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# Some Generalized Aggregation Operators for Cubic Hesitant Fuzzy Sets and Their Applications to Multi Criteria Decision Making

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**Abstract.** In our study, we define aggregation operators for cubic hesitant fuzzy sets which includes generalized cubic hesitant fuzzy averaging (geometric) operator, cubic hesitant fuzzy ordered weighted averaging (geometric) operator, generalized cubic hesitant fuzzy ordered weighted averaging (geometric) operator, cubic hesitant fuzzy hybrid averaging (geometric) operator, cubic hesitant fuzzy arithmetical averaging (geometric) operator and generalized cubic hesitant fuzzy hybrid averaging (geometric) operator. We also solve a multi criteria decision making argument by using generalized cubic hesitant fuzzy hybrid averaging operator and generalized cubic hesitant fuzzy hybrid geometric operator. We choose the best alternative amongst those alternatives suggested by decision makers.

**AMS (MOS) Subject Classification Codes: 03E72; 03B52; 68T37; 97D05**

**Key Words:** cubic hesitant fuzzy sets, generalized cubic hesitant fuzzy hybrid averaging operator, generalized cubic hesitant fuzzy hybrid geometric operator, multi criteria decision making.

## 1. Introduction

Since Zadeh [30] originated fuzzy set theory in 1965, it has become an importrant tool to handle inaccurate and ambiguous information in different fields of prevailing civilization. Such inaccuracies are associated with the membership function that belongs to [0, 1]. Through membership function, we get information which makes possible for us to reach the conclusion. In order to handle this information many extensions have been added to the theory of fuzzy sets. These include intuitionistic fuzzy sets (IFSs) [1, 5], interval valued fuzzy sets (IVFSs) [6, 31], hesitant fuzzy sets (HFSs) [17-18], interval valued hesitant fuzzy sets (IVHFSs) [2-3], cubic sets (CSs) [10], cubic hesitant fuzzy sets (CHFSs) [15], etc. Torra [17-18] initially gave the notion of HFSs that allows membership grade to be the finite set of feasible values between $0$ and $1$. IVFSs [31] allows the membership grade of an element to closed subinterval of the [0, 1]. Jun et al. [10] defined cubic sets which include an IVFs [31] with the fuzzy set [30]. Chen and Xu [2] introduced a structure which generalizes the concept of HFS [17-18] to IVHFS that enables the membership grade of an element into many feasible interval nembers. Multi criteria decision making (MCDM) [8, 9, 11, 12, 14, 16, 19, 29, 32] has to do with making being a higher position decision over the available attributes that are represented by multiple normally conflicting properties.

When a decision maker has provided the information about a certain statement that he may be sure about the statement is $\{0.3, 0.5, 0.8\}$ and $\{[0.25, 0.35], [0.4, 0.65], [0.55, 0.85]\}$, such kind of information cannot be presented by using other defined tools. Therefore, to overcome this drawback of the existing tools Mahmood et al. [15] defined CHFSs by combining IVHFSs [2] and HFSs [17-18] and defined some basic operations, properties, aggregation operators (AOs) and practiced them to solve a MCDM probvlem under cubic hesitant fuzzy information. Aggregation of information is an important part in different fields such as economics, social and management sciences, information technology and medical diagnosis, etc. In general prospect, we claim that the aggregation of information is a process in which we treat different sorts of information obtained from many sources in order to get a final decision. Yager [27] provided weighted averaging (WA) operator and the ordered weighted averaging (OWA) operator that are two noted aggregation expert ways. The point or amount unlike of the WA and OWA operators is that the WA operator only weights the given information origin in connection with their authenticity, but the OWA operator weights the given information with respect to their ordered position. As an outcome of that, weights show different features in both WA and OWA operators. Another aggregation operator (AO) is the generalized OWA (GOWA) operator [28] which is formulated through generalized mean [7]. Chiclana et al. [4] proposed ordered weighted geometric (OWG) operator through geometric mean. The drawback of OWA and OWG operators is that they only rank the arguments and do not consider the given significance of the argument itself, for the sake of improvement of this fact Xu and Da [23] defined hybrid weighted averaging (HWA) operator. Lindahl et al. [13] gave the generalization of the hybrid averaging (HA) operator. Recently, many aggregation approaches for fuzzy information have been defined. Xu [25-26], Xu and Yager [24] gave AOs for IFSs. Zhao et al. [34] defined generalized AOs for IFSs. Xia and Xu [20-22] established AOs for the facts obtained from HFSs. Zhang Z [33] developed AOs for IVIHFS and practiced it to solve a MCDM problem.

The article is arranged as: Section 2 contains the primary Definitions that are used in this paper. Section 3 defines AOs for CHFSs. Section 4 describes the algorithm to solve a MCDM problem by using defined operators. Section 5 solves a MCDM problem by applying the algorithm. Section 6 consists of the conclusion of the presented work. Section 7 consists of acknowledgments.

## 2. NOTATIONS AND PRELIMINARIES

**Definition 2.1 [30]** A fuzzy set (FS) on a non-empty set $X$ is defined to be a function from $X$ to $P = [0, 1]$ as $\alpha : X \to P$.

**Definition 2.2 [31]** An interval valued fuzzy set (IVFS) on a non-empty set $X$ is defined by the function $F$ from $X$ to the set of closed intervals in [0, 1].

**Definition 2.3 [10]** A cubic set (CS) on a non-empty set $X$ is defined by $\Delta = \{< x, D(x), \alpha(x) > /x \in X\}$, where $D(x)$ is an IVFS in $X$ and $\alpha(x)$ is a FS in $X$. A cubic set is simply denoted by $\Delta =< D, \alpha >$ .

**Definition 2.4 [17, 18]** A hesitant fuzzy set (HFS) on a non-empty set $X$ is a mapping that while enforced on $X$ yields a finite subset of [0, 1], which is denoted and defined by $h = \{< x, \eta(x) > /x \in X\}$, where $\eta(x)$ is a set of a few different values in [0, 1], that shows the feasible membership values of the element $x \in X$.

**Definition 2.5 [2]** Let $X$ be a non-empty set and $S[0, 1]$ denote the collection of closed subintervals of $[0, 1]$. An interval valued hesitant fuzzy set (IVHFS) on $X$ is denoted and defined by $\Omega = \{< x_j, C(x_j) > /x_j \in X, j = 1, 2, \ldots, n\}$, where $C(x_j) : X \to S[0, 1]$ expresses entire feasible interval valued membership values of the element $x_j \in X$ to $\Omega$.

**Definition 2.6 [15]** Assume $X$ is a non-empty set. A cubic hesitant fuzzy set (for short, CHFS) is defined by $\Gamma = \{< x, C(x), \eta(x) > /x \in X\}$, where $C(x)$ is an IVHFE and $\eta(x)$ is HFE. A CHFS is simply denoted by $\Gamma =< C, \eta >$.

**Definition 2.7 [15]** Let $X$ be a non-empty set $\Gamma = \{< x, C(x), \eta(x) > /x \in X\}$ and $\delta = \{< x, A(x), \xi(x) > /x \in X\}$ be any two CHFSs on $X$ then the addition of $\Gamma$ and $\delta$ are denoted and defined as,

$\Gamma \oplus \delta = \{x, \gamma \in C(x) + A(x), \rho \in \eta(x) + \xi(x)/\{[\mu_i^- + \sigma_i^- - \mu_i^- \sigma_i^-, \mu_i^+ + \sigma_i^+ - \mu_i^+ \sigma_i^+]\}, \{h_i k_i\} >\}$.

**Definition 2.8 [15]** Let $X$ be a non-empty set $\Gamma = \{< x, C(x), \eta(x) > /x \in X\}$ and $\delta = \{< x, A(x), \xi(x) > /x \in X\}$ be any two CHFSs on $X$ then multiplication of $\Gamma$ and $\delta$ are denoted and defined as,

$\Gamma \otimes \delta = \{x, \gamma \in C(x) \times A(x), \rho \in \eta(x) \times \xi(x)/\{[\mu_i^- \sigma_i^-, \mu_i^+ \sigma_i^+]\}, \{h_i + k_i - h_i k_i\} >\}$.

**Definition 2.9 [15]** Assume $X$ is a non-empty set and $\Gamma = \{< x, C(x), \eta(x) > /x \in X\}$ is the CHFS on $X$ with $\lambda > 0$, then we have the following operations,

$\lambda\Gamma = \{< x, \lambda\gamma \in (\lambda C)(x), \lambda\rho \in (\lambda\eta)(x)/\{[1-(1-\mu_i^-)^\lambda, 1-(1-\mu_i^+)^\lambda]\}, \{h_i^\lambda\} >\}$,

$\Gamma^\lambda = \{< x, \gamma^\lambda \in C^\lambda(x), \rho^\lambda \in \eta^\lambda(x)/\{[(\mu_i^-)^\lambda, (\mu_i^+)^\lambda]\}, \{1 - (1 - h_i)^\lambda\} >\}$,

$\Gamma^c = \{< x, \gamma^c \in C^c(x), \rho^c \in \eta^c(x)/\{[1 - \mu_i^+, 1 - \mu_i^-]\}, \{1 - h_i\} >\}$.

**Definition 2.10 [15]** Let $\Gamma = \{< x, C(x), \eta(x) > /x \in X\}$ be a CHFS. The cubic hesitant fuzzy element on a non-empty set $X$ is defined to be $ch = \{< \mu_i = [\mu_i^-, \mu_i^+] \in C(x), \rho_i \in \eta(x)/\{[\mu_i^-, \mu_i^+]\}, \{\rho_i\} >\}$, where $C(x)$ represents IVHFE and $\eta(x)$ represents HFE. Denotes $G$ by the set of all CHFEs.

**Definition 2.11 [15]** Let $X$ be a non-empty set and $ch = \{< \mu_i = [\mu_i^-, \mu_i^+] \in C(x), \rho_i \in \eta(x)/\{[\mu_i^-, \mu_i^+]\}, \{h_i\} >\}$ be a CHFE on $X$, the score of $ch$ is denoted and defined by $T(ch) = \frac{1}{\circledast(ch)}(\mu_i^- + \mu_i^+ - \frac{\circledast(ch)}{2} + h_i)$, where $\mu_i = [\mu_i^-, \mu_i^+] \in C(x)$ (an IVHFE), $h_i \in \eta(x)$ (HFE) for all $x \in X$, $\circledast(ch)$ is the number of elements in $ch$.

**Theorem 2.12 [15]** Let $ch, ch_1, ch_2$ be three CHFEs on a non-empty set $X$ and $\lambda > 0$ then $ch_1 \oplus ch_2, ch_1 \otimes ch_2, \lambda ch, ch^\lambda$ are also CHFEs.

