}^{m}_{\bar{j}}\widetilde{A}^{s}_{i}A^{h}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{ms} + \widetilde{A}^{\bar{m}}_{\bar{j}}\widetilde{A}^{s}_{i}A^{h}_{k}\;{}^{C}\Gamma^{k}_{\bar{m}s} + \widetilde{A}^{m}_{\bar{j}}\widetilde{A}^{\bar{s}}_{i}A^{h}_{k}\;{}^{C}\Gamma^{k}_{m\bar{s}} \\
&+\widetilde{A}^{\bar{m}}_{\bar{j}}\widetilde{A}^{\bar{s}}_{i}A^{h}_{k}\;{}^{C}\Gamma^{k}_{\overline{ms}} + \widetilde{A}^{\bar{m}}_{\bar{j}}\widetilde{A}^{s}_{i}A^{h}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{\bar{m}s} + \widetilde{A}^{m}_{\bar{j}}\widetilde{A}^{\bar{s}}_{i}A^{h}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{m\bar{s}} + \widetilde{A}^{\bar{m}}_{\bar{j}}\widetilde{A}^{\bar{s}}_{i}A^{h}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{\overline{ms}} \\
&+A^{h}_{k}\widetilde{A}^{k}_{\bar{j}i} + A^{h}_{\bar{k}}\widetilde{A}^{\bar{k}}_{\bar{j}i} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
{}^{G}\Gamma^{\bar{h}}_{j\bar{i}} &= \widetilde{A}^{m}_{j}\widetilde{A}^{s}_{\bar{i}}A^{\bar{h}}_{k}\;{}^{C}\Gamma^{k}_{ms} + \widetilde{A}^{m}_{j}\widetilde{A}^{s}_{\bar{i}}A^{\bar{h}}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{ms} + \widetilde{A}^{\bar{m}}_{j}\widetilde{A}^{s}_{\bar{i}}A^{\bar{h}}_{k}\;{}^{C}\Gamma^{k}_{\bar{m}s} + \widetilde{A}^{m}_{j}\widetilde{A}^{\bar{s}}_{\bar{i}}A^{\bar{h}}_{k}\;{}^{C}\Gamma^{k}_{m\bar{s}} \\
&+\widetilde{A}^{\bar{m}}_{j}\widetilde{A}^{\bar{s}}_{\bar{i}}A^{\bar{h}}_{k}\;{}^{C}\Gamma^{k}_{\overline{ms}} + \widetilde{A}^{\bar{m}}_{j}\widetilde{A}^{s}_{\bar{i}}A^{\bar{h}}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{\bar{m}s} + \widetilde{A}^{m}_{j}\widetilde{A}^{\bar{s}}_{\bar{i}}A^{\bar{h}}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{m\bar{s}} + \widetilde{A}^{\bar{m}}_{j}\widetilde{A}^{\bar{s}}_{\bar{i}}A^{\bar{h}}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{\overline{ms}} \\
&+A^{\bar{h}}_{k}\widetilde{A}^{k}_{j\bar{i}} + A^{\bar{h}}_{\bar{k}}\widetilde{A}^{\bar{k}}_{j\bar{i}} \\
&= \delta^{m}_{j}g^{si}g_{hk}\Gamma^{k}_{ms} + g_{hk}\partial_{j}g^{ki} \\
&= g^{si}\tfrac{1}{2}\left(\partial_{j}g_{sh} + \partial_{s}g_{hj} - \partial_{h}g_{js}\right) - g^{ki}\partial_{j}g_{hk} \\
&= \tfrac{1}{2}g^{si}\left(-\partial_{j}g_{sh} - \partial_{h}g_{js} + \partial_{s}g_{hj}\right) \\
&= -\tfrac{1}{2}g^{si}\left(\partial_{j}g_{sh} + \partial_{h}g_{js} - \partial_{s}g_{hj}\right) \\
&= -\Gamma^{i}_{jh}
\end{aligned}
$$

$$
\begin{aligned}
{}^{G}\overset{*}{\Gamma}{}^{\bar{h}}_{\bar{j}i} &= \widetilde{A}^{m}_{\bar{j}}\widetilde{A}^{s}_{i}A^{\bar{h}}_{k}\;{}^{C}\Gamma^{k}_{ms} + \widetilde{A}^{m}_{\bar{j}}\widetilde{A}^{s}_{i}A^{\bar{h}}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{ms} + \widetilde{A}^{\bar{m}}_{\bar{j}}\widetilde{A}^{s}_{i}A^{\bar{h}}_{k}\;{}^{C}\Gamma^{k}_{\bar{m}s} + \widetilde{A}^{m}_{\bar{j}}\widetilde{A}^{\bar{s}}_{i}A^{\bar{h}}_{k}\;{}^{C}\Gamma^{k}_{m\bar{s}} \\
&+\widetilde{A}^{\bar{m}}_{\bar{j}}\widetilde{A}^{\bar{s}}_{i}A^{\bar{h}}_{k}\;{}^{C}\Gamma^{k}_{\overline{ms}} + \widetilde{A}^{\bar{m}}_{\bar{j}}\widetilde{A}^{s}_{i}A^{\bar{h}}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{\bar{m}s} + \widetilde{A}^{m}_{\bar{j}}\widetilde{A}^{\bar{s}}_{i}A^{\bar{h}}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{m\bar{s}} + \widetilde{A}^{\bar{m}}_{\bar{j}}\widetilde{A}^{\bar{s}}_{i}A^{\bar{h}}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{\overline{ms}} \\
&+A^{\bar{h}}_{k}\widetilde{A}^{k}_{\bar{j}i} + A^{\bar{h}}_{\bar{k}}\widetilde{A}^{\bar{k}}_{\bar{j}i} \\
&= g^{mj}\delta^{s}_{i}g_{hk}\Gamma^{k}_{ms} + g_{hk}\partial_{i}g^{kj} \\
&= g^{mj}g_{hk}\Gamma^{k}_{mi} - g^{kj}\partial_{i}g_{hk} \\
&= g^{kj}g_{hm}\Gamma^{m}_{ki} - g^{kj}\partial_{i}g_{hk} \\
&= g^{kj}\tfrac{1}{2}\left(\partial_{k}g_{hi} + \partial_{i}g_{hk} - \partial_{h}g_{ki}\right) - g^{kj}\partial_{i}g_{hk} \\
&= -\tfrac{1}{2}g^{kj}\left(\partial_{h}g_{ki} + \partial_{i}g_{hk} - \partial_{k}g_{hi}\right) \\
&= -\Gamma^{j}_{hi}
\end{aligned}
$$

