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Numerical Solution of Nonlocal Parabolic Partial Differential Equation via Bernstein Polynomial Method

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Abstract. In this paper we apply an efficient approaches based on Bernstein polynomials to solve one-dimensional partial differential equations (PDEs) subject to the given nonlocal conditions. The main idea is based on collocation and transforming the considered PDEs into their associated algebraic equations. Numerical results are presented through the illustrative graphs which demonstrate good accuracy.

AMS (MOS) Subject Classification Codes: 35-XX; 41AXX; 65-XX
Key Words: Parabolic partial differential equations, Non-local boundary conditions, Bernstein basis, Operational matrices.

1. Introduction

The development of numerical techniques for solving parabolic partial differential equations in physics subject to non-classical conditions is a subject of considerable interest. Numerical solutions of such PDEs together with traditional conditions were studied deeply by researchers in literature. However, these PDEs subject to nonclassical conditions were investigated by mathematicians, but improvements of the existing methods should be done to get more accurate solutions. There are many papers that deal with nonclassical conditions e.g.[4, 5, 6, 14, 8, 12]. Dehghan in [7], applied some numerical schemes to approximate. The usual numerical methods for PDEs subject to these nonclassical conditions are finite difference methods (FDMs), Galerkin techniques [3], collocation approaches [11], and Tau schemes [15]. Moreover, one can point out to the new methods such as Bernstein

Tau technique [16], Sinc collocation method [1]. This work is aimed at applying a very efficient method (Bernstein spectral method), for solving the following non-local boundary value problem:

$$u_t(x,t) - u_{xx}(x,t) = g(x,t),\ a < x < b,\ 0 < t \leq T, \tag{1. 1}$$

with initial condition

$$u(x,0) = f(x),\ a \leq x \leq b, \tag{1. 2}$$

and the non-classical conditions:

$$\lambda_0 u(0,t) = \int_0^1 p_0(x)u(x,t)dx + q_0(t),\ 0 < t \leq T, \tag{1. 3}$$

$$\lambda_1 u(1,t) = \int_0^1 p_1(x)u(x,t)dx + q_1(t),\ 0 < t \leq T, \tag{1. 4}$$

where x and t are the spatial and time coordinates respectively, $u(x,t)$ is unknown function to be determined, λ_0 and λ_1 are given constants and $g(x,t), f(x), p_0(x), p_1(x)$, $q_0(x)$ and $q_1(x)$ are suitably prescribed functions. The organization of this article is as follows: In Section 2, we describe Bernstein basis functions and its properties. In Section 3, the use of these basis is discussed for solving nonlocal parabolic equations. In Section 4 the proposed method is applied to several examples. The conclusions are discussed in Section 5.

2. The properties of Bernstein polynomials

The polynomials determined in the Bernstein basis enjoy considerable popularity in many different applications. For example in computer-aided design (CAD) applications [13, 9]. Bernstein polynomials (B-polynomials), have advantage of the continuity and unity partition properties of the basis set of B-polynomials over an interval [0,R]. The B-polynomial bases vanish except the first polynomial at x = 0, which is equal to 1 and the last polynomial at x = R, which is also equal to 1 over the interval [0,R]. Therefore, a greater flexibilty can be achieved using the imposed boundary conditions at both ends of the interval. In this section some definitions and formulas for Bernstein polynomials are summarized as following:

$$B_{k,n}(t) = \left(\begin{array}{c} n \\ k \end{array}\right) t^k (1-t)^{n-k},\ 0 \leq t \leq 1, \tag{2. 5}$$

where

$$\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n!}{k!(n-k)!}. \tag{2. 6}$$

By using the binomial expansion

$$(1-t)^{n-k} = \sum_{i=0}^{n-k} (-1)^i t^i \left(\begin{array}{c} n-k \\ i \end{array}\right), \tag{2. 7}$$

we have:

$$B_{k,n} = \sum_{i=0}^{n-k} (-1)^i t^i \left(\begin{array}{c} n-k \\ i \end{array}\right) \left(\begin{array}{c} n \\ k \end{array}\right) t^{k+i}. \tag{2. 8}$$