## 3. Aggregation Operators for CHFSs

**Definition 3.1 (GCHFWA operator)** Assume $ch_k (k = 1, 2, \ldots, n)$ are the collections of CHFEs and let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of CHFEs $ch_k (k = 1, 2, \ldots, n)$, where $w_k \in [0, 1]$, $\Sigma_{k=1}^n w_k = 1$ and $\sigma > 0$. Then generalized cubic hesitant fuzzy weighted averaging (for short, GCHFWA) operator is a mapping $G^n \to G$ defined by,

$GCHFWA_\sigma(ch_1, ch_2, \ldots, ch_n) = \beta_k^\sigma = ((\oplus)_{k=1}^n (w_k ch_k^\sigma))^{\frac{1}{\sigma}}$.

**Theorem 3.2** Let $ch_k(k=1,2,\ldots,n)$ be a collection of CHFEs. Then the aggregated result is gained through applying GCHFWA operator is also a CHFE, and:
$GCHFWA_\sigma(ch_1,ch_2,\ldots,ch_n)=\beta_k^\sigma=((\oplus)_{k=1}^n(w_kch_k^\sigma))^{\frac{1}{\sigma}}$
$=\{<\{[(1-\Pi_{k=1}^n(1-(\mu_{i_k}^-)^\sigma)^{w_k})^{\frac{1}{\sigma}},(1-\Pi_{k=1}^n(1-(\mu_{i_k}^+)^\sigma)^{w_k})^{\frac{1}{\sigma}}]\},$
$\{1-(1-\Pi_{k=1}^n(1-(1-h_{i_k})^\sigma)^{w_k})^{\frac{1}{\sigma}}\}>\}.$

**Proof** By Definition 3.1 and Theorem 2.12 we have the GCHFWA operator is a CHFE. We prove the theorem by using mathematical induction.

First, we show that the result holds for $n=2$.

$ch_1^\sigma=\{<\{[(\mu_{i_1}^-)^\sigma,(\mu_{i_1}^+)^\sigma]\},\{1-(1-h_{i_1})^\sigma\}>\},$
$ch_2^\sigma=\{<\{[(\mu_{i_2}^-)^\sigma,(\mu_{i_2}^+)^\sigma]\},\{1-(1-h_{i_2})^\sigma\}>\},$
$w_1ch_1^\sigma=\{<\{[1-(1-(\mu_{i_1}^-)^\sigma)^{w_1},1-(1-(\mu_{i_1}^+)^\sigma)^{w_1}]\},$
$\{(1-(1-h_{i_1})^\sigma)^{w_1}\}>\},$
$w_2ch_2^\sigma=\{<\{[1-(1-(\mu_{i_2}^-)^\sigma)^{w_2},1-(1-(\mu_{i_2}^+)^\sigma)^{w_2}]\},$
$\{(1-(1-h_{i_2})^\sigma)^{w_2}\}>\},$
$w_1ch_1^\sigma\oplus w_2ch_2^\sigma=\{<\{[1-(1-(\mu_{i_1}^-)^\sigma)^{w_1}+1-(1-(\mu_{i_2}^-)^\sigma)^{w_2}-(1-(1-(\mu_{i_1}^-)^\sigma)^{w_1})(1-(1-(\mu_{i_2}^-)^\sigma)^{w_2}),$
$1-(1-(\mu_{i_1}^+)^\sigma)^{w_1}+1-(1-(\mu_{i_2}^+)^\sigma)^{w_2}-(1-(1-(\mu_{i_1}^+)^\sigma)^{w_1})(1-(1-(\mu_{i_2}^+)^\sigma)^{w_2})]\},$
$\{(1-(1-h_{i_1})^\sigma)^{w_1}(1-(1-h_{i_2})^\sigma)^{w_2}\}>\},$
$w_1ch_1^\sigma\oplus w_2ch_2^\sigma=\{<\{[1-(1-(1-(1-(\mu_{i_1}^-)^\sigma)^{w_1}))(1-(1-(1-(\mu_{i_2}^-)^\sigma)^{w_2})),1-(1-(1-(1-(\mu_{i_1}^+)^\sigma)^{w_1}))(1-(1-(1-(\mu_{i_2}^+)^\sigma)^{w_2}))]\},\{(1-(1-h_{i_1})^\sigma)^{w_1}(1-(1-h_{i_2})^\sigma)^{w_2}\}>\},$
$w_1ch_1^\sigma\oplus w_2ch_2^\sigma=\{<\{[1-((1-(\mu_{i_1}^-)^\sigma)^{w_1})((1-(\mu_{i_2}^-)^\sigma)^{w_2}),1-((1-(\mu_{i_1}^+)^\sigma)^{w_1})((1-(\mu_{i_2}^+)^\sigma)^{w_2})]\},\{(1-(1-h_{i_1})^\sigma)^{w_1}(1-(1-h_{i_2})^\sigma)^{w_2}\}>\},$
$(w_1ch_1^\sigma\oplus w_2ch_2^\sigma)^{\frac{1}{\sigma}}=\{<\{[(1-((1-(\mu_{i_1}^-)^\sigma)^{w_1})((1-(\mu_{i_2}^-)^\sigma)^{w_2}))^{\frac{1}{\sigma}},(1-((1-(\mu_{i_1}^+)^\sigma)^{w_1})((1-(\mu_{i_2}^+)^\sigma)^{w_2}))^{\frac{1}{\sigma}}]\},\{1-(1-((1-(1-h_{i_1})^\sigma)^{w_1}(1-(1-h_{i_2})^\sigma)^{w_2}))^{\frac{1}{\sigma}}\}>\},$
$(w_1ch_1^\sigma\oplus w_2ch_2^\sigma)^{\frac{1}{\sigma}}=\{<\{[(1-\Pi_{k=1}^2(1-(\mu_{i_k}^-)^\sigma)^{w_k})^{\frac{1}{\sigma}},(1-\Pi_{k=1}^2(1-(\mu_{i_k}^+)^\sigma)^{w_k})^{\frac{1}{\sigma}}]\},\{1-(1-\Pi_{k=1}^2(1-(1-h_{i_k})^\sigma)^{w_k})^{\frac{1}{\sigma}}\}>\}.$

Now suppose that the result holds for $n=t$, then

$GCHFWA_\sigma(ch_1,ch_2,\ldots,ch_t)=((\oplus)_{k=1}^t(w_kch_k^\sigma))^{\frac{1}{\sigma}}=\{<\{[(1-\Pi_{k=1}^t(1-(\mu_{i_k}^-)^\sigma)^{w_k})^{\frac{1}{\sigma}},(1-\Pi_{k=1}^t(1-(\mu_{i_k}^+)^\sigma)^{w_k})^{\frac{1}{\sigma}}]\},\{1-(1-\Pi_{k=1}^t(1-(1-h_{i_k})^\sigma)^{w_k})^{\frac{1}{\sigma}}\}>\}.$

We prove that the result holds for $n=t+1$,

$(\oplus)_{k=1}^t(w_kck_k^\sigma)\oplus(w_{t+1}ch_{t+1}^\sigma)=\{<\{[1-\Pi_{k=1}^t(1-(\mu_{i_k}^-)^\sigma)^{w_k}+1-(1-(\mu_{i_{t+1}}^-)^\sigma)^{w_{t+1}}-(1-\Pi_{k=1}^t(1-(\mu_{i_k}^-)^\sigma)^{w_k})(1-(1-(\mu_{i_{t+1}}^-)^\sigma)^{w_{t+1}}),1-\Pi_{k=1}^t(1-(\mu_{i_k}^+)^\sigma)^{w_k}+1-(1-(\mu_{i_{t+1}}^+)^\sigma)^{w_{t+1}}-(1-\Pi_{k=1}^t(1-(\mu_{i_k}^+)^\sigma)^{w_k})(1-(1-(\mu_{i_{t+1}}^+)^\sigma)^{w_{t+1}})]\},\{(\Pi_{k=1}^t(1-(1-h_{i_k})^\sigma)^{w_k})(1-(1-h_{i_{t+1}})^\sigma)^{w_k}))\}>\},$

$(\oplus)_{k=1}^t(w_kck_k^\sigma)\oplus(w_{t+1}ch_{t+1}^\sigma)=\{<\{[1-(1-(1-\Pi_{k=1}^t(1-(\mu_{i_k}^-)^\sigma)^{w_k}))(1-(1-(1-(\mu_{i_{t+1}}^-)^\sigma)^{w_k})),1-(1-(1-\Pi_{k=1}^t(1-(\mu_{i_k}^+)^\sigma)^{w_k}))(1-(1-(1-(\mu_{i_{t+1}}^+)^\sigma)^{w_k}))]\},\{\Pi_{k=1}^{t+1}(1-(1-h_{i_k})^\sigma)^{w_k})\}>\},$

$(\oplus)_{k=1}^t(w_kck_k^\sigma)\oplus(w_{t+1}ch_{t+1}^\sigma)=\{<\{[1-(\Pi_{k=1}^t(1-(\mu_{i_k}^-)^\sigma)^{w_k})(1-(\mu_{i_{t+1}}^-)^\sigma)^{w_k}),1-(\Pi_{k=1}^t(1-(\mu_{i_k}^+)^\sigma)^{w_k})(1-(\mu_{i_{t+1}}^+)^\sigma)^{w_k})]\},\{\Pi_{k=1}^{t+1}(1-(1-h_{i_k})^\sigma)^{w_k})\}>\},$

$(\oplus)_{k=1}^t(w_kck_k^\sigma)\oplus(w_{t+1}ch_{t+1}^\sigma)=\{<\{[1-(\Pi_{k=1}^{t+1}(1-(\mu_{i_k}^-)^\sigma)^{w_k}),1-(\Pi_{k=1}^{t+1}(1-(\mu_{i_k}^+)^\sigma)^{w_k})]\},\{\Pi_{k=1}^{t+1}(1-(1-h_{i_k})^\sigma)^{w_k})\}>\},$

$((\oplus)_{k=1}^t(w_kck_k^\sigma)\oplus(w_{t+1}ch_{t+1}^\sigma))^{\frac{1}{\sigma}}=\{<\{[(1-(\Pi_{k=1}^{t+1}(1-(\mu_{i_k}^-)^\sigma)^{w_k}))^{\frac{1}{\sigma}},(1-(\Pi_{k=1}^{t+1}(1-(\mu_{i_k}^+)^\sigma)^{w_k}))^{\frac{1}{\sigma}}]\},\{1-(1-\Pi_{k=1}^{t+1}(1-(1-h_{i_k})^\sigma)^{w_k})^{\frac{1}{\sigma}}\}>\},$

$GCHFWA_\sigma(ch_1, ch_2, \ldots, ch_t, ch_{t+1}) = ((\oplus)_{k=1}^{t+1}(w_k ch_k^\sigma))^{\frac{1}{\sigma}} = ((\oplus)_{k=1}^{t}(w_k ck_k^\sigma) \oplus (w_{t+1} ch_{t+1}^\sigma))^{\frac{1}{\sigma}} = \{< \{[(1-(\Pi_{k=1}^{t+1}(1-(\mu_{i_k}^-)^\sigma)^{w_k}))^{\frac{1}{\sigma}}, (1-(\Pi_{k=1}^{t+1}(1-(\mu_{i_k}^+)^\sigma)^{w_k}))^{\frac{1}{\sigma}}]\}, \{1-(1-\Pi_{k=1}^{t+1}(1-(1-h_{i_k})^\sigma)^{w_k})^{\frac{1}{\sigma}}\} >\}$,

so the result holds for $n = t + 1$.