$$
\begin{aligned}
{}^{G}\overset{*}{\Gamma}{}^{h}_{\overline{ji}} &= \widetilde{A}^{m}_{\bar{j}}\widetilde{A}^{s}_{\bar{i}}A^{h}_{k}\;{}^{C}\Gamma^{k}_{ms} + \widetilde{A}^{m}_{\bar{j}}\widetilde{A}^{s}_{\bar{i}}A^{h}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{ms} + \widetilde{A}^{\bar{m}}_{\bar{j}}\widetilde{A}^{s}_{\bar{i}}A^{h}_{k}\;{}^{C}\Gamma^{k}_{\bar{m}s} + \widetilde{A}^{m}_{\bar{j}}\widetilde{A}^{\bar{s}}_{\bar{i}}A^{h}_{k}\;{}^{C}\Gamma^{k}_{m\bar{s}} \\
&+\widetilde{A}^{\bar{m}}_{\bar{j}}\widetilde{A}^{\bar{s}}_{\bar{i}}A^{h}_{k}\;{}^{C}\Gamma^{k}_{\overline{ms}} + \widetilde{A}^{\bar{m}}_{\bar{j}}\widetilde{A}^{s}_{\bar{i}}A^{h}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{\bar{m}s} + \widetilde{A}^{m}_{\bar{j}}\widetilde{A}^{\bar{s}}_{\bar{i}}A^{h}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{m\bar{s}} + \widetilde{A}^{\bar{m}}_{\bar{j}}\widetilde{A}^{\bar{s}}_{\bar{i}}A^{h}_{\bar{k}}\;{}^{C}\Gamma^{\bar{k}}_{\overline{ms}} \\
&+A^{h}_{k}\widetilde{A}^{k}_{\overline{ji}} + A^{h}_{\bar{k}}\widetilde{A}^{\bar{k}}_{\overline{ji}} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
&{}^{G}\overset{*}{\Gamma}{}^{\bar h}_{\bar j \bar i} = \widetilde{A}^{m}_{\bar j}\widetilde{A}^{s}_{\bar i}A^{\bar h}_{k}\,{}^{C}\Gamma^{k}_{ms} + \widetilde{A}^{m}_{\bar j}\widetilde{A}^{s}_{\bar i}A^{\bar h}_{\bar k}\,{}^{C}\Gamma^{\bar k}_{ms} + \widetilde{A}^{\bar m}_{\bar j}\widetilde{A}^{s}_{\bar i}A^{\bar h}_{k}\,{}^{C}\Gamma^{k}_{\bar m s} + \widetilde{A}^{m}_{\bar j}\widetilde{A}^{\bar s}_{\bar i}A^{\bar h}_{k}\,{}^{C}\Gamma^{k}_{m\bar s} \\
&+\widetilde{A}^{\bar m}_{\bar j}\widetilde{A}^{\bar s}_{\bar i}A^{\bar h}_{k}\,{}^{C}\Gamma^{k}_{\overline{ms}} + \widetilde{A}^{\bar m}_{\bar j}\widetilde{A}^{s}_{\bar i}A^{\bar h}_{\bar k}\,{}^{C}\Gamma^{\bar k}_{\bar m s} + \widetilde{A}^{m}_{\bar j}\widetilde{A}^{\bar s}_{\bar i}A^{\bar h}_{\bar k}\,{}^{C}\Gamma^{\bar k}_{m\bar s} + \widetilde{A}^{\bar m}_{\bar j}\widetilde{A}^{\bar s}_{\bar i}A^{\bar h}_{\bar k}\,{}^{C}\Gamma^{\bar k}_{\overline{ms}} \\
&+A^{\bar h}_{k}\widetilde{A}^{k}_{\overline{ji}} + A^{\bar h}_{\bar k}\widetilde{A}^{\bar k}_{\overline{ji}} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
&{}^{G}\overset{*}{\Gamma}{}^{\bar h}_{ji} = \widetilde{A}^{m}_{j}\widetilde{A}^{s}_{i}A^{\bar h}_{k}\,{}^{C}\Gamma^{k}_{ms} + \widetilde{A}^{m}_{j}\widetilde{A}^{s}_{i}A^{\bar h}_{\bar k}\,{}^{C}\Gamma^{\bar k}_{ms} + \widetilde{A}^{\bar m}_{j}\widetilde{A}^{s}_{i}A^{\bar h}_{k}\,{}^{C}\Gamma^{k}_{\bar m s} + \widetilde{A}^{m}_{j}\widetilde{A}^{\bar s}_{i}A^{\bar h}_{k}\,{}^{C}\Gamma^{k}_{m\bar s} \\
&+\widetilde{A}^{\bar m}_{j}\widetilde{A}^{\bar s}_{i}A^{\bar h}_{k}\,{}^{C}\Gamma^{k}_{\overline{ms}} + \widetilde{A}^{\bar m}_{j}\widetilde{A}^{s}_{i}A^{\bar h}_{\bar k}\,{}^{C}\Gamma^{\bar k}_{\bar m s} + \widetilde{A}^{m}_{j}\widetilde{A}^{\bar s}_{i}A^{\bar h}_{\bar k}\,{}^{C}\Gamma^{\bar k}_{m\bar s} + \widetilde{A}^{\bar m}_{j}\widetilde{A}^{\bar s}_{i}A^{\bar h}_{\bar k}\,{}^{C}\Gamma^{\bar k}_{\overline{ms}} \\
&+A^{\bar h}_{k}\widetilde{A}^{k}_{ji} + A^{\bar h}_{\bar k}\widetilde{A}^{\bar k}_{ji} \\
&= \delta^{m}_{j}\delta^{s}_{i}y^{t}\partial_{k}g_{th}\Gamma^{k}_{ms} + \delta^{m}_{j}\delta^{s}_{i}g_{hk}y^{t}\partial_{t}\Gamma^{k}_{ms} + p_{t}\partial_{j}g^{mt}\delta^{s}_{i}g_{hk}\Gamma^{k}_{ms} \\
&+\delta^{m}_{j}p_{t}\partial_{i}g^{st}g_{hk}\Gamma^{k}_{ms} + g_{hk}p_{t}\partial^{2}_{ji}g^{tk} \\
&= y^{t}\left(\partial_{k}g_{th}\right)\Gamma^{k}_{ji} + g_{hk}y^{t}\partial_{t}\Gamma^{k}_{ji} + p_{t}\partial_{j}g^{mt}g_{hk}\Gamma^{k}_{mi} + p_{t}\partial_{i}g^{st}g_{hk}\Gamma^{k}_{js} \\
&+g_{hk}p_{t}\partial^{2}_{ji}g^{tk} \\
&= y^{t}\left(\partial_{k}g_{th}\right)\Gamma^{k}_{ji} + g_{hk}y^{t}\partial_{t}\Gamma^{k}_{ji} + p_{t}\left(\partial_{j}g^{mt}\right)g_{hk}\Gamma^{k}_{mi} + p_{t}\left(\partial_{i}g^{st}\right)g_{hk}\Gamma^{k}_{js} \\
&-p_{t}\left(\partial_{j}\Gamma^{t}_{ih}\right) - g_{hk}p_{t}\Gamma^{t}_{is}\partial_{j}g^{sk} - g_{hk}p_{t}\left(\partial_{j}\Gamma^{k}_{is}\right)g^{ts} - g_{hk}p_{t}\Gamma^{k}_{is}\partial_{j}g^{ts} \\
&= -p_{t}\left(\partial_{j}\Gamma^{t}_{ih}\right) + y^{t}\left(\partial_{k}g_{th}\right)\Gamma^{k}_{ji} + g_{hk}y^{t}\partial_{t}\Gamma^{k}_{ji} + p_{t}\left(\partial_{i}g^{st}\right)g_{hk}\Gamma^{k}_{js} \\
&-g_{hk}p_{t}\Gamma^{t}_{is}\partial_{j}g^{sk} - g_{hk}p_{t}\left(\partial_{j}\Gamma^{k}_{is}\right)g^{ts} \\
&= -p_{t}\left(\partial_{j}\Gamma^{t}_{ih}\right) + y^{t}\left(\partial_{k}g_{th}\right)\Gamma^{k}_{ji} + g_{hk}y^{t}\partial_{t}\Gamma^{k}_{ji} - g_{hk}p_{t}\left(\partial_{j}\Gamma^{k}_{is}\right)g^{ts} \\
&+p_{t}\left(\partial_{i}g^{st}\right)g_{hk}\Gamma^{k}_{js} - g_{hk}p_{t}\Gamma^{t}_{is}\left(\partial_{j}g^{sk}\right) \\
&= -p_{t}\left(\partial_{j}\Gamma^{t}_{ih}\right) + y^{t}\left(\partial_{k}g_{th}\right)\Gamma^{k}_{ji} + g_{hk}\left(g^{ts}p_{s}\partial_{t}\Gamma^{k}_{ji} - p_{t}g^{ts}\partial_{j}\Gamma^{k}_{is}\right) \\
&+p_{t}\left(-\Gamma^{s}_{im}g^{mt} - \Gamma^{t}_{im}g^{sm}\right)g_{hk}\Gamma^{k}_{js} - g_{hk}p_{t}\Gamma^{t}_{is}\left(-\Gamma^{s}_{jm}g^{mk} - \Gamma^{k}_{jm}g^{ms}\right) \\
&= -p_{t}\left(\partial_{j}\Gamma^{t}_{ih}\right) + y^{t}\left(\partial_{k}g_{th}\right)\Gamma^{k}_{ji} + g_{hk}g^{ts}p_{s}\left(\partial_{t}\Gamma^{k}_{ji} - \partial_{j}\Gamma^{k}_{it}\right) \\
&-g_{hk}p_{s}\left(\Gamma^{t}_{im}\Gamma^{k}_{jt}g^{ms} + \Gamma^{s}_{im}\Gamma^{k}_{jt}g^{tm}\right) + g_{hk}p_{s}\left(\Gamma^{s}_{it}\Gamma^{t}_{jm}g^{mk} + \Gamma^{s}_{it}\Gamma^{k}_{jm}g^{mt}\right) \\
&= -p_{t}\left(\partial_{j}\Gamma^{t}_{ih}\right) + g^{ts}p_{s}\left(\Gamma^{m}_{kt}g_{mh} + \Gamma^{m}_{kh}g_{tm}\right)\Gamma^{k}_{ji} + g_{hk}g^{ts}p_{s}\left(\partial_{t}\Gamma^{k}_{ji} - \partial_{j}\Gamma^{k}_{it}\right) \\
&-g_{hk}p_{s}\left(\Gamma^{t}_{im}\Gamma^{k}_{jt}g^{ms} + \Gamma^{s}_{im}\Gamma^{k}_{jt}g^{tm}\right) + g_{hk}p_{s}\left(\Gamma^{s}_{it}\Gamma^{t}_{jm}g^{mk} + \Gamma^{s}_{it}\Gamma^{k}_{jm}g^{mt}\right) \\
&= -p_{t}\left(\partial_{j}\Gamma^{t}_{ih}\right) + p_{s}\Gamma^{s}_{hk}\Gamma^{k}_{ji} + g^{ts}g_{mh}p_{s}\Gamma^{m}_{kt}\Gamma^{k}_{ji} + g_{hk}g^{ts}p_{s}\left(\partial_{t}\Gamma^{k}_{ji} - \partial_{j}\Gamma^{k}_{it}\right) \\
&-g_{hk}p_{s}\left(\Gamma^{t}_{im}\Gamma^{k}_{jt}g^{ms} + \Gamma^{s}_{im}\Gamma^{k}_{jt}g^{tm}\right) + g_{hk}p_{s}\left(\Gamma^{s}_{it}\Gamma^{t}_{jm}g^{mk} + \Gamma^{s}_{it}\Gamma^{k}_{jm}g^{mt}\right) \\
&+p_{t}R^{t}_{hij} - p_{t}R^{t}_{hij} \\
&= p_{s}\left(\partial_{h}\Gamma^{s}_{ij} - \partial_{i}\Gamma^{s}_{jh} - \partial_{j}\Gamma^{s}_{ih} + 2\Gamma^{s}_{hk}\Gamma^{k}_{ij}\right) + g_{hk}g^{ts}p_{s}\left(\partial_{t}\Gamma^{k}_{ji} - \partial_{j}\Gamma^{k}_{it}\right) \\
&-p_{t}R^{t}_{hij} - g_{hk}g^{ms}p_{s}\Gamma^{t}_{im}\Gamma^{k}_{jt} - g_{hk}g^{mt}p_{s}\Gamma^{s}_{im}\Gamma^{k}_{jt} \\
&+g_{hk}g^{mt}p_{s}\Gamma^{s}_{it}\Gamma^{t}_{jm} + g_{mh}g^{ts}p_{s}\Gamma^{m}_{kt}\Gamma^{k}_{ji} \\
&= p_{s}\left(\partial_{h}\Gamma^{s}_{ij} - \partial_{i}\Gamma^{s}_{jh} - \partial_{j}\Gamma^{s}_{ih} + 2\Gamma^{s}_{hk}\Gamma^{k}_{ij}\right) - p_{t}R^{t}_{hij} + g_{hk}g^{ms}p_{s}\left(\Gamma^{k}_{mt}\Gamma^{t}_{ji} - \Gamma^{t}_{im}\Gamma^{k}_{jt}\right) \\
&+g_{hk}g^{mt}p_{s}\left(\Gamma^{s}_{it}\Gamma^{k}_{jm} - \Gamma^{s}_{im}\Gamma^{k}_{jt}\right) + g_{hk}g^{ms}p_{s}\partial_{m}\Gamma^{k}_{ji} - g_{hk}g^{ms}p_{s}\partial_{j}\Gamma^{k}_{im}
\end{aligned}
$$

