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Non Standard Finite Difference Method for Quadratic Riccati Differential Equation

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Abstract. In this paper, we proposed an unconditionally stable Non-Standard Finite Difference (NSFD) scheme to solve nonlinear Riccati differential equation. The accuracy and efficiency of the proposed scheme is verified by comparing the results with other numerical techniques such as Euler and RK-4 and semi analytical technique DTM. The obtained results show that the performance of NSFD scheme is more accurate and reliable. Unlike other schemes the proposed NSFD scheme preserves all the essential features of continuous model.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words:: Non linear differential equations, Riccati problem, Non-standard finite difference (NSFD) method, Differential transformation method (DTM), RK-4.

1. Introduction and main results

The Riccati differential equation is a well known non-linear differential equation named after the Jacopo Francessco Riccati, an Italian Nobel Laureate. Riccati differential equation is very important and has many applications in important engineering and sciences domains, such as robust stabilization, stochastic realization theory, network synthesis, optimal control and most recently used in financial mathematics [14]- [11]. Many researchers have studied the solutions of non linear differential equation using different variational and perturbation approaches (VAPA), such as Adomian decomposition method (ADM), differential transformation method (DTM), Homotopy perturbation methods (HPM), and variational iterative methods [6]- [4]. It is important to note that many of these authors describe the power series solutions of such problem that are unable to provide the reasonable dynamics of the system [12].These methods, sometimes, lose some very important properties of the continuous system like the boundedness and positivity of solutions for certain values of step size or with the change in initial conditions. F. M. Fernndez in [9] has mentioned that all these (VAPA) methods are not appropriate for the non-linear differential equations. In his paper, he analyzed the VAPA methods for some non-linear problems such as HPM for Riccati equation and prey predator model. For all of these models, he mentioned that the convergence of these methods is limited to the initial conditions and step sizes. Moreover, time power series solutions for non linear dynamics produce singularities with the change in initial conditions. The purpose of this paper is to provide unconditionally stable scheme for the Riccati differential equation solved by means of HPM [2], using ADM [6] and using DTM [8]. The limitations of HPM for Riccati differential equation [13] regarding the singularities, with the change in initial conditions have been discussed in detail in [9]. The discrete models obtained by numerical methods cause a major difficulty in the calculation of numerical solutions, thus they yield the numerical instabilities. Numerical instabilities are solutions to discrete equations that do not correspond to any solution of the original differential equations. Discrete models contain the same parameter as that of the continuous model such as time and space step sizes. Variations in these parameters may cause instabilities. For example some finite difference schemes such as forward Euler, Rung-e-Kutta, sometimes, generate artificial chaos, non-physical oscillations, produce fictitious bifurcations, and false steady states. Therefore, how to choose discrete schemes that guarantee the global dynamics of the models is very important issue. The approach we introduced here is the Non-standard Finite Difference (NSFD) method. Mickens, in 1989 has given the idea of NSFD scheme for the numerical approximations of differential equations [12]. This method can be applied more accurately and efficiently to the non-linear differential equation. It preserves significant properties of exact solutions of the differential equations and shows a good convergence behavior. This method maintains the dynamical consistency and numerical stability with respect to initial conditions and variable step sizes, whereas other numerical techniques lose the stability and convergence towards right solution with the increase in step size. We will show in the section of results and discussion that with the increase in step size some numerical methods do not remain stable and do not converge to the original solution such as RK-4 or Euler's method. In section 2, NSFD method and its adaptation to Riccati Equation are briefly presented. In section 3, numerical results and comparisons have been shown for the Riccati differential equation. Finally, the conclusion of our results is presented.

2. Numerical Method

Nonstandard Finite difference (NSFD) methods for the numerical solutions of differential equations were first introduced by Mickens in 1989 [7]. Other numerical schemes sometimes produce unnecessary oscillation, false steady states chaos and bifurcations such as forward Euler scheme, Runge-kutta methods [7]. These types of numerical instabilities can be avoided by the construction of numerical scheme: *'Non-standard finite difference (NSFD) method'*. A non-standard finite difference scheme is a discrete representation of a system of differential equations that is constructed based on the following rules:

- The orders of the discrete derivatives should be equal to the orders of the corresponding derivatives appearing in the differential equations.
- Discrete representations for derivatives can be replaced by non-trivial denominator functions $\phi(h) = h + O(h^2)$ with the property that, $h \to 0$ implies $\phi(h) \to 0$.
- Nonlinear terms should, in general, be replaced by nonlocal discrete representations.
- Special conditions that hold for either the differential equation or its solutions should also hold for the discretized model and its solutions.