**Definition 3.3 (GCHFWG operator)** Let $ch_k(k = 1, 2, \ldots, n)$ be the collection of CHFEs and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $ch_k(k = 1, 2, \ldots, n)$, where $w_k \in [0, 1], \Sigma_{k=1}^n w_k = 1$ and $\sigma > 0$. Then the generalized cubic hesitant fuzzy weighted geometric (for short, GCHFWG) operator is a mapping $G^n \to G$ defined by,

$GCHFWG_\sigma(ch_1, ch_2, \ldots, ch_n) = \tau_k^\sigma = \frac{1}{\sigma}((\otimes)_{k=1}^n(\sigma ch_k)^{w_k})$.

**Theorem 3.4** Let $ch_k(k = 1, 2, \ldots, n)$ be the collection of CHFEs. Then the aggregated result is obtained by applying GCHFWG operator is also a CHFE, and:

$GCHFWG_\sigma(ch_1, ch_2, \ldots, ch_n) = \tau_k^\sigma = \frac{1}{\sigma}((\otimes)_{k=1}^n(\sigma ch_k)^{w_k}) = \{< \{[1-(1-\Pi_{k=1}^n(1-(1-\mu_{i_k}^-)^\sigma)^{w_k})^{\frac{1}{\sigma}}, 1-(1-\Pi_{k=1}^n(1-(1-\mu_{i_k}^+)^\sigma)^{w_k})^{\frac{1}{\sigma}}]\}, \{(1-\Pi_{k=1}^n(1-(h_{i_k})^\sigma)^{w_k})^{\frac{1}{\sigma}}\} >\}$.

**Proof** By Definition 3.3 and Theorem 2.12 we have the GCHFWG operator is a CHFE.

We prove the theorem by using mathematical induction.

First, we show that the result holds for $n = 2$.

$\sigma ch_1 = \{< \{[1-(1-\mu_{i_1}^-)^\sigma, 1-(1-\mu_{i_1}^+)^\sigma]\}, \{h_{i_1}^\sigma\} >\}$,

$\sigma ch_2 = \{< \{[1-(1-\mu_{i_2}^-)^\sigma, 1-(1-\mu_{i_1}^+)^\sigma]\}, \{h_{i_2}^\sigma\} >\}$,

$(\sigma ch_1)^{w_1} = \{< \{[(1-(1-\mu_{i_1}^-)^\sigma)^{w_1}, (1-(1-\mu_{i_1}^+)^\sigma)^{w_1}]\}, \{1-(1-h_{i_1}^\sigma)^{w_1}\} >\}$,

$(\sigma ch_2)^{w_2} = \{< \{[(1-(1-\mu_{i_2}^-)^\sigma)^{w_2}, (1-(1-\mu_{i_2}^+)^\sigma)^{w_2}]\}, \{1-(1-h_{i_2}^\sigma)^{w_2}\} >\}$,

$(\sigma ch_1)^{w_1} \otimes (\sigma ch_2)^{w_2} = \{< \{[(1-(1-\mu_{i_1}^-)^\sigma)^{w_1}(1-(1-\mu_{i_2}^-)^\sigma)^{w_2}, (1-(1-\mu_{i_1}^+)^\sigma)^{w_1}(1-(1-\mu_{i_2}^+)^\sigma)^{w_2}]\}, \{1-(1-h_{i_1}^\sigma)^{w_1} + 1-(1-h_{i_2}^\sigma)^{w_2} - (1-(1-h_{i_1}^\sigma)^{w_1})(1-(1-h_{i_2}^\sigma)^{w_2})\} >\}$,

$(\sigma ch_1)^{w_1} \otimes (\sigma ch_2)^{w_2} = \{< \{[(1-(1-\mu_{i_1}^-)^\sigma)^{w_1}(1-(1-\mu_{i_2}^-)^\sigma)^{w_2}, (1-(1-\mu_{i_1}^+)^\sigma)^{w_1}(1-(1-\mu_{i_2}^+)^\sigma)^{w_2}]\}, \{1-(1-(1-(1-h_{i_1}^\sigma)^{w_1}))(1-(1-(1-h_{i_2}^\sigma)^{w_2}))\} >\}$,

$(\sigma ch_1)^{w_1} \otimes (\sigma ch_2)^{w_2} = \{< \{[(1-(1-\mu_{i_1}^-)^\sigma)^{w_1}(1-(1-\mu_{i_2}^-)^\sigma)^{w_2}, (1-(1-\mu_{i_1}^+)^\sigma)^{w_1}(1-(1-\mu_{i_2}^+)^\sigma)^{w_2}]\}, \{1-(1-h_{i_1}^\sigma)^{w_1}(1-h_{i_2}^\sigma)^{w_2}\} >\}$,

$\frac{1}{\sigma}((\sigma ch_1)^{w_1} \otimes (\sigma ch_2)^{w_2}) = \{< \{[1-(1-(1-(1-\mu_{i_1}^-)^\sigma)^{w_1}(1-(1-\mu_{i_2}^-)^\sigma)^{w_2})^{\frac{1}{\sigma}}, 1-(1-(1-(1-\mu_{i_1}^+)^\sigma)^{w_1}(1-(1-\mu_{i_2}^+)^\sigma)^{w_2})^{\frac{1}{\sigma}}]\}, \{(1-(1-h_{i_1}^\sigma)^{w_1}(1-h_{i_2}^\sigma)^{w_2})^{\frac{1}{\sigma}}\} >\}$,

$\frac{1}{\sigma}((\sigma ch_1)^{w_1} \otimes (\sigma ch_2)^{w_2}) = \{< \{[1-(1-\Pi_{k=1}^2(1-(1-\mu_{i_k}^-)^\sigma)^{w_k})^{\frac{1}{\sigma}}, 1-(1-\Pi_{k=1}^2(1-(1-\mu_{i_k}^+)^\sigma)^{w_k})^{\frac{1}{\sigma}}]\}, \{(1-\Pi_{k=1}^2(1-(h_{i_k})^\sigma)^{w_k})^{\frac{1}{\sigma}}\} >\}$.

Now suppose that the result holds for $n = t$, then

$GCHFWG_\sigma(ch_1, ch_2, \ldots, ch_t) = \tau_k^\sigma = \frac{1}{\sigma}((\otimes)_{k=1}^t(\sigma ch_k)^{w_k}) = \{< \{[1-(1-\Pi_{k=1}^t(1-(1-\mu_{i_k}^-)^\sigma)^{w_k})^{\frac{1}{\sigma}}, 1-(1-\Pi_{k=1}^t(1-(1-\mu_{i_k}^+)^\sigma)^{w_k})^{\frac{1}{\sigma}}]\}, \{(1-\Pi_{k=1}^t(1-(h_{i_k})^\sigma)^{w_k})^{\frac{1}{\sigma}}\} >\}$.

We prove that the result is valid for $n = t + 1$,

$(\otimes)_{k=1}^t(\sigma ch_k)^{w_k} \otimes (\sigma ch_{t+1})^{w_{t+1}} = \{< \{[(\Pi_{k=1}^t(1-(1-\mu_{i_k}^-)^\sigma)^{w_k})(1-(1-\mu_{i_{t+1}}^-)^\sigma)^{w_{t+1}}, (\Pi_{k=1}^t(1-(1-\mu_{i_k}^+)^\sigma)^{w_k})(1-(1-\mu_{i_{t+1}}^+)^\sigma)^{w_{t+1}}]\}, \{1-\Pi_{k=1}^t(1-h_{i_k}^\sigma)^{w_k} + 1-(1-h_{i_{t+1}}^\sigma)^{w_{t+1}} - (1-\Pi_{k=1}^t(1-h_{i_k}^\sigma)^{w_k})(1-(1-h_{i_{t+1}}^\sigma)^{w_{t+1}})\} >\}$,

$(\otimes)_{k=1}^t(\sigma ch_k)^{w_k} \otimes (\sigma ch_{t+1})^{w_{t+1}} = \{< \{[\Pi_{k=1}^{t+1}(1-(1-\mu_{i_k}^-)^\sigma)^{w_k}, \Pi_{k=1}^{t+1}(1-(1-\mu_{i_k}^+)^\sigma)^{w_k}]\}, \{1-\Pi_{k=1}^{t+1}(1-h_{i_k}^\sigma)^{w_k}\} >\}$,

$\frac{1}{\sigma}((\otimes)_{k=1}^{t}(\sigma ch_k)^{w_k} \otimes (\sigma ch_{t+1})^{w_{t+1}}) = \{< \{[1-(1-\Pi_{k=1}^{t+1}(1-(1-\mu_{i_k}^{-})^{\sigma})^{w_k})^{\frac{1}{\sigma}}, 1-(1-\Pi_{k=1}^{t+1}(1-(1-\mu_{i_k}^{+})^{\sigma})^{w_k})^{\frac{1}{\sigma}}]\}, \{(1-\Pi_{k=1}^{t+1}(1-h_{i_k}^{\sigma})^{w_k})^{\frac{1}{\sigma}}\} >\},$

$GCHFWG_{\sigma}(ch_1, ch_2, \ldots, ch_t, ch_{t+1}) = \tau_k^{\sigma} = \frac{1}{\sigma}((\otimes)_{k=1}^{t+1}(\sigma ch_k)^{w_k})$
$= \frac{1}{\sigma}((\otimes)_{k=1}^{t}(\sigma ch_k)^{w_k} \otimes (\sigma ch_{t+1})^{w_{t+1}})$
$= \{< \{[1-(1-\Pi_{k=1}^{t+1}(1-(1-\mu_{i_k}^{-})^{\sigma})^{w_k})^{\frac{1}{\sigma}}, 1-(1-\Pi_{k=1}^{t+1}(1-(1-\mu_{i_k}^{+})^{\sigma})^{w_k})^{\frac{1}{\sigma}}]\},$
$\{(1-\Pi_{k=1}^{t+1}(1-(h_{i_k})^{\sigma})^{w_k})^{\frac{1}{\sigma}}\} >\},$
thus the result holds for $n = t+1$.