$$
\begin{aligned}
&= p_s\left(\partial_h\Gamma^s_{ij} - \partial_i\Gamma^s_{jh} - \partial_j\Gamma^s_{ih} + 2\Gamma^s_{hk}\Gamma^k_{ij}\right) - p_t R^t_{hij} \\
&+ g_{hk} g^{ms} p_s \left(\partial_m\Gamma^k_{ji} - \partial_j\Gamma^k_{im} + \Gamma^k_{mt}\Gamma^t_{ji} - \Gamma^t_{im}\Gamma^k_{jt}\right) \\
&+ g_{hk} g^{mt} p_s \left(\Gamma^s_{it}\Gamma^k_{jm} - \Gamma^s_{im}\Gamma^k_{jt}\right) \\
&= p_s\left(\partial_h\Gamma^s_{ij} - \partial_i\Gamma^s_{jh} - \partial_j\Gamma^s_{ih} + 2\Gamma^s_{hk}\Gamma^k_{ij}\right) - p_t R^t_{hij} + p_s R^s_{hji} \\
&= p_s\left(\partial_h\Gamma^s_{ij} - \partial_i\Gamma^s_{jh} - \partial_j\Gamma^s_{ih} + 2\Gamma^s_{hk}\Gamma^k_{ij}\right) - y^t g_{ts} R^s_{hij} + y^t g_{ts} R_{jih}{}^s \\
&= p_s\left(\partial_h\Gamma^s_{ij} - \partial_i\Gamma^s_{jh} - \partial_j\Gamma^s_{ih} + 2\Gamma^s_{hk}\Gamma^k_{ij}\right) + y^t(-R_{hijt} + R_{tjih}) \\
&= p_s\left(\partial_h\Gamma^s_{ij} - \partial_i\Gamma^s_{jh} - \partial_j\Gamma^s_{ih} + 2\Gamma^s_{hk}\Gamma^k_{ij}\right) + (-R_{hijt} + R_{ihtj}) \\
&= p_s\left(\partial_h\Gamma^s_{ij} - \partial_i\Gamma^s_{jh} - \partial_j\Gamma^s_{ih} + 2\Gamma^s_{hk}\Gamma^k_{ij}\right) + (-R_{hijt} + R_{hijt}) \\
&= p_s\left(\partial_h\Gamma^s_{ij} - \partial_i\Gamma^s_{jh} - \partial_j\Gamma^s_{ih} + 2\Gamma^s_{hk}\Gamma^k_{ij}\right)
\end{aligned}
$$