To support our proposed NSFD scheme, we apply this method on Riccati differential equation. Consider the quadratic Riccati differential equation which is solved by HPM [15] and a criticism for this method is given in [14].

$$\frac{dy(t)}{dt} = 1 + 2y(t) - y^2(t), \tag{2. 1}$$

with initial condition $y(0) = 0$ and the exact solution can be written as [15].

$$y(t) = 1 + \sqrt{2}\tanh\Big(\sqrt{2}t + \frac{1}{2}\log\Big(\frac{\sqrt{2}-1}{\sqrt{2}+1}\Big)\Big). \tag{2. 2}$$

In this paper, we proposed the NSFD approach to approximate the solution of this equation. The NSFD scheme with non-local approximation of non linear term for the Riccati problem can be written as

$$\frac{y^{n+1} - y^n}{h} = 1 + 2y^n - y^n y^{n+1}, \tag{2. 3}$$

which gives

$$y^{n+1} = \frac{h + (1+2h)y^n}{(1+hy^n)} \tag{2. 4}$$

where y^{n+1} is the value of the solution at $(n+1)^{th}$ time step and y^n is the value of the solution at n^{th} time step, h is the time stepping parameter. In this formulation the non-linear term is replaced by non-local approximation whereas the time step parameter h is approximated in usual way as in standard finite difference scheme. Another scheme can be developed by replacing the denominator $'h'$ with a function $\phi(h)$ so that $\phi(h) \to 0$ as $h \to 0$. This non trivial denominator helps in maintaining the positivity and stability of the solution. For the Riccati problem, we approximated the denominator h as,

$$\phi(h) = 1 - e^{-h}$$

using this denominator our scheme becomes

$$y^{n+1} = \frac{(1-e^{-h})[1 + 2(1-e^{-h})]y^n}{1 + y^n(1-e^{-h})}, \tag{2. 5}$$

(2. 5) with initial condition $y^0 = 0$.
We will see in the next section that how this non trivial denominator will overcome the un-stable behavior of the NSFD scheme (2. 4).

3. Results and Discussions

In this section, we provide the numerical simulations to support the method proposed in previous section. Firstly, we compute the numerical solutions of Riccati equation using NSFD schemes (2. 4) and (2. 5) and compare with exact solution.
Figure (1), shows that the NSFD (with trivial denominator) and NSFD with non trivial denominator are in good agreement with the exact solution for small h, whereas when we

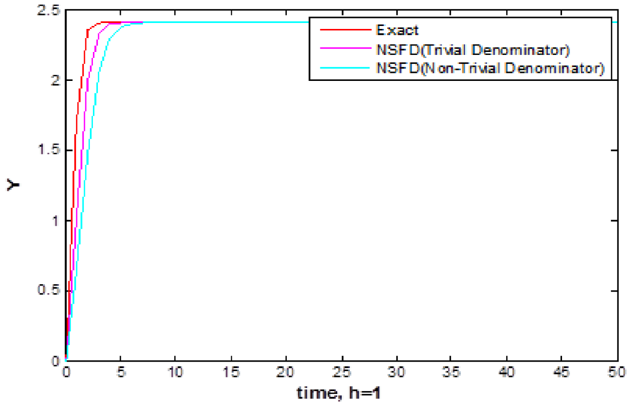

FIGURE 1. Comparison of two NSFD schemes with exact solution.

increased $h = 1$ to $h = 3$ the NSFD scheme with non-trivial denominator shows the oscillation. It is shown in figure (2) that unstable behavior of NSFD scheme (2. 4) can be overcome by applying the NSFD with non trivial denominator. Next we compared the results obtained by proposed NSFD scheme with a semi analytical technique, the Differential Transformation Method DTM which is used for Riccati equation in [9].
Figure (3) shows that the DTM, converges for a very small interval of time almost up to 7 seconds and after that it started diverging from the exact solution while the NSFD scheme remains convergent. Finally, we make a comparison between Euler method, 4^{th} order Rung-e-Kutta method and the NSFD schemes (2. 4) and (2. 5).
It can be seen from figure (4), that the 4^{th} order Rung-e-Kutta and Euler methods converge for small step size $h = 0.1$.
It can be seen in figure (5) as we increased step size from $h = 1$ to $h = 2$, RK-4 did not converge, while the proposed NSFD scheme remains convergent.

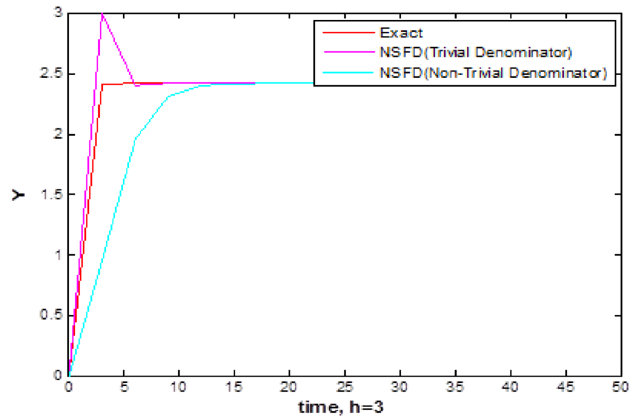

FIGURE 2. Comparison of NSFD schemes with increased step size.