**Definition 3.5 (CHFOWA)** Let $ch_k(k = 1, 2, \ldots, n)$ be a collection of CHFEs. The cubic hesitant fuzzy ordered weighted averaging (briefly, CHFOWA) operator is denoted and defined by the mapping $G^n \to G$ such that,

$CHFOWA(ch_1, ch_2, \ldots, ch_n) = A_{k\phi}^{o} = \beta_k^{o} = (\oplus)_{l=1}^{n}\phi_l ch_l^{\sim},$

where $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T$ is a weight vector affiliated with the mapping $G^n \to G$ having $\phi_l \in [0,1]$ and $\Sigma_{l=1}^{n}\phi_l = 1$. $ch_l^{\sim}$ is obtained as $lth$ largest of $n$-cubic hesitant fuzzy elements $ch_k(k = 1, 2, \ldots, n)$ by Definition 2.11 as score function ranking method.

**Theorem 3.6** Let $ch_k(k = 1, 2, \ldots, n)$ be a collection of CHFEs. Then the aggregated result is gained through applying CHFOWA operator is also a CHFE, and:

$CHFOWA(ch_1, ch_2, \ldots, ch_n) = A_{k\phi}^{o} = \beta_k^{o} = (\oplus)_{l=1}^{n}\phi_l ch_l^{\sim} = \{< \{[1-\Pi_{l=1}^{n}(1-\mu_{i_l}^{-\sim})^{\phi_l}, 1-\Pi_{l=1}^{n}(1-\mu_{i_l}^{+\sim})^{\phi_l}]\}, \{\Pi_{l=1}^{n}(h_{i_l}^{\sim})^{\phi_l}\} >\},$

where $ch_l^{\sim} = \{< \{[\mu_{i_l}^{-\sim}, \mu_{i_l}^{+\sim}]\}, \{h_{i_l}^{\sim}\} >\}$ is the $lth$ largest of $n-$cubic hesitant fuzzy elements $ch_k(k = 1, 2, \ldots, n)$ which is obtained by using Definition 2.11 as score function ranking method.

**Proof** Omitted (same as Theorem 3.2).

**Definition 3.7 (CHFOWG)** Let $ch_k(k = 1, 2, \ldots, n)$ be a collection of CHFEs. The cubic hesitant fuzzy ordered weighted geometric (briefly, CHFOWG) operator is defined by the function $G^n \to G$ as follows,

$CHFOWG(ch_1, ch_2, \ldots, ch_n) = D_{k\phi}^{o} = \tau_k^{o} = (\otimes)_{l=1}^{n}ch_l^{\sim\phi_l},$

where $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T$ is a weight vector affiliated with the mapping $G^n \to G$ having $\phi_l \in [0,1]$ and $\Sigma_{l=1}^{n}\phi_l = 1$. $ch_l^{\sim}$ is obtained as $lth$ largest of $n-$cubic hesitant fuzzy elements $ch_k(k = 1, 2, \ldots, n)$ by Definition 2.11 as score function ranking method.

**Theorem 3.8** Let $ch_k(k = 1, 2, \ldots, n)$ be a collection of CHFEs. Then the aggregated result is gained through applying CHFOWG operator is also a CHFE, and:

$CHFOWG(ch_1, ch_2, \ldots, ch_n) = D_{k\phi}^{o} = \tau_k^{o} = (\otimes)_{l=1}^{n}ch_l^{\sim\phi_l}$
$= \{< \{[\Pi_{l=1}^{n}(\mu_{i_l}^{-\sim})^{\phi_l}, \Pi_{l=1}^{n}(\mu_{i_l}^{+\sim})^{\phi_l}]\}, \{1-\Pi_{l=1}^{n}(1-h_{i_l}^{\sim})^{\phi_l}\} >\},$

where $ch_l^{\sim} = \{< \{[\mu_{i_l}^{-\sim}, \mu_{i_l}^{+\sim}]\}, \{h_{i_l}^{\sim}\} >\}$ is the $lth$ largest of $n-$cubic hesitant fuzzy elements $ch_k(k = 1, 2, \ldots, n)$ which is obtained by using Definition 2.11 as score function ranking method.

**Proof** Omitted (same as Theorem 3.4).

**Definition 3.9 (GCHFOWA operator)** Let $ch_k(k = 1, 2, \ldots, n)$ be the collection of CHFEs. Let $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T$ be the weighting vector affiliated with the mapping $G^n \to G$ having $\phi_k \in [0,1]$ and $\Sigma_{k=1}^{n}\phi_k = 1$. Then the generalized cubic hesitant fuzzy ordered weighted averaging (for short, GCHFOWA) operator is a mapping $G^n \to G$ defined by,

$GCHFOWA_{\sigma}(ch_1, ch_2, \ldots, ch_n) = \beta_k^{o\sigma} = ((\oplus)_{l=1}^{n}(\phi_l ch_l^{\sim\sigma}))^{\frac{1}{\sigma}},$

where $\sigma > 0$ and $ch_l^{\sim} = \{< \{[\mu_{i_l}^{-\sim}, \mu_{i_l}^{+\sim}]\}, \{h_{i_l}^{\sim}\} >\}$ is the $lth$ largest of $n-$cubic hesitant fuzzy elements $ch_k(k = 1, 2, \ldots, n)$ which is obtained by using Definition 2.11 as score function ranking method.

**Theorem 3.10** Let $ch_k(k = 1, 2, \ldots, n)$ be the collection of CHFEs. Then the aggregated result is obtained by applying GCHFOWA operator is also a CHFE, and:

$GCHFOWA_\sigma(ch_1, ch_2, \ldots, ch_n) = \beta_k^{o\sigma} = ((\oplus)_{l=1}^n(\phi_l ch_l^{\sim\sigma}))^{\frac{1}{\sigma}} = \{< \{[(1 - \Pi_{l=1}^n(1-(\mu_{i_l}^{-\sim})^\sigma)^{\phi_l})^{\frac{1}{\sigma}}, (1-\Pi_{l=1}^n(1-(\mu_{i_l}^{+\sim})^\sigma)^{\phi_l})^{\frac{1}{\sigma}}]\}, \{1-(1-\Pi_{l=1}^n(1-(1-h_{i_l}^\sim)^\sigma)^{\phi_l})^{\frac{1}{\sigma}}\} > \}$,

where $\sigma > 0$ and $ch_l^\sim = \{< \{[\mu_{i_l}^{-\sim}, \mu_{i_l}^{+\sim}]\}, \{h_{i_l}^\sim\} >\}$ is the $lth$ largest of $n-$cubic hesitant fuzzy elements $ch_k(k = 1, 2, \ldots, n)$ which is obtained by using Definition 2.11 as score function ranking method.

**Proof** Omitted (same as Theorem 3.2).

**Definition 3.11 (GCHFOWG operator)** Let $ch_k(k = 1, 2, \ldots, n)$ be the collection of CHFEs. Let $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T$ be the weighting vector affiliated with the function $G^n \to G$ having $\phi_k \in [0, 1]$ and $\Sigma_{k=1}^n \phi_k = 1$. Then the generalized cubic hesitant fuzzy ordered weighted geometric (for short, GCHFOWG) operator is a mapping $G^n \to G$ defined by,

$GCHFOWG_\sigma(ch_1, ch_2, \ldots, ch_n) = \tau_k^{o\sigma} = \frac{1}{\sigma}((\otimes)_{l=1}^n(\sigma ch_l^\sim)^{\phi_l})$,

where $\sigma > 0$ and $ch_l^\sim = \{< \{[\mu_{i_l}^{-\sim}, \mu_{i_l}^{+\sim}]\}, \{h_{i_l}^\sim\} >\}$ is the $lth$ largest of $n-$cubic hesitant fuzzy elements $ch_k(k = 1, 2, \ldots, n)$ which is obtained by using Definition 2.11 as score function ranking method.

**Theorem 3.12** Let $ch_k(k = 1, 2, \ldots, n)$ be the collection of CHFEs. Then the aggregated result is obtained by applying GCHFOWG operator is also a CHFE, and:

$GCHFOWG_\sigma(ch_1, ch_2, \ldots, ch_n) = \tau_k^{o\sigma} = \frac{1}{\sigma}((\otimes)_{l=1}^n(\sigma ch_l^\sim)^{\phi_l}) = \{< \{[1 - (1 - \Pi_{l=1}^n(1 - (1 - \mu_{i_l}^{-\sim})^\sigma)^{\phi_l})^{\frac{1}{\sigma}}, 1 - (1 - \Pi_{l=1}^n(1 - (1 - \mu_{i_l}^{+\sim})^\sigma)^{\phi_l})^{\frac{1}{\sigma}}]\}, \{(1 - \Pi_{l=1}^n(1 - (h_{i_l}^\sim)^\sigma)^{\phi_l})^{\frac{1}{\sigma}}\} >\}$,

where $\sigma > 0$ and $ch_l^\sim = \{< \{[\mu_{i_l}^{-\sim}, \mu_{i_l}^{+\sim}]\}, \{h_{i_l}^\sim\} >\}$ is the $lth$ largest of $n-$cubic hesitant fuzzy elements $ch_k(k = 1, 2, \ldots, n)$ which is obtained by using Definition 2.11 as score function ranking method.

**Proof** Omitted (same as Theorem 3.4).

**Definition 3.13 (CHFHA operator)** Let $ch_k(k = 1, 2, \ldots, n)$ be the collection of CHFEs.The cubic hesitant fuzzy hybrid averaging (for short, CHFHA) operator is defined by the mapping $G^n \to G$ such that,

$CHFHA(ch_1, ch_2, \ldots, ch_n) = \varkappa_{k\phi,w}^H = \psi_k = (\oplus)_{l=1}^n(\phi_l ch_l^{o^\sim})$,

where $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T$ weighting vector is associated with the mapping $G^n \to G$ with $\phi_k \in [0, 1]$ and $\Sigma_{k=1}^n \phi_k = 1$, $ch_l^{o^\sim}$ is obtained as $lth$ largest of $n-$cubic hesitant fuzzy elements $ch_k^o = nw_k ch_k$ by the ranking method defined by Definition 2.11 as scoring function ranking method. Here $n$ is a balance factor and $w = (w_1, w_2, \ldots, w_n)^T$ is a weight vector of CHFEs $ch_k(k = 1, 2, \ldots, n)$ with $w_k \in [0, 1]$ and $\Sigma_{k=1}^n w_k = 1$.