We have seen that the components of the complete lift ${}^C\overset{*}{\nabla}$ to the cotangent bundle T^*M are given in the form (2.4) according to the induced coordinates in T^*M. So we obtain

$$ {}^G\overset{*}{\nabla} = {}^C\overset{*}{\nabla}. $$

□

3. The G-lift of Curvature tensor

Let R be a curvature tensor of ∇ affine connection on M manifold. The components R^h_{kji} of R are given by

$$ R^h_{kji} = \partial_k\Gamma^h_{ji} - \partial_j\Gamma^h_{ki} + \Gamma^h_{kt}\Gamma^t_{ji} - \Gamma^h_{jt}\Gamma^t_{ki} \ . $$

There exists a complete lift of curvature tensor ${}^C R$ in TM which satisfies

$$ {}^C R\left({}^C X, {}^C Y\right) {}^C Z = {}^C\left(R\left(X,Y\right)Z\right) $$

for any $X, Y, Z \in \Im^1_0(M)$.

Let ${}^C R^M_{TSN}$ be components of ${}^C R$ according to the induced coordinates $\left(x^h, y^h\right)$ in TM. The components of the ${}^C R$ are given by

$$
\begin{aligned}
{}^C R^m_{tsn} &= R^m_{tsn}, \quad {}^C R^{\overline{m}}_{tsn} = y^s\partial_s R^m_{tsn}, \\
{}^C R^{\overline{m}}_{ts\overline{n}} &= R^m_{tsn}, \quad {}^C R^{\overline{m}}_{t\overline{s}n} = R^m_{tsn}, \quad {}^C R^{\overline{m}}_{\overline{t}sn} = R^m_{tsn}
\end{aligned}
\tag{3. 6}
$$

according to the induced coordinates $\left(x^h, y^h\right)$ in TM. And the other components are zero.