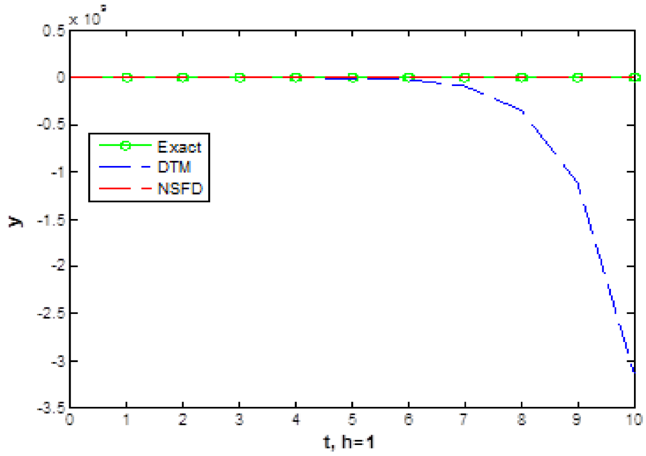

FIGURE 3. comparison of NSFD with DTM.

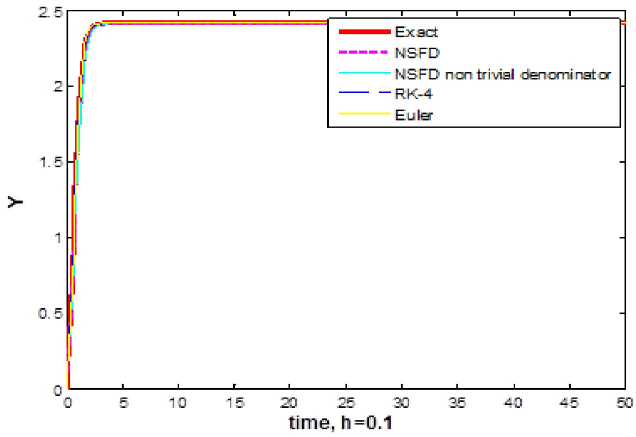

FIGURE 4. comparison of NSFD with RK-4 and Euler for h=0.1

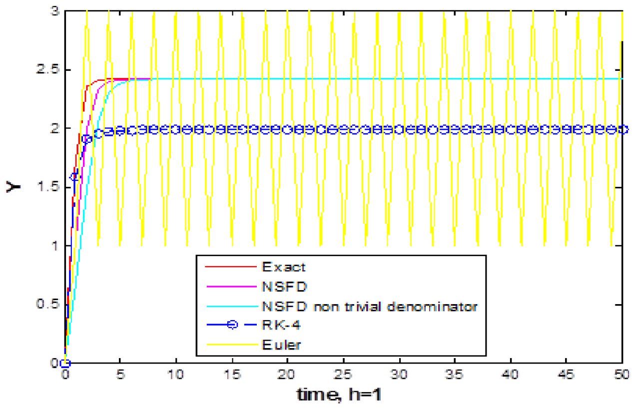

FIGURE 5. comparison of NSFD with RK-4 and Euler for $h = 1$

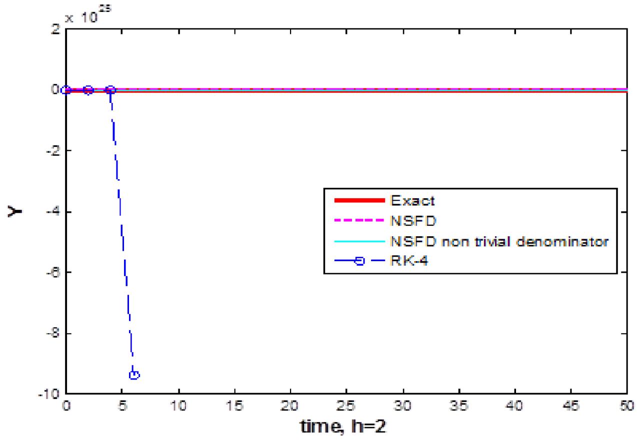

FIGURE 6. diverged behavior of RK-4 for step size $h = 2$

4. CONCLUSION

In this paper, we proposed an unconditionally stable Non-Standard Finite Difference (NSFD) scheme for non-linear quadratic Riccati differential equation. The robustness of the NSFD (with non-trivial denominator) scheme is shown in terms of stability for large step size. It is shown that the other schemes show unnecessary oscillations or converge towards false steady state as we increase the step size. It can be concluded from this discussion that the proposed NSFD schemes have some very important advantages over the standard finite difference methods and semi-analytic techniques to solve non-linear differential equations.

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