**Theorem 3.14** Let $ch_k(k = 1, 2, \ldots, n)$ be the collection of CHFEs. Then the aggregated result of CHFEs calculated through applying CHFHA operator is also a CHFE, and:

$CHFHA(ch_1, ch_2, \ldots, ch_n) = \varkappa_{k\phi,w}^H = \psi_k = (\oplus)_{l=1}^n(\phi_l ch_l^{o^\sim}) = \{< \{[1-\Pi_{l=1}^n(1-\mu_{i_l}^{-o^\sim})^{\phi_l}, 1 - \Pi_{l=1}^n(1 - \mu_{i_l}^{+o^\sim})^{\phi_l}]\}, \{\Pi_{l=1}^n(h_{i_l}^{o^\sim})^{\phi_l}\} >\}$,

where $ch_l^{o^\sim} = \{< \{[\mu_{i_l}^{-o^\sim}, \mu_{i_l}^{+o^\sim}]\}, \{h_{i_l}^{o^\sim}\} >\}$ is the $lth$ largest of $n-$cubic hesitant fuzzy elements $ch_k^o = nw_k ch_k(k = 1, 2, \ldots, n)$ which is obtained by using Definition 2.11 defined as scoring function ranking method.

**Proof** Omitted (same Theorem 3.2).

**Definition 3.15 (CHFHG operator)** Let $ch_k(k = 1, 2, \ldots, n)$ be the collection of CHFEs.The cubic hesitant fuzzy hybrid geometric (for short, CHFHG) operator is defined by the function $G^n \to G$ such that,

$CHFHG(ch_1, ch_2, \ldots, ch_n) = G^H_{k\phi,w} = \chi_k = (\otimes)^n_{l=1}((ch^{o\sim}_l)^{\phi_l}),$

where $\phi= (\phi_1, \phi_2, \ldots, \phi_n)^T$ weighting vector is associated with the mapping $G^n \to G$ with $\phi_k \in [0, 1]$ and $\Sigma^n_{k=1}\phi_k = 1$, $ch^{o\sim}_l$ is obtained as $lth$ largest of $n-$cubic hesitant fuzzy elements $ch^o_k = ch^{nw_k}_k$ by the ranking method defined by Definition 2.11 as scoring function ranking method. Here $n$ is a balance factor and $w = (w_1, w_2, \ldots, w_n)^T$ is a weight vector of CHFEs $ch_k(k = 1, 2, \ldots, n)$ with $w_k \in [0, 1]$ and $\Sigma^n_{k=1}w_k = 1$.

**Theorem 3.16** Let $ch_k(k = 1, 2, \ldots, n)$ be a collection of CHFEs. Then the aggregated result is gained through applying CHFHG operator is also a CHFE, and:

$CHFHG(ch_1, ch_2, \ldots, ch_n) = G^H_{k\phi,w} = \chi_k = (\otimes)^n_{l=1}((ch^{o\sim}_l)^{\phi_l})$

$= \{< \{[\Pi^n_{l=1}(\mu^{-o\sim}_{i_l})^{\phi_l}, \Pi^n_{l=1}(\mu^{+o\sim}_{i_l})^{\phi_l}]\}, \{1 - \Pi^n_{l=1}(1 - h^{o\sim}_{i_l})^{\phi_l}\} >\},$

where $ch^{o\sim}_l = \{< \{[\mu^{-o\sim}_{i_l}, \mu^{+o\sim}_{i_l}]\}, \{h^{o\sim}_{i_l}\} >\}$ is the $lth$ largest of $n-$cubic hesitant fuzzy elements $ch^o_k = ch^{nw_k}_k(k = 1, 2, \ldots, n)$ which is obtained by using Definition 2.11 defined as scoring function ranking method.

**Proof** Omitted (same as Theorem 3.4).

**Definition 3.17 (GCHFHA operator)** Let $ch_k(k = 1, 2, \ldots, n)$ be the collection of CHFEs. The generalized cubic hesitant fuzzy hybrid averaging (for short, GCHFHA) operator is a mapping $G^n \to G$ defined by,

$GCHFHA_\sigma(ch_1, ch_2, \ldots, ch_n) = \psi^\sigma_k = ((\oplus)^n_{l=1}(\phi_l(ch^{o\sim}_l)^\sigma))^{\frac{1}{\sigma}},$

where $\sigma > 0, \phi = (\phi_1, \phi_2, \ldots, \phi_n)^T$ weighting vector is associated with the mapping $G^n \to G$ with $\phi_k \in [0, 1]$ and $\Sigma^n_{k=1}\phi_k = 1$, $ch^{o\sim}_l$ is obtained as $lth$ largest of $n-$cubic hesitant fuzzy elements $ch^o_k = nw_kch_k$ by using Definition 2.11 as scoring function ranking method. Here $n$ is a balance factor and $w = (w_1, w_2, \ldots, w_n)^T$ is a weight vector of CHFEs $ch_k(k = 1, 2, \ldots, n)$ with $w_k \in [0, 1]$ and $\Sigma^n_{k=1}w_k = 1$.

**Remark 3.18** In special cases when $\phi = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ then the GCHFHA operator becomes GCHFWA operator, if $w =(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ then GCHFHA operator becomes GCHFOWA operator and if $\sigma = 1$ then GCHFHA operator becomes CHFHA operator.

**Theorem 3.19** Let $ch_k(k = 1, 2, \ldots, n)$ be a collection of CHFEs. Then the aggregated result is gained through applying GCHFHA operator is also a CHFE, and:

$GCHFHA_\sigma(ch_1, ch_2, \ldots, ch_n) = \psi^\sigma_k = ((\oplus)^n_{l=1}(\phi_l(ch^{o\sim}_l)^\sigma))^{\frac{1}{\sigma}} = \{< \{[(1 - \Pi^n_{l=1}(1 - (\mu^{-o\sim}_{i_l})^\sigma)^{\phi_l})^{\frac{1}{\sigma}}, (1 - \Pi^n_{l=1}(1 - (\mu^{+o\sim}_{i_l})^\sigma)^{\phi_l})^{\frac{1}{\sigma}}]\}, \{1 - (1 - \Pi^n_{l=1}(1 - (1 - h^{o\sim}_{i_l})^\sigma)^{\phi_l})^{\frac{1}{\sigma}}\} >\},$

where $ch^{o\sim}_l = \{< \{[\mu^{-o\sim}_{i_l}, \mu^{+o\sim}_{i_l}]\}, \{h^{o\sim}_{i_l}\} >\}$ is the $lth$ largest of $n-$cubic hesitant fuzzy elements $ch^o_k = nw_kch_k(k = 1, 2, \ldots, n)$ which is obtained by using Definition 2.11 as scoring function ranking method.

**Proof** Omitted (same as Theorem 3.2).

**Definition 3.20 (GCHFHG operator)** Let $ch_k(k = 1, 2, \ldots, n)$ be the collection of CHFEs. The generalized cubic hesitant fuzzy hybrid geometric (for short, GCHFHG) operator is a mapping $G^n \to G$ defined by,

$GCHFHG_\sigma(ch_1, ch_2, \ldots, ch_n) = \chi^\sigma_k = \frac{1}{\sigma}((\otimes)^n_{l=1}(\sigma ch^{o\sim}_l)^{\phi_l}),$

where $\sigma > 0, \phi = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ weighting vector is associated with the mapping $G^n \to G$ with $\phi_k \in [0, 1]$ and $\Sigma^n_{k=1}\phi_k = 1$, $ch^{o\sim}_l$ is obtained as $lth$ largest of $n-$cubic hesitant fuzzy elements $ch^o_k = ch^{nw_k}_k$ by using Definition 2.11 as scoring function ranking method. Here $n$ is a balance factor and $w = (w_1, w_2, \ldots, w_n)^T$ is a weight vector of CHFEs $ch_k(k = 1, 2, \ldots, n)$ with $w_k \in [0, 1]$ and $\Sigma^n_{k=1}w_k = 1$.

**Remark 3.21** In special cases when $\phi = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ then the GCHFHG operator becomes GCHFWG operator, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ then GCHFHG operator becomes GCHFOWG operator and if $\sigma = 1$ then GCHFHG operator becomes CHFHG operator.

**Theorem 3.22** Let $ch_k (k = 1, 2, \ldots, n)$ be a collection of CHFEs. Then the aggregated result calculated through GCHFHG operator is also a CHFE, and:

$GCHFHG_\sigma(ch_1, ch_2, \ldots, ch_n) = \chi_k^\sigma = \frac{1}{\sigma}((\otimes)_{l=1}^n (\sigma ch_l^{o\sim})^{\phi_l}) = \{< \{[1 - (1 - \Pi_{l=1}^n (1 - (1 - \mu_{i_l}^{-o\sim})^\sigma)^{\phi_l})^{\frac{1}{\sigma}}, 1 - (1 - \Pi_{l=1}^n (1 - (1 - \mu_{i_l}^{+o\sim})^\sigma)^{\phi_l})^{\frac{1}{\sigma}}]\}, \{(1 - \Pi_{l=1}^n (1 - (h_{i_l}^{o\sim})^\sigma)^{\phi_l})^{\frac{1}{\sigma}}\} >\}$,

where $ch_l^{o\sim} = \{< \{[\mu_{i_l}^{-o\sim}, \mu_{i_l}^{+o\sim}]\}, \{h_{i_l}^{o\sim}\} >\}$ is the $lth$ largest of $n-$cubic hesitant fuzzy elements $ch_k^o = ch_k^{nw_k} (k = 1, 2, \ldots, n)$ which is obtained by using Definition 2.11 as scoring function ranking method.

**Proof** Omitted (same as Theorem 3.4).

CHFHA and CHFHG operators do not satisfy the property of idempotency for aggregation operators, so in order to remove this deficiency we developed new operators as shown below in Definition 3.23 and Definition 3.26.

**Definition 3.23 (CHFHAA operator)** Let $ch_k (k = 1, 2, \ldots, n)$ be the collection of CHFEs. The cubic hesitant fuzzy hybrid arithmetical averaging (for short, CHFHAA) operator is a function $G^n \rightarrow G$ denoted and defined as,

$CHFHAA(ch_1, ch_2, \ldots, ch_n) = \psi_k^+ = \frac{(\oplus)_{l=1}^n (w_l \phi_l ch_l^\sim)}{\Sigma_{l=1}^n w_l \phi_l}$,

where $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T$ is a weight vector that is associated with the mapping $G^n \rightarrow G, \phi_k \in [0, 1]$ and $\Sigma_{k=1}^n \phi_k = 1$, $ch_l^\sim$ is obtained as $lth$ largest of $n-$cubic hesitant fuzzy elements $ch_k$ by using Definition 2.11 as scoring function ranking method. Here $n$ is a balance factor and $w = (w_1, w_2, \ldots, w_n)^T$ is a weight vector of CHFEs $ch_k (k = 1, 2, \ldots, n)$ with $w_k \in [0, 1]$ and $\Sigma_{k=1}^n w_k = 1$.