Let ${}^C\overset{*}{R}{}^M_{TSN}$ be components of ${}^C\overset{*}{R}$ according to the induced coordinates $\left(x^h, p_h\right)$ in T^*M. According to the induced coordinates $\left(x^h, p_h\right)$ in T^*M, the components of the ${}^C\overset{*}{R}$ are given by

$$\begin{aligned}
{}^{C}\overset{*}{R}{}^{\ m}_{\ tsn} &= R^{m}_{tsn}, \\
{}^{C}\overset{*}{R}{}^{\ \overline{m}}_{\ tsn} &= p_a(\nabla_m R^{a}_{tsn} - \nabla_n R^{a}_{tsm} \\
&\quad +\Gamma^{a}_{mb}R^{b}_{tsn} + \Gamma^{a}_{tb}R^{b}_{nms} + \Gamma^{a}_{sb}R^{b}_{mnt} + \Gamma^{a}_{nb}R^{b}_{tsm}) \\
{}^{C}\overset{*}{R}{}^{\ \overline{m}}_{\ ts\overline{n}} &= -R^{n}_{tsm}, \quad {}^{C}\overset{*}{R}{}^{\ \overline{m}}_{\ t\overline{s}n} = -R^{s}_{tmn}, \quad {}^{C}\overset{*}{R}{}^{\ \overline{m}}_{\ \overline{t}sn} = -R^{t}_{msn}
\end{aligned} \tag{3. 7}$$

where $\nabla_{\partial_m} = \nabla_m$. And the other components are zero [15].

Theorem 3.1. *Let M be a $n-$dimensional pseudo Riemannian manifold with pseudo Riemannian metric g. Let ${}^{C}R$ and ${}^{C}\overset{*}{R}$ be complete lifts of R curvature tensor to TM and $T^{*}M$, respectively. Then the differential of ${}^{C}R$ by $g^{\flat}$, i.e. a $g-$lift ${}^{G}\overset{*}{R}$ in the cotangent bundle $T^{*}M$, coincides with the complete lift ${}^{C}\overset{*}{R}$ if ∇ is a Riemannian connection which is a metric connection with vanishing torsion.*

Proof. Using (1.1) , (1.2) and (3.6) the components of ${}^{G}\overset{*}{R}$ are obtained from equation

$$g^{\flat}_{*}\,{}^{C}R = \left({}^{G}\overset{*}{R}{}^{\ H}_{\ KJI} \right) = \left(\widetilde{A}^{H}_{M}\ A^{T}_{K}\ A^{S}_{J}\ A^{N}_{I}\ {}^{C}R{}^{\ M}_{\ TSN} \right)$$

where $H, I, ... = 1, ..., 2n$:

$$\begin{aligned}
{}^{G}\overset{*}{R}{}^{\ h}_{\ kji} &= \widetilde{A}^{h}_{m}A^{t}_{k}\ A^{s}_{j}A^{n}_{i}\ {}^{C}R^{m}_{tsn} + \widetilde{A}^{h}_{\overline{m}}A^{t}_{k}\ A^{s}_{j}A^{n}_{i}\ {}^{C}R^{\overline{m}}_{tsn} + \widetilde{A}^{h}_{m}A^{\overline{t}}_{k}\ A^{s}_{j}A^{n}_{i}\ {}^{C}R^{m}_{\overline{t}sn} \\
&\quad +\widetilde{A}^{h}_{m}A^{t}_{k}\ A^{\overline{s}}_{j}A^{n}_{i}\ {}^{C}R^{m}_{t\overline{s}n} + \widetilde{A}^{h}_{m}A^{t}_{k}\ A^{s}_{j}A^{\overline{n}}_{i}\ {}^{C}R^{m}_{ts\overline{n}} + \widetilde{A}^{h}_{\overline{m}}A^{\overline{t}}_{k}\ A^{s}_{j}A^{n}_{i}\ {}^{C}R^{\overline{m}}_{\overline{t}sn} \\
&\quad +\widetilde{A}^{h}_{\overline{m}}A^{t}_{k}\ A^{\overline{s}}_{j}A^{n}_{i}\ {}^{C}R^{\overline{m}}_{t\overline{s}n} + \widetilde{A}^{h}_{\overline{m}}A^{t}_{k}\ A^{s}_{j}A^{\overline{n}}_{i}\ {}^{C}R^{\overline{m}}_{ts\overline{n}} + \widetilde{A}^{h}_{m}A^{\overline{t}}_{k}\ A^{\overline{s}}_{j}A^{n}_{i}\ {}^{C}R^{m}_{\overline{t}\overline{s}n} \\
&\quad +\widetilde{A}^{h}_{m}A^{t}_{k}\ A^{\overline{s}}_{j}A^{\overline{n}}_{i}\ {}^{C}R^{m}_{t\overline{s}\overline{n}} + \widetilde{A}^{h}_{m}A^{\overline{t}}_{k}\ A^{s}_{j}A^{\overline{n}}_{i}\ {}^{C}R^{m}_{\overline{t}s\overline{n}} + \widetilde{A}^{h}_{m}A^{\overline{t}}_{k}\ A^{\overline{s}}_{j}A^{\overline{n}}_{i}\ {}^{C}R^{m}_{\overline{t}\overline{s}\overline{n}} \\
&\quad +\widetilde{A}^{h}_{\overline{m}}A^{\overline{t}}_{k}\ A^{\overline{s}}_{j}A^{n}_{i}\ {}^{C}R^{\overline{m}}_{\overline{t}\overline{s}n} + \widetilde{A}^{h}_{\overline{m}}A^{\overline{t}}_{k}\ A^{s}_{j}A^{\overline{n}}_{i}\ {}^{C}R^{\overline{m}}_{\overline{t}s\overline{n}} + \widetilde{A}^{h}_{\overline{m}}A^{t}_{k}\ A^{\overline{s}}_{j}A^{\overline{n}}_{i}\ {}^{C}R^{\overline{m}}_{t\overline{s}\overline{n}} \\
&\quad +\widetilde{A}^{h}_{\overline{m}}A^{\overline{t}}_{k}\ A^{\overline{s}}_{j}A^{\overline{n}}_{i}\ {}^{C}R^{\overline{m}}_{\overline{t}\overline{s}\overline{n}} \\
&= \delta^{h}_{m}\delta^{t}_{k}\delta^{s}_{j}\delta^{n}_{i}R^{m}_{tsn} \\
&= R^{h}_{kji}
\end{aligned}$$