So, there are $n+1$ n-th degree B-polynomials. A polynomial $h(x)$ of degree m can be expressed as

$$h(x) = \sum_{i=0}^{n} d_i B_{i,n}(x) = d^T \phi(x), \tag{2. 9}$$

where the Bernstein coefficient vector d and the Bernstein vector $\phi(x)$ are given by

$$d^T = [d_0, d_1, ..., d_n], \tag{2. 10}$$

and

$$\phi^T(x) = [B_{0,n}(x), B_{1,n}(x), ..., B_{n,n}(x)]. \tag{2. 11}$$

Lemma 1: Let $\phi(x)$ be Bernstein polynomial then

$$\frac{d\phi(x)}{dx} = D_b \phi(x), \tag{2. 12}$$

where D_b is the $(n+1) \times (n+1)$ operational matrix of derivative given by

$$D_b = A \Lambda V, \tag{2. 13}$$

such that A is a $(n+1) \times (n+1)$ upper triangular matrix where

$$A_{i+1,j+1} = \begin{cases} 0, & \text{for} \quad i > j \\ (-1)^{j-i} \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} n-i \\ j-i \end{pmatrix}, & \text{for} \quad i \leq j \end{cases} \tag{2. 14}$$

$i, j = 0, 1, ..., n$, Λ is $(n+1) \times (n)$ matrix as follows

$$\Lambda_{i+1,j+1} = \begin{cases} j, & \text{for} \quad i = j+1, \\ 0, & \text{for} \quad \text{otherwise}, \end{cases} \tag{2. 15}$$

$i = 0, ..., n$, $j = 0, ..., n-1$. And V is $(n) \times (n+1)$ matrix can be expressed by

$$V_{k+1} = A^{-1}_{k+1}, \ k = 0, 1, ..., n-1, \tag{2. 16}$$

where A^{-1}_{k+1} is $(k+1)$th row of A^{-1}.

3. Solution of the problem

We consider Eqs. (1. 1)-(1. 4), and suppose $\phi(x)$ and $\phi(t)$ are vectors of Bernstein polynomials on [0,1]. we consider approximate solution of the form

$$U_n(x,t) = \sum_{i=0}^{n} \sum_{j=0}^{n} u_{i,j} B_{i,n}(x) B_{j,n}(t) = \phi_n^T(t) U \phi_n(x), \tag{3. 17}$$

where

$$U = [U_0, ..., U_n],$$

with

$$U_i = [u_{0i}, ..., u_{ni}]^T.$$

Also, we approximate $g(x,t)$ and $f(x)$ by (n+1)terms of the Bernstein series, thus we get

$$g(x,t) \simeq \sum_{i=0}^{n} \sum_{j=0}^{n} g_{i,j} B_{i,n}(t) B_{j,n}(x) = \phi_n^T(t) G \phi_n(x), \tag{3. 18}$$

where

$$G = [G_0, ..., G_n], G_i = [g_{0i}, ..., g_{ni}]^T, \ i = 0, 1, ..., n.$$

$$f(x) \simeq \sum_{j=0}^{n} f_j B_{j,n}(x) = F \phi_n(x), \tag{3. 19}$$

$$F = [f_0, ..., f_n],$$

Also, we can write:

$$u_t(x,t) = \phi^T(x)UD_b\phi(t). \tag{3. 20}$$

Also, we have

$$u_{xx}(x,t) = \phi^T(x)(D_b^2)^T U\phi(t), \tag{3. 21}$$

Using Eqs. (3. 20) and (3. 21) in Eq. (1. 1) we obtain

$$\phi^T(x)UD_b\phi(t) = \phi^T(x)(D_b^2)^T U\phi(t) + g(x,t), \tag{3. 22}$$

we now collocate Eq. (3. 22) in $(n-1)\times(n)$ points (x_i, t_j), $i = 2, ..., n,\ j = 2, ..., n+1$ and hence the residual is as following:

$$R(x_i, t_j) = \phi^T(x_i)UD_b\phi(t_j) - \phi^T(x_i)(D_b^2)^T U\phi(t_j) - g(x_i, t_j) = 0, \tag{3. 23}$$

$$i = 2, ..., m,\ j = 2, ..., m+1.$$

where $x_i, i = 1, ..., n$ and $t_j, j = 1, ..., n+1$, are shifted points of chebyshev polynomial. Collocating Eqs. (1. 2)-(1. 4) in $n+1$ points x_i, $i = 1, ..., n+1$ and n points t_j, $j = 1, ..., m$ we obtain:

$$u(x_i, 0) = f(x_i),\ i = 1, 2, ..., n+1, \tag{3. 24}$$

$$\lambda_0 u(0, t_j) = \int_0^1 p_0(x)u(x, t_j)dx + q_0(t_j),\ j = 1, ..., n,, \tag{3. 25}$$

$$\lambda_1 u(1, t_j) = \int_0^1 p_1(x)u(x, t_j)dx + q_1(t_j),\ j = 1, ..., n,, \tag{3. 26}$$

(3. 23) together with (3. 24)-(3. 26) give a system of equations, Now u(x,t) can be calculated.

4. Numerical results

In this section, for testing the accuracy and efficiency of described method we solve two test examples.