**Theorem 3.24** Let $ch_k (k = 1, 2, \ldots, n)$ be a collection of CHFEs. Then the aggregated result calculated through CHFHAA operator is also a CHFE, and:

$CHFHAA(ch_1, ch_2, \ldots, ch_n) = \psi_k^+ = \frac{(\oplus)_{l=1}^n (w_l \phi_l ch_l^\sim)}{\Sigma_{l=1}^n w_l \phi_l} = \{< \{[1 - \Pi_{l=1}^n (1 - \mu_{i_l}^{-\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^n w_l \phi_l}}, 1 - \Pi_{l=1}^n (1 - \mu_{i_l}^{+\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^n w_l \phi_l}}]\}, \{\Pi_{l=1}^n (h_{i_l}^\sim)^{\frac{w_l \phi_l}{\Sigma_{l=1}^n w_l \phi_l}}\} >\}$,

where $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T$ is a weight vector that is associated with the mapping $G^n \rightarrow G, \phi_k \in [0, 1]$ and $\Sigma_{k=1}^n \phi_k = 1$, $ch_l^\sim$ is obtained as $lth$ largest of $n-$cubic hesitant fuzzy elements $ch_k$ by using Definition 2.11 as scoring function ranking method. Here $n$ is a balance factor and $w = (w_1, w_2, \ldots, w_n)^T$ is a weight vector of CHFEs $ch_k (k = 1, 2, \ldots, n)$ with $w_k \in [0, 1]$ and $\Sigma_{k=1}^n w_k = 1$.

**Proof** By Theorem 2.12, it is clear that the aggregated result obtained by applying CHFHAA operator is a CHFE.

$\frac{w_l \phi_l ch_l^\sim}{\Sigma_{l=1}^n w_l \phi_l} = \{< \{[1 - (1 - \mu_{i_l}^{-\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^n w_l \phi_l}}, 1 - (1 - \mu_{i_l}^{+\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^n w_l \phi_l}}]\},$
$\{(h_{i_l}^\sim)^{\frac{w_l \phi_l}{\Sigma_{l=1}^n w_l \phi_l}}\} >\},$

$CHFHAA(ch_1, ch_2, \ldots, ch_n) = \frac{(\oplus)_{l=1}^n (w_l \phi_l ch_l^\sim)}{\Sigma_{l=1}^n w_l \phi_l}$
$= (\oplus)_{l=1}^n \{< \{[1 - (1 - \mu_{i_l}^{-\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^n w_l \phi_l}}, 1 - (1 - \mu_{i_l}^{+\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^n w_l \phi_l}}]\},$
$\{(h_{i_l}^\sim)^{\frac{w_l \phi_l}{\Sigma_{l=1}^n w_l \phi_l}}\} >\},$

$CHFHAA(ch_1, ch_2, \ldots, ch_n)$
$= \{< \{[1 - \Pi_{l=1}^n (1 - \mu_{i_l}^{-\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^n w_l \phi_l}}, 1 - \Pi_{l=1}^n (1 - \mu_{i_l}^{+\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^n w_l \phi_l}}]\},$
$\{\Pi_{l=1}^n (h_{i_l}^\sim)^{\frac{w_l \phi_l}{\Sigma_{l=1}^n w_l \phi_l}}\} >\},$

**Theorem 3.25 (Idempotency)** If $ch_l^{\sim} = ch^{\sim}(l = 1, 2 \dots, n)$, then $CHFHAA(ch_1, ch_2, \dots, ch_n) = ch^{\sim}$.

**Proof** $CHFHAA(ch_1, ch_2, \dots, ch_n) = \frac{(\oplus)_{l=1}^{n}(w_l \phi_l ch_l^{\sim})}{\Sigma_{l=1}^{n} w_l \phi_l}$
$= \frac{(\oplus)_{l=1}^{n}(w_l \phi_l ch^{\sim})}{\Sigma_{l=1}^{n} w_l \phi_l} = ch^{\sim} \frac{(\oplus)_{l=1}^{n}(w_l \phi_l)}{\Sigma_{l=1}^{n} w_l \phi_l} = ch^{\sim} \frac{\Sigma_{l=1}^{n} w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l} = ch^{\sim}$.

**Definition 3.26 (CHFHAG operator)** Let $ch_k(k = 1, 2, \dots, n)$ be the collection of CHFEs. The cubic hesitant fuzzy hybrid arithmetical geometric (for short, CHFHAG) operator is a function $G^n \to G$ denoted and defined as,

$CHFHAG(ch_1, ch_2, \dots, ch_n) = \chi_k^{+} = (\otimes)_{l=1}^{n}(ch_l^{\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}$,

where $\phi = (\phi_1, \phi_2, \dots, \phi_n)^T$ is a weight vector that is affiliated with the mapping $G^n \to G, \phi_k \in [0, 1]$ and $\Sigma_{k=1}^{n} \phi_k = 1$, $ch_l^{\sim}$ is obtained as $lth$ largest of $n-$cubic hesitant fuzzy elements $ch_k$ by using Definition 2.11 as scoring function ranking method. Here $n$ is a balance factor and $w = (w_1, w_2, \dots, w_n)^T$ is a weight vector of CHFEs $ch_k(k = 1, 2, \dots, n)$ with $w_k \in [0, 1]$ and $\Sigma_{k=1}^{n} w_k = 1$.

**Theorem 3.27** Suppose we have a collection of CHFEs $ch_k(k = 1, 2, \dots, n)$. Then the aggregated result is gained through applying CHFHAG operator is also a CHFE, and:

$CHFHAG(ch_1, ch_2, \dots, ch_n) = \chi_k^{+} = (\otimes)_{l=1}^{n}(ch_l^{\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}$
$= \{< \{[\Pi_{l=1}^{n}(\mu_{i_l}^{-\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}, \Pi_{l=1}^{n}(\mu_{i_l}^{+\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}]\},$
$\{1 - \Pi_{l=1}^{n}(1 - h_{i_l}^{\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}\} >\},$

where $ch_l^{\sim} = \{< \{[\mu_{i_l}^{-\sim}, \mu_{i_l}^{+\sim}]\}, \{h_{i_l}^{\sim}\} >\}$ is the $lth$ largest of $n-$cubic hesitant fuzzy elements $ch_k(k = 1, 2, \dots, n)$ which is obtained by using Definition 2.11 as score function ranking method.

**Proof** From Theorem 2.12 it is clear that the aggregated result obtained by applying CHFHAG operator is a CHFE.

$(ch_l^{\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}} = \{< \{[(\mu_{i_l}^{-\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}, (\mu_{i_l}^{+\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}]\},$
$\{1 - (1 - h_{i_l}^{\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}\} >\},$
$CHFHAG(ch_1, ch_2, \dots, ch_n) = (\otimes)_{l=1}^{n}(ch_l^{\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}$
$= (\otimes)_{l=1}^{n}\{< \{[(\mu_{i_l}^{-\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}, (\mu_{i_l}^{+\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}]\}, \{1 - (1 - h_{i_l}^{\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}\} >\},$
$CHFHAG(ch_1, ch_2, \dots, ch_n) = (\otimes)_{l=1}^{n}(ch_l^{\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}$
$= \{< \{[\Pi_{l=1}^{n}(\mu_{i_l}^{-\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}, \Pi_{l=1}^{n}(\mu_{i_l}^{+\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}]\},$
$\{1 - \Pi_{l=1}^{n}(1 - h_{i_l}^{\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}\} >\}.$

**Theorem 3.28 (Idempotency)** If $ch_l^{\sim} = ch^{\sim}(l = 1, 2 \dots, n)$, then $CHFHAG(ch_1, ch_2, \dots, ch_n) = ch^{\sim}$.

**Proof**

$CHFHAG(ch_1, ch_2, \dots, ch_n) = (\otimes)_{l=1}^{n}(ch_l^{\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}$
$= \{< \{[\Pi_{l=1}^{n}(\mu_{i_l}^{-\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}, \Pi_{l=1}^{n}(\mu_{i_l}^{+\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}]\},$
$\{1 - \Pi_{l=1}^{n}(1 - h_{i_l}^{\sim})^{\frac{w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}\} >\},$

since $ch_l^{\sim} = ch^{\sim}(l = 1, 2 \dots, n)$ so $\mu_{i_l}^{-\sim} = \mu_i^{-\sim}, \mu_{i_l}^{+\sim} = \mu_i^{+\sim}$ and $h_{i_l}^{\sim} = h_i^{\sim}$,

$CHFHAG(ch_1, ch_2, \dots, ch_n) = \{< \{[(\mu_{i_l}^{-\sim})^{\frac{\Sigma_{l=1}^{n} w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}, (\mu_{i_l}^{+\sim})^{\frac{\Sigma_{l=1}^{n} w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}]\}, \{1 - (1 - h_i^{\sim})^{\frac{\Sigma_{l=1}^{n} w_l \phi_l}{\Sigma_{l=1}^{n} w_l \phi_l}}\} >\} = \{< \{[\mu_{i_l}^{-\sim}, \mu_{i_l}^{+\sim}]\}, \{h_{i_l}^{\sim}\} >\} = ch_l^{\sim}$.

## 4. MULTI CRITERIA DECISION MAKING USING GENERALIZED HYBRID AGGREGATION OPERATORS BASED ON CUBIC HESITANT FUZZY SETS

We are giving a method for tackling a MCDM problem based on CHFSs by applying GCHFHA and GCHFHG operators. We are allocating the weights to both CHFEs and the vector associated with the mapping $G^n \rightarrow G$. Suppose that $ch = \{ch_1, ch_2, \ldots, ch_n\}$ be collection of alternatives and also assume that $R = \{R_1, R_2, \ldots, R_m\}$ be a set of criteria. Considering the criteria $R_j(1 \leq j \leq m)$, recommended by the decision maker having weights $w_k \in [0,1]$ and $\phi_k \in [0,1]$. Here $w_k$ is the weight vector of CHFEs $ch_k(k = 1, 2, \ldots, n)$ and $\phi_k(1 \leq k \leq n)$ is the weighting vector associated with the mapping $G^n \rightarrow G$, also we have $\Sigma_{k=1}^{n} w_k = 1$ and $\Sigma_{k=1}^{n} \phi_k = 1$. Thus $ch_k = \{< \{R_j, \{[\mu^-_{i_{kj}}(R_j), \mu^+_{i_{kj}}(R_j)]\}, \{h_{i_{kj}}(R_j)\} >\}$. The CHFE which is a pair of IVHFE and HFE is denoted by $ch_k = d_{kj} = \{< \{[\mu^-_{i_{kj}}, \mu^+_{i_{kj}}]\}, \{h_{i_{kj}}\} >\}$. Hence we obtain a decision matrix $D = d_{kj}$. We are giving the notation to generalized cubic hesitant fuzzy hybrid averaging CHFE as $\psi^\sigma_k$ for $ch^{o\sim}_{kl}$ $(1 \leq k \leq n, 1 \leq l \leq m)$, here $ch^{o\sim}_{kl}$ is obtained as $lth$ largest of $m-$CHFEs $ch^o_{kj} = ch^{mw_k}_{kj}$ by using score function (see Definition 2.11) ranking method. These CHFEs are obtained by using GCHFHA operator and GCHFHG operator through each row in the decision matrix. First, we weight the CHFEs by using the weights of CHFEs assigned to each criterion and then rank the CHFEs through a score function (see Definition 2.11) by using $kth$ row and $jth$ column in the matrix. Then aggregate the CHFEs by applying GCHFHA or GCHFHG operator through weighting vector associated with the mapping $G^n \rightarrow G$ on each $kth$ row in the matrix. Then after we use score function (see Definition 2.11) in order to obtain $T(\psi^\sigma_k)$ for $\psi^\sigma_k$ or $T(\chi^\sigma_k)$ for $\chi^\sigma_k$. In view of the score function, we rank $ch_k(k = 1, 2, \ldots, n)$ and choose the optimum alternative. The decision making process is followed through the steps given below.