$$\begin{aligned}
{}^{G}\overset{*}{R}{}^{\ h}_{\ kj\overline{i}} &= \widetilde{A}^{h}_{m}A^{t}_{k}\ A^{s}_{j}A^{n}_{\overline{i}}\ {}^{C}R^{m}_{tsn} + \widetilde{A}^{h}_{\overline{m}}A^{t}_{k}\ A^{s}_{j}A^{n}_{\overline{i}}\ {}^{C}R^{\overline{m}}_{tsn} + \widetilde{A}^{h}_{m}A^{\overline{t}}_{k}\ A^{s}_{j}A^{n}_{\overline{i}}\ {}^{C}R^{m}_{\overline{t}sn} \\
&\quad +\widetilde{A}^{h}_{m}A^{t}_{k}\ A^{\overline{s}}_{j}A^{n}_{\overline{i}}\ {}^{C}R^{m}_{t\overline{s}n} + \widetilde{A}^{h}_{m}A^{t}_{k}\ A^{s}_{j}A^{\overline{n}}_{\overline{i}}\ {}^{C}R^{m}_{ts\overline{n}} + \widetilde{A}^{h}_{\overline{m}}A^{\overline{t}}_{k}\ A^{s}_{j}A^{n}_{\overline{i}}\ {}^{C}R^{\overline{m}}_{\overline{t}sn} \\
&\quad +\widetilde{A}^{h}_{\overline{m}}A^{t}_{k}\ A^{\overline{s}}_{j}A^{n}_{\overline{i}}\ {}^{C}R^{\overline{m}}_{t\overline{s}n} + \widetilde{A}^{h}_{\overline{m}}A^{t}_{k}\ A^{s}_{j}A^{\overline{n}}_{\overline{i}}\ {}^{C}R^{\overline{m}}_{ts\overline{n}} + \widetilde{A}^{h}_{m}A^{\overline{t}}_{k}\ A^{\overline{s}}_{j}A^{n}_{\overline{i}}\ {}^{C}R^{m}_{\overline{t}\overline{s}n} \\
&\quad +\widetilde{A}^{h}_{m}A^{t}_{k}\ A^{\overline{s}}_{j}A^{\overline{n}}_{\overline{i}}\ {}^{C}R^{m}_{t\overline{s}\overline{n}} + \widetilde{A}^{h}_{m}A^{\overline{t}}_{k}\ A^{s}_{j}A^{\overline{n}}_{\overline{i}}\ {}^{C}R^{m}_{\overline{t}s\overline{n}} + \widetilde{A}^{h}_{m}A^{\overline{t}}_{k}\ A^{\overline{s}}_{j}A^{\overline{n}}_{\overline{i}}\ {}^{C}R^{m}_{\overline{t}\overline{s}\overline{n}} \\
&\quad +\widetilde{A}^{h}_{\overline{m}}A^{\overline{t}}_{k}\ A^{\overline{s}}_{j}A^{n}_{\overline{i}}\ {}^{C}R^{\overline{m}}_{\overline{t}\overline{s}n} + \widetilde{A}^{h}_{\overline{m}}A^{\overline{t}}_{k}\ A^{s}_{j}A^{\overline{n}}_{\overline{i}}\ {}^{C}R^{\overline{m}}_{\overline{t}s\overline{n}} + \widetilde{A}^{h}_{\overline{m}}A^{t}_{k}\ A^{\overline{s}}_{j}A^{\overline{n}}_{\overline{i}}\ {}^{C}R^{\overline{m}}_{t\overline{s}\overline{n}} \\
&\quad +\widetilde{A}^{h}_{\overline{m}}A^{\overline{t}}_{k}\ A^{\overline{s}}_{j}A^{\overline{n}}_{\overline{i}}\ {}^{C}R^{\overline{m}}_{\overline{t}\overline{s}\overline{n}} \\
&= 0
\end{aligned}$$

$$ {}^{G}\overset{*}{R}{}^{h}_{k\bar{j}i} = 0, \quad {}^{G}\overset{*}{R}{}^{h}_{\bar{k}ji} = 0 $$

$$
\begin{aligned}
{}^{G}\overset{*}{R}{}^{\bar{h}}_{kji} &= \widetilde{A}^{\bar{h}}_{m}A^{t}_{k}\,A^{s}_{j}A^{n}_{i}\,{}^{C}R^{m}_{tsn} + \widetilde{A}^{\bar{h}}_{\bar{m}}A^{t}_{k}\,A^{s}_{j}A^{n}_{i}\,{}^{C}R^{\bar{m}}_{tsn} + \widetilde{A}^{\bar{h}}_{m}A^{\bar{t}}_{k}\,A^{s}_{j}A^{n}_{i}\,{}^{C}R^{m}_{\bar{t}sn} \\
&+\widetilde{A}^{\bar{h}}_{m}A^{t}_{k}\,A^{\bar{s}}_{j}A^{n}_{i}\,{}^{C}R^{m}_{t\bar{s}n} + \widetilde{A}^{\bar{h}}_{m}A^{t}_{k}\,A^{s}_{j}A^{\bar{n}}_{i}\,{}^{C}R^{m}_{ts\bar{n}} + \widetilde{A}^{\bar{h}}_{\bar{m}}A^{\bar{t}}_{k}\,A^{s}_{j}A^{n}_{i}\,{}^{C}R^{\bar{m}}_{\bar{t}sn} \\
&+\widetilde{A}^{\bar{h}}_{\bar{m}}A^{t}_{k}\,A^{\bar{s}}_{j}A^{n}_{i}\,{}^{C}R^{\bar{m}}_{t\bar{s}n} + \widetilde{A}^{\bar{h}}_{\bar{m}}A^{t}_{k}\,A^{s}_{j}A^{\bar{n}}_{i}\,{}^{C}R^{\bar{m}}_{ts\bar{n}} + \widetilde{A}^{\bar{h}}_{m}A^{\bar{t}}_{k}\,A^{\bar{s}}_{j}A^{n}_{i}\,{}^{C}R^{m}_{\bar{t}\bar{s}n} \\
&+\widetilde{A}^{\bar{h}}_{m}A^{t}_{k}\,A^{\bar{s}}_{j}A^{\bar{n}}_{i}\,{}^{C}R^{m}_{t\bar{s}\bar{n}} + \widetilde{A}^{\bar{h}}_{m}A^{\bar{t}}_{k}\,A^{s}_{j}A^{\bar{n}}_{i}\,{}^{C}R^{m}_{\bar{t}s\bar{n}} + \widetilde{A}^{\bar{h}}_{m}A^{\bar{t}}_{k}\,A^{\bar{s}}_{j}A^{\bar{n}}_{i}\,{}^{C}R^{m}_{\bar{t}\bar{s}\bar{n}} \\
&+\widetilde{A}^{\bar{h}}_{\bar{m}}A^{\bar{t}}_{k}\,A^{\bar{s}}_{j}A^{n}_{i}\,{}^{C}R^{\bar{m}}_{\bar{t}\bar{s}n} + \widetilde{A}^{\bar{h}}_{\bar{m}}A^{\bar{t}}_{k}\,A^{s}_{j}A^{\bar{n}}_{i}\,{}^{C}R^{\bar{m}}_{\bar{t}s\bar{n}} + \widetilde{A}^{\bar{h}}_{\bar{m}}A^{t}_{k}\,A^{\bar{s}}_{j}A^{\bar{n}}_{i}\,{}^{C}R^{\bar{m}}_{t\bar{s}\bar{n}} \\
&+\widetilde{A}^{\bar{h}}_{\bar{m}}A^{\bar{t}}_{k}\,A^{\bar{s}}_{j}A^{\bar{n}}_{i}\,{}^{C}R^{\bar{m}}_{\bar{t}\bar{s}\bar{n}} \\
&= y^{c}\partial_{m}g_{hc}\delta^{t}_{k}\delta^{s}_{j}\delta^{n}_{i}R^{m}_{tsn} + g_{hm}\delta^{t}_{k}\delta^{s}_{j}\delta^{n}_{i}y^{d}\partial_{d}R^{m}_{tsn} + g_{hm}p_{e}\partial_{k}g^{te}\delta^{s}_{j}\delta^{n}_{i}R^{m}_{tsn} \\
&+g_{hm}\delta^{t}_{k}p_{a}\partial_{j}g^{sa}\delta^{n}_{i}R^{m}_{tsn} + g_{hm}\delta^{t}_{k}\delta^{s}_{j}p_{b}\partial_{i}g^{nb}R^{m}_{tsn} \\
&= p_{s}\left(\nabla_{h}R^{s}_{kji} - \nabla_{i}R^{s}_{kjh} + \Gamma^{s}_{ht}R^{t}_{kji} + \Gamma^{s}_{kt}R^{t}_{ihj} + \Gamma^{s}_{jt}R^{t}_{hik} + \Gamma^{s}_{it}R^{t}_{kjh}\right)
\end{aligned}
$$