Example 1. For the first example, we consider Eqs. (1. 1)-(1. 4) with

$$g(x,t) = exp(t)(x^2 - 2),\ 0 < x < 1,\ 0 < t \le 1, f(x) = x^2,$$

$$\lambda_0 = 1, p_0(x) = 0, q_0(t) = 0,$$

$$\lambda_1 = 0, p_1(x) = 1, q_1(t) = -\frac{exp(t)}{3},$$

The theoretical solution of this problem is $u(x,t) = exp(t)(x^2)$.

We compare the absolute errors at grid points of the computed solution are given for different values of time levels with result in [2], in Tables 1. As the numerical results in this table show the proposed method is very effective.

Example 2. We consider Eqs. (1. 1)-(1. 4) with:

$$g(x,t) = 0,\ 0 < x < 1,\ 0 < t \le 1, f(x) = \cos(\frac{\pi x}{2}),$$

$$\lambda_0 = 1, p_0(x) = 0, q_0(t) = exp(\frac{-\pi^2 t}{4}),$$

$$\lambda_1 = 0, p_1(x) = 1, q_1(t) = -(\frac{2}{\pi})exp(\frac{-\pi^2 t}{4}),$$

The theoretical solution of this problem is $exp(\frac{-\pi^2 t}{4})\cos(\frac{\pi x}{2})$.

Similar to the previous example, the values of absolute error for different values of x and t are given in Tables 2. The obtained results are seen to be very reliable and accurate. For more investigation, the absolute errors for $0 < t < 1$ for examples 1 and 2 are plotted in Fig.1 and Fig.2. As we observe, there is very good agreement between the approximate solution obtained by the spectral collocation method and the exact solution.

Table 1: Comparison the absolute error of the peresented method and method in [2] for $u(x,t)$ from Example 1.

(x,t)	$presented\ method$	[2]
$(0.1, 01)$	-7.0398×10^{-20}	1.19×10^{-08}
$(0.2, 0.2)$	3.0217×10^{-19}	2.81×10^{-11}
$(0.4, 04)$	1.7972×10^{-18}	3.98×10^{-11}
$(0.6, 0.6)$	6.0223×10^{-18}	2.52×10^{-11}
$(0.8, 0.8)$	1.5018×10^{-17}	1.38×10^{-13}
$(1, 0.8)$	-1.2493×10^{-16}	1.19×10^{-11}

Table 2: Comparison the absolute error of the peresented method and method in [2] for $u(x,t)$ from Example 2.

(x,t)	$presentedmethod$	[2]
$(0.1, 01)$	1.0186×10^{-12}	1.08×10^{-08}
$(0.2, 0.2)$	6.5900×10^{-13}	2.49×10^{-11}
$(0.4, 04)$	-5.5424×10^{-14}	5.59×10^{-11}
$(0.6, 0.6)$	-3.1669×10^{-13}	1.45×10^{-10}
$(0.8, 0.8)$	-5.9724×10^{-13}	1.38×10^{-13}
$(1, 0.8)$	-1.2493×10^{-16}	4.27×10^{-11}

absolute error of u(x,t) for m=n=12, example1.

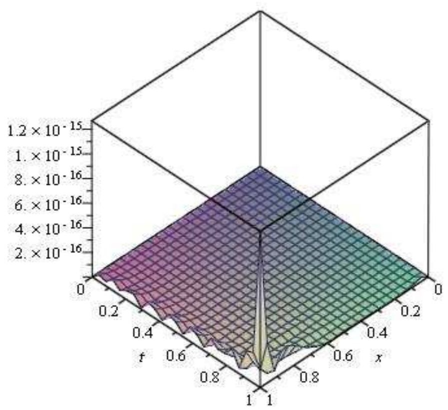

Fig. 1: Absolute error of u(x,t) for example 1.

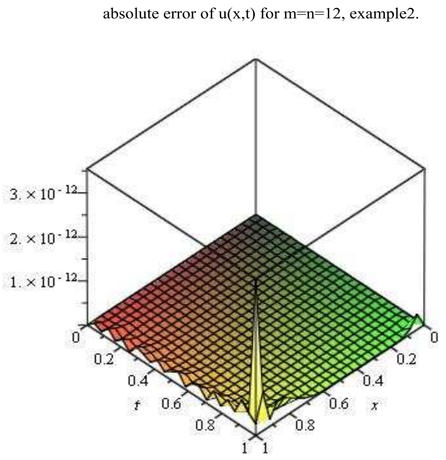

Fig. 2: Absolute error of u(x,t) for example 2.

5. Conclusion

In this paper, the spectral method with Bernstein polynomials has been successfully used to obtain the approximate solutions to the non-local parabolic partial differential equations. Based on the numerical experiments, we conclude that our method is a practical and effective numerical technique for solving the non-local parabolic partial differential equations.

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