**Step 1** Weight the CHFEs by using the weights of CHFEs assigned to each criterion through $kth$ row and $jth$ columns in the matrix.

**Step 2** Rank the CHFEs through a score function (see Definition 2.11).

**Step 3** Calculate GCHFHA or GCHFHG values by applying Theorem 3.19 or Theorem 3.22 respectively through weighting vector associated with the mapping $G^n \rightarrow G$ on every $kth$ row in the matrix.

**Step 4** Determine the accuracy by using a scoring function of CHFE (see Definition 2.11) to get $T(\psi^\sigma_k)$ or $T(\chi^\sigma_k)$, rank $ch_k(k = 1, 2, \ldots, n)$ and choose the best alternative by comparing each $T(\psi^\sigma_k)$ or $T(\chi^\sigma_k)$.

## 5. DESCRIPTIVE EXAMPLE

We are providing an example, in order to solve a multi criteria decision making (MCDM) problem for CHFSs using generalized cubic hesitant fuzzy hybrid aggregation operators. Suppose that there is a group of people those are planning to invest in an international bank. The four alternatives are $(c_1)$ National Australia Bank $(c_2)$ Barclays $(c_3)$ Royal Bank of Canada $(c_4)$ Bank of New York Mellon, under four criteria $(h_1)$ country Risk $(h_2)$ credit Risk $(h_3)$ banking performance $(h_4)$ portfolio selection and management. Now we will find the best alternative through the algorithm as defined in Section 4. We are considering the matrix $M_1$ consisting of CHFS values as follows,

| | $h_1$ | $h_2$ | $h_3$ | $h_4$ |
|---|---|---|---|---|
| $c_1$ | $\{<\{[0.1,0.4],[0.5,0.7]\},\{0.3,0.6\}>\}$ | $\{<\{[0.2,0.4],[0.5,0.6],[0.7,0.8]\},\{0.3,0.55,0.75\}>\}$ | $\{<\{[0.1,0.3]\},\{0.2\}>\}$ | $\{<\{[0.15,0.35],[0.4,0.7]\},\{0.2,0.6\}>\}$ |
| $c_2$ | $\{<\{[0.4,0.7]\},\{0.5\}>\}$ | $\{<\{[0.2,0.3],[0.4,0.6]\},\{0.25,0.5\}>\}$ | $\{<\{[0.1,0.2],[0.3,0.5],[0.6,0.7],[0.8,0.9]\},\{0.15,0.4,0.65,0.85\}>\}$ | $\{<\{[0.4,0.5],[0.55,0.6],[0.7,0.9]\},\{0.45,0.55,0.8\}>\}$ |
| $c_3$ | $\{<\{[0.1,0.3],[0.4,0.6],[0.7,0.8]\},\{0.2,0.5,0.75\}>\}$ | $\{<\{[0.6,0.9]\},\{0.8\}>\}$ | $\{<\{[0.4,0.6],[0.7,0.9]\},\{0.5,0.8\}>\}$ | $\{<\{[0.2,0.4],[0.5,0.6],[0.7,0.9],[0.92,1]\},\{0.3,0.55,0.8,0.95\}>\}$ |
| $c_4$ | $\{<\{[0.3,0.4],[0.5,0.6],[0.7,0.8],[0.9,1]\},\{0.35,0.55,0.75,0.95\}>\}$ | $\{<\{[0.1,0.2],[0.3,0.4],[0.6,0.8]\},\{0.15,0.35,0.7\}>\}$ | $\{<\{[0.5,0.7]\},\{0.6\}>\}$ | $\{<\{[0.3,0.5],[0.6,0.9]\},\{0.4,0.7\}>\}$ |

Suppose the weight vector of $h_1, h_2, h_3$ and $h_4$ is $w = (0.1, 0.2, 0.3, 0.4)^T$. Considering the aspect that different alternatives can spotlight some different characteristics, in order to demonstrate this concern, we have one more weighting vector of each criterion as $\phi = (0.15, 0.25, 0.5, 1)^T$ and we use the algorithm as follows,

**Step 1**

$ch_{11} = \{<\{[0.1,0.4],[0.5,0.7]\},\{0.3,0.6\}>\}$,
$ch_{12} = \{<\{[0.2,0.4],[0.5,0.6],[0.7,0.8]\},\{0.3,0.55,0.75\}>\}$,
$ch_{13} = \{<\{[0.1,0.3]\},\{0.2\}>\}$,
$ch_{14} = \{<\{[0.15,0.35],[0.4,0.7]\},\{0.2,0.6\}>\}$,
$ch^o_{11} = nw_1 ch_{11} = \{<\{[0.0413,0.1848],[0.2421,0.3822]\},$
$\{0.6178,0.8152\}>\}$,
$ch^o_{12} = nw_2 ch_{12} = \{<\{[0.1635,0.3355],[0.4257,0.5196],[0.6183,0.7241]\},$
$\{0.3817,0.6199,0.7944\}>\}$,
$ch^o_{13} = nw_3 ch_{13} = \{<\{[0.1188,0.3482]\},\{0.1450\}>\}$,
$ch^o_{14} = nw_4 ch_{14} = \{<\{[0.2290,0.4980],[0.5584,0.8543]\},$
$\{0.0761,0.4416\}>\}$,
$ch_{21} = \{<\{[0.4,0.7]\},\{0.5\}>\}$,
$ch_{22} = \{<\{[0.2,0.3],[0.4,0.6]\},\{0.25,0.5\}>\}$,
$ch_{23} = \{<\{[0.1,0.2],[0.3,0.5],[0.6,0.7],[0.8,0.9]\},$
$\{0.15,0.4,0.65,0.85\}>\}$,
$ch_{24} = \{<\{[0.4,0.5],[0.55,0.6],[0.7,0.9]\},\{0.45,0.55,0.8\}>\}$,

$ch^{o}_{21} = nw_1 ch_{21} = \{< \{[0.1848, 0.3822]\}, \{0.7579\} >\},$
$ch^{o}_{22} = nw_2 ch_{22} = \{< \{[0.1635, 0.2482], [0.3355, 0.5196]\},$
$\{0.3299, 0.5743\} >\},$
$ch^{o}_{23} = nw_3 ch_{23} = \{< \{[0.1188, 0.2349], [0.3482, 0.5647], [0.6670, 0.7642],$
$[0.8550, 0.9369]\}, \{0.1026, 0.3330, 0.5963, 0.8228\} >\},$
$ch^{o}_{24} = nw_4 ch_{24} = \{< \{[0.5584, 0.6701], [0.7213, 0.7692], [0.8543, 0.9749]\},$
$\{0.2787, 0.3842, 0.6998\} >\},$
$ch_{31} = \{< \{[0.1, 0.3], [0.4, 0.6], [0.7, 0.8]\}, \{0.2, 0.5, 0.75\} >\},$
$ch_{32} = \{< \{[0.6, 0.9]\}, \{0.8\} >\},$
$ch_{33} = \{< \{[0.4, 0.6], [0.7, 0.9]\}, \{0.5, 0.8\} >\},$
$ch_{34} = \{< \{[0.2, 0.4], [0.5, 0.6], [0.7, 0.9], [0.92, 1]\}, \{0.3, 0.55, 0.8, 0.95\} >\},$
$ch^{o}_{31} = nw_1 ch_{31} = \{< \{[0.0413, 0.1330], [0.1848, 0.3069], [0.3822, 0.4747]\},$
$\{0.5253, 0.7579, 0.8913\} >\},$
$ch^{o}_{32} = nw_2 ch_{32} = \{< \{[0.5196, 0.8415]\}, \{0.8365\} >\},$
$ch^{o}_{33} = nw_3 ch_{33} = \{< \{[0.4583, 0.6670], [0.7642, 0.9369]\},$
$\{0.4353, 0.7651\} >\},$
$ch^{o}_{34} = nw_4 ch_{34} = \{< \{[0.3002, 0.5584], [0.6701, 0.7692], [0.8543, 0.9749], [0.9824, 1]\},$
$\{0.1457, 0.3842, 0.6998, 0.9212\} >\},$
$ch_{41} = \{< \{[0.3, 0.4], [0.5, 0.6], [0.7, 0.8], [0.9, 1]\},$
$\{0.35, 0.55, 0.75, 0.95\} >\},$
$ch_{42} = \{< \{[0.1, 0.2], [0.3, 0.4], [0.6, 0.8]\}, \{0.15, 0.35, 0.7\} >\},$
$ch_{43} = \{< \{[0.5, 0.7]\}, \{0.6\} >\},$
$ch_{44} = \{< \{[0.3, 0.5], [0.6, 0.9]\}, \{0.4, 0.7\} >\},$
$ch^{o}_{41} = nw_1 ch_{41} = \{< \{[0.1330, 0.1848], [0.2421, 0.3069], [0.3822, 0.4747], [0.6019, 1]\},$
$\{0.6571, 0.7873, 0.8913, 0.9797\} >\},$
$ch^{o}_{42} = nw_2 ch_{42} = \{< \{[0.0808, 0.1635], [0.2482, 0.3355], [0.5196, 0.7241]\},$
$\{0.2192, 0.4318, 0.7518\} >\},$
$ch^{o}_{43} = nw_3 ch_{43} = \{< \{[0.5647, 0.7642]\}, \{0.5417\} >\},$
$ch^{o}_{44} = nw_4 ch_{44} = \{< \{[0.4349, 0.6701], [0.7692, 0.9749]\}, \{0.2308, 0.5651\} >\},$

**Step 2**

$T(ch^{o}_{11}) = 0.0709, T(ch^{o}_{12}) = 0.2638, T(ch^{o}_{13}) = -0.1940, T(ch^{o}_{14}) = 0.1644,$
clearly we have by ranking method $ch^{o}_{12} > ch^{o}_{14} > ch^{o}_{11} > ch^{o}_{13}$.
Thus we have, $ch^{o\sim}_{11} = ch^{o}_{12}, ch^{o\sim}_{12} = ch^{o}_{14}, ch^{o\sim}_{13} = ch^{o}_{11}, ch^{o\sim}_{14} = ch^{o}_{13}$.
Thus $ch^{o\sim}_{11} = \{< \{[0.1635, 0.3355], [0.4257, 0.5196], [0.6183, 0.7241]\},$
$\{0.3817, 0.6199, 0.7944\} >\},$
$ch^{o\sim}_{12} = \{< \{[0.2290, 0.4980], [0.5584, 0.8543]\}, \{0.0761, 0.4416\} >\},$
$ch^{o\sim}_{13} = \{< \{[0.0413, 0.1848], [0.2421, 0.3822]\}, \{0.6178, 0.8152\} >\},$
$ch^{o\sim}_{14} = \{< \{[0.1188, 0.3482]\}, \{0.1450\} >\},$
$T(ch^{o}_{21}) = 0.1625, T(ch^{o}_{22}) = 0.0428, T(ch^{o}_{23}) = 0.2931, T(ch^{o}_{24}) = 0.4852,$
clearly we have by ranking method $ch^{o}_{24} > ch^{o}_{23} > ch^{o}_{21} > ch^{o}_{22}$.
Thus we have, $ch^{o\sim}_{21} = ch^{o}_{24}, ch^{o\sim}_{22} = ch^{o}_{23}, ch^{o\sim}_{23} = ch^{o}_{21}, ch^{o\sim}_{24} = ch^{o}_{22}$.
Thus $ch^{o\sim}_{21} = \{< \{[0.5584, 0.6701], [0.7213, 0.7692], [0.8543, 0.9749]\},$
$\{0.2787, 0.3842, 0.6998\} >\},$
$ch^{o\sim}_{22} = \{< \{[0.1188, 0.2349], [0.3482, 0.5647], [0.6770, 0.7642], [0.8550, 0.9369]\},$
$\{0.1026, 0.3330, 0.5963, 0.8228\} >\},$
$ch^{o\sim}_{23} = \{< \{[0.1848, 0.3822]\}, \{0.7579\} >\},$
$ch^{o\sim}_{24} = \{< \{[0.1635, 0.2482], [0.3355, 0.5196]\}, \{0.3299, 0.5743\} >\},$
$T(ch^{o}_{31}) = 0.1162, T(ch^{o}_{32}) = 0.5988, T(ch^{o}_{33}) = 0.5067, T(ch^{o}_{34}) = 0.5326,$

clearly we have by ranking method $ch^o_{32} > ch^o_{34} > ch^o_{33} > ch^o_{31}$.
Thus we have, $ch^{o\sim}_{31} = ch^o_{32}, ch^{o\sim}_{32} = ch^o_{34}, ch^{o\sim}_{33} = ch^o_{33}, ch^{o\sim}_{34} = ch^o_{31}$.
So $ch^{o\sim}_{31} = \{<\{[0.5196, 0.8415]\}, \{0.8365\}>\}$,
$ch^{o\sim}_{32} = \{<\{[0.3002, 0.5584], [0.6701, 0.7692], [0.8543, 0.9749], [0.9824, 1]\}$,
$\{0.1457, 0.3842, 0.6998, 0.9212\}>\}$,
$ch^{o\sim}_{33} = \{<\{[0.4583, 0.6770], [0.7642, 0.9369]\}, \{0.4353, 0.7651\}>\}$,
$ch^{o\sim}_{34} = \{<\{[[0.0413, 0.1330], [0.1848, 0.3069], [0.3822, 0.4747]\}$,
$\{0.5253, 0.7579, 0.8913\}>\}$,
$T(ch^o_{41}) = 0.3301, T(ch^o_{42}) = 0.0791, T(ch^o_{43}) = 0.4353, T(ch^o_{44}) = 0.4113$,
clearly we have by ranking method $ch^o_{43} > ch^o_{44} > ch^o_{41} > ch^o_{42}$.
Thus we have, $ch^{o\sim}_{41} = ch^o_{43}, ch^{o\sim}_{42} = ch^o_{44}, ch^{o\sim}_{43} = ch^o_{41}, ch^{o\sim}_{44} = ch^o_{42}$.
Hence $ch^{o\sim}_{41} = \{<\{[0.5647, 0.7642]\}, \{0.5417\}>\}$,
$ch^{o\sim}_{42} = \{<\{[0.4349, 0.6701], [0.7692, 0.9749]\}, \{0.2308, 0.5651\}>\}$,
$ch^{o\sim}_{43} = \{<\{[0.1330, 0.1848], [0.2421, 0.3069], [0.3822, 0.4747], [0.6019, 1]\}$,
$\{0.6571, 0.7873, 0.8913, 0.9797\}>\}$,
$ch^{o\sim}_{44} = \{<\{[0.0808, 0.1635], [0.2482, 0.3355], [0.5196, 0.7241]\}$,
$\{0.2192, 0.4318, 0.7518\}>\}$,

**Step 3**

By Theorem 3.19 we have,
$GCHFHA_\sigma(ch_{kl})^n_{k,l=1} = \psi^\sigma_k = (((\oplus)^n_{l=1})^n_{k=1}(\phi_l(ch^{o\sim}_{kl})^\sigma))^{\frac{1}{\sigma}} = \{<\{[(1-\Pi^n_{k,l=1}(1-(\mu^{-o\sim}_{i_{kl}})^\sigma)^{\phi_l})^{\frac{1}{\sigma}}, (1-\Pi^n_{k,l=1}(1-(\mu^{+o\sim}_{i_{kl}})^\sigma)^{\phi_l})^{\frac{1}{\sigma}}]\}, \{1-(1-\Pi^n_{k,l=1}(1-(1-h^{o\sim}_{i_{kl}})^\sigma)^{\phi_l})^{\frac{1}{\sigma}}\}>\}$,
for $\sigma = 1$ we have,
$GCHFHA_1(ch_{11}, ch_{12}, ch_{13}, ch_{14}) = \psi^1_1 = ((\oplus)^4_{l=1})_{k=1}(\phi_l ch^{o\sim}_{kl}) = \{<\{[1 - \Pi^4_{l=1}(1-\mu^{-o\sim}_{i_{1l}})^{\phi_l}, 1-\Pi^4_{l=1}(1-\mu^{+o\sim}_{i_{1l}})^{\phi_l}]\}, \{\Pi^4_{l=1}(h^{o\sim}_{i_{1l}})^{\phi_l}\}>\}$,
$GCHFHA_1(ch_{11}, ch_{12}, ch_{13}, ch_{14}) = \psi^1_1 = \{<\{[0.1180, 0.3152], [0.2158, 0.4038]$,
$[0.8509, 0.4973], [0.3178, 0.5624], [0.1664, 0.3477], [0.2588, 0.4321], [0.2748, 0.5212]$,
$[0.3552, 0.5832], [0.2159, 0.3997], [0.3028, 0.4775], [0.3179, 0.5594], [0.3935, 0.6165]\}$,
$\{0.2946, 0.3384, 0.4572, 0.5252, 0.3168, 0.3639, 0.4917, 0.5648, 0.3288, 0.3777$,
$0.5103, 0.5862\}>\}$,
$GCHFHA_1(ch_{21}, ch_{22}, ch_{23}, ch_{24}) = \psi^1_2 = ((\oplus)^4_{l=1})_{k=2}(\phi_l ch^{o\sim}_{kl}) = \{<\{[1 - \Pi^4_{l=1}(1-\mu^{-o\sim}_{i_{2l}})^{\phi_l}, 1-\Pi^4_{l=1}(1-\mu^{+o\sim}_{i_{2l}})^{\phi_l}]\}, \{\Pi^4_{l=1}(h^{o\sim}_{i_{2l}})^{\phi_l}\}>\}$,
$GCHFHA_1(ch_{21}, ch_{22}, ch_{23}, ch_{24}) = \psi^1_2 = \{<\{[0.2398, 0.3950], [0.2571, 0.4215]$,
$[0.2906, 0.4266], [0.3067, 0.4517], [0.3563, 0.5889], [0.3710, 0.6069], [0.2950, 0.4746]$,
$[0.3111, 0.4976], [0.3412, 0.5020], [0.3570, 0.5237], [0.4031, 0.6430], [0.4167, 0.6586]$,
$[0.4040, 0.5493], [0.4176, 0.5690], [0.4438, 0.5728], [0.4564, 0.5915], [0.4953, 0.6937]$,
$[0.5068, 0.7071], [0.5159, 0.6758], [0.5269, 0.6900], [0.5482, 0.6927], [0.5584, 0.7062]$,
$[0.5900, 0.7797], [0.5994, 0.7894]\}, \{0.3641, 0.3848, 0.3820, 0.4038, 0.4180, 0.4418$,
$0.4887, 0.5165, 0.5128, 0.5420, 0.5610, 0.5930, 0.5653, 0.5975, 0.5932, 0.6270$,
$0.6490, 0.5930, 0.6127, 0.6476, 0.6429, 0.6796, 0.7034, 0.7435\}>\}$,
$GCHFHA_1(ch_{31}, ch_{32}, ch_{33}, ch_{34}) = \psi^1_3 = ((\oplus)^4_{l=1})_{k=3}(\phi_l ch^{o\sim}_{kl}) = \{<\{[1 - \Pi^4_{l=1}(1-\mu^{-o\sim}_{i_{3l}})^{\phi_l}, 1-\Pi^4_{l=1}(1-\mu^{+o\sim}_{i_{3l}})^{\phi_l}]\}, \{\Pi^4_{l=1}(h^{o\sim}_{i_{3l}})^{\phi_l}\}>\}$,
$GCHFHA_1(ch_{31}, ch_{32}, ch_{33}, ch_{34}) = \psi^1_3 = \{<\{[0.3995, 0.6482], [0.4091, 0.6560]$,
$[0.4253, 0.6654], [0.6038, 0.8469], [0.6102, 0.8503], [0.6208, 0.8543], [0.5024, 0.7009]$,
$[0.5104, 0.7075], [0.5238, 0.7155], [0.6717, 0.8698], [0.6770, 0.8727], [0.6858, 0.8762]$,
$[0.5943, 0.8282], [0.6009, 0.8320], [0.6118, 0.8366], [0.7324, 0.