$$
\begin{aligned}
{}^{G}\overset{*}{R}{}^{\bar{h}}_{\bar{k}ji} &= \widetilde{A}^{\bar{h}}_{m}A^{t}_{\bar{k}}\,A^{s}_{j}A^{n}_{i}\,{}^{C}R^{m}_{tsn} + \widetilde{A}^{\bar{h}}_{\bar{m}}A^{t}_{\bar{k}}\,A^{s}_{j}A^{n}_{i}\,{}^{C}R^{\bar{m}}_{tsn} + \widetilde{A}^{\bar{h}}_{m}A^{\bar{t}}_{\bar{k}}\,A^{s}_{j}A^{n}_{i}\,{}^{C}R^{m}_{\bar{t}sn} \\
&+\widetilde{A}^{\bar{h}}_{m}A^{t}_{\bar{k}}\,A^{\bar{s}}_{j}A^{n}_{i}\,{}^{C}R^{m}_{t\bar{s}n} + \widetilde{A}^{\bar{h}}_{m}A^{t}_{\bar{k}}\,A^{s}_{j}A^{\bar{n}}_{i}\,{}^{C}R^{m}_{ts\bar{n}} + \widetilde{A}^{\bar{h}}_{\bar{m}}A^{\bar{t}}_{\bar{k}}\,A^{s}_{j}A^{n}_{i}\,{}^{C}R^{\bar{m}}_{\bar{t}sn} \\
&+\widetilde{A}^{\bar{h}}_{\bar{m}}A^{t}_{\bar{k}}\,A^{\bar{s}}_{j}A^{n}_{i}\,{}^{C}R^{\bar{m}}_{t\bar{s}n} + \widetilde{A}^{\bar{h}}_{\bar{m}}A^{t}_{\bar{k}}\,A^{s}_{j}A^{\bar{n}}_{i}\,{}^{C}R^{\bar{m}}_{ts\bar{n}} + \widetilde{A}^{\bar{h}}_{m}A^{\bar{t}}_{\bar{k}}\,A^{\bar{s}}_{j}A^{n}_{i}\,{}^{C}R^{m}_{\bar{t}\bar{s}n} \\
&+\widetilde{A}^{\bar{h}}_{m}A^{t}_{\bar{k}}\,A^{\bar{s}}_{j}A^{\bar{n}}_{i}\,{}^{C}R^{m}_{t\bar{s}\bar{n}} + \widetilde{A}^{\bar{h}}_{m}A^{\bar{t}}_{\bar{k}}\,A^{s}_{j}A^{\bar{n}}_{i}\,{}^{C}R^{m}_{\bar{t}s\bar{n}} + \widetilde{A}^{\bar{h}}_{m}A^{\bar{t}}_{\bar{k}}\,A^{\bar{s}}_{j}A^{\bar{n}}_{i}\,{}^{C}R^{m}_{\bar{t}\bar{s}\bar{n}} \\
&+\widetilde{A}^{\bar{h}}_{\bar{m}}A^{\bar{t}}_{\bar{k}}\,A^{\bar{s}}_{j}A^{n}_{i}\,{}^{C}R^{\bar{m}}_{\bar{t}\bar{s}n} + \widetilde{A}^{\bar{h}}_{\bar{m}}A^{\bar{t}}_{\bar{k}}\,A^{s}_{j}A^{\bar{n}}_{i}\,{}^{C}R^{\bar{m}}_{\bar{t}s\bar{n}} + \widetilde{A}^{\bar{h}}_{\bar{m}}A^{t}_{\bar{k}}\,A^{\bar{s}}_{j}A^{\bar{n}}_{i}\,{}^{C}R^{\bar{m}}_{t\bar{s}\bar{n}} \\
&+\widetilde{A}^{\bar{h}}_{\bar{m}}A^{\bar{t}}_{\bar{k}}\,A^{\bar{s}}_{j}A^{\bar{n}}_{i}\,{}^{C}R^{\bar{m}}_{\bar{t}\bar{s}\bar{n}} \\
&= g_{hm}g^{tk}\delta^{s}_{j}\delta^{n}_{i}R^{m}_{tsn} \\
&= R^{k}_{hji}
\end{aligned}
$$

$$ {}^{G}\overset{*}{R}{}^{\bar{h}}_{k\bar{j}i} = R^{j}_{khi}, \quad {}^{G}\overset{*}{R}{}^{\bar{h}}_{kj\bar{i}} = R^{i}_{kjh} $$

$$
\begin{aligned}
{}^{G}\overset{*}{R}{}^{h}_{\bar{k}\bar{j}i} &= \widetilde{A}^{h}_{m}A^{t}_{\bar{k}}\,A^{s}_{\bar{j}}A^{n}_{i}\,{}^{C}R^{m}_{tsn} + \widetilde{A}^{h}_{\bar{m}}A^{t}_{\bar{k}}\,A^{s}_{\bar{j}}A^{n}_{i}\,{}^{C}R^{\bar{m}}_{tsn} + \widetilde{A}^{h}_{m}A^{\bar{t}}_{\bar{k}}\,A^{s}_{\bar{j}}A^{n}_{i}\,{}^{C}R^{m}_{\bar{t}sn} \\
&+\widetilde{A}^{h}_{m}A^{t}_{\bar{k}}\,A^{\bar{s}}_{\bar{j}}A^{n}_{i}\,{}^{C}R^{m}_{t\bar{s}n} + \widetilde{A}^{h}_{m}A^{t}_{\bar{k}}\,A^{s}_{\bar{j}}A^{\bar{n}}_{i}\,{}^{C}R^{m}_{ts\bar{n}} + \widetilde{A}^{h}_{\bar{m}}A^{\bar{t}}_{\bar{k}}\,A^{s}_{\bar{j}}A^{n}_{i}\,{}^{C}R^{\bar{m}}_{\bar{t}sn} \\
&+\widetilde{A}^{h}_{\bar{m}}A^{t}_{\bar{k}}\,A^{\bar{s}}_{\bar{j}}A^{n}_{i}\,{}^{C}R^{\bar{m}}_{t\bar{s}n} + \widetilde{A}^{h}_{\bar{m}}A^{t}_{\bar{k}}\,A^{s}_{\bar{j}}A^{\bar{n}}_{i}\,{}^{C}R^{\bar{m}}_{ts\bar{n}} + \widetilde{A}^{h}_{m}A^{\bar{t}}_{\bar{k}}\,A^{\bar{s}}_{\bar{j}}A^{n}_{i}\,{}^{C}R^{m}_{\bar{t}\bar{s}n} \\
&+\widetilde{A}^{h}_{m}A^{t}_{\bar{k}}\,A^{\bar{s}}_{\bar{j}}A^{\bar{n}}_{i}\,{}^{C}R^{m}_{t\bar{s}\bar{n}} + \widetilde{A}^{h}_{m}A^{\bar{t}}_{\bar{k}}\,A^{s}_{\bar{j}}A^{\bar{n}}_{i}\,{}^{C}R^{m}_{\bar{t}s\bar{n}} + \widetilde{A}^{h}_{m}A^{\bar{t}}_{\bar{k}}\,A^{\bar{s}}_{\bar{j}}A^{\bar{n}}_{i}\,{}^{C}R^{m}_{\bar{t}\bar{s}\bar{n}} \\
&+\widetilde{A}^{h}_{\bar{m}}A^{\bar{t}}_{\bar{k}}\,A^{\bar{s}}_{\bar{j}}A^{n}_{i}\,{}^{C}R^{\bar{m}}_{\bar{t}\bar{s}n} + \widetilde{A}^{h}_{\bar{m}}A^{\bar{t}}_{\bar{k}}\,A^{s}_{\bar{j}}A^{\bar{n}}_{i}\,{}^{C}R^{\bar{m}}_{\bar{t}s\bar{n}} + \widetilde{A}^{h}_{\bar{m}}A^{t}_{\bar{k}}\,A^{\bar{s}}_{\bar{j}}A^{\bar{n}}_{i}\,{}^{C}R^{\bar{m}}_{t\bar{s}\bar{n}} \\
&+\widetilde{A}^{h}_{\bar{m}}A^{\bar{t}}_{\bar{k}}\,A^{\bar{s}}_{\bar{j}}A^{\bar{n}}_{i}\,{}^{C}R^{\bar{m}}_{\bar{t}\bar{s}\bar{n}} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
{}^{G}\overset{*}{R}{}_{k\overline{j}\overline{i}}^{\,h} &= 0, \quad {}^{G}\overset{*}{R}{}_{\overline{k}j\overline{i}}^{\,h} = 0, \quad {}^{G}\overset{*}{R}{}_{\overline{k}\overline{j}i}^{\,\overline{h}} = 0, \quad {}^{G}\overset{*}{R}{}_{k\overline{j}\overline{i}}^{\,\overline{h}} = 0, \\
{}^{G}\overset{*}{R}{}_{\overline{k}j\overline{i}}^{\,\overline{h}} &= 0, \quad {}^{G}\overset{*}{R}{}_{\overline{k}\overline{j}\overline{i}}^{\,h} = 0, \quad {}^{G}\overset{*}{R}{}_{\overline{k}\overline{j}\overline{i}}^{\,\overline{h}} = 0 \ .
\end{aligned}
$$

We have seen that the components of the complete lift ${}^{C}\overset{*}{R}$ to the cotangent bundle $T^{*}M$ is given in the form (3.7) according to the induced coordinates in $T^{*}M$. So we obtain

$$
{}^{G}\overset{*}{R} = {}^{C}\overset{*}{R}.
$$

□

4. CONCLUSION

In this paper, we have studied the g-lift of affine connection and curvature tensor in the $T^{*}M$ cotangent bundle. Obtained the $g-$lift ${}^{G}\overset{*}{\nabla}$ and compared with the complete lift ${}^{C}\overset{*}{\nabla}$ in the $T^{*}M$ cotangent bundle. Obtained the $g-$lift ${}^{G}\overset{*}{R}$ and compared with the complete lift ${}^{C}\overset{*}{R}$ in the $T^{*}M$ cotangent bundle. It was seen that the $g-$lifts ${}^{G}\overset{*}{\nabla}$ and ${}^{G}\overset{*}{R}$ coincide with the complete lifts ${}^{C}\overset{*}{\nabla}$ and ${}^{C}\overset{*}{R}$, respectively, if ∇ is a Riemannian connection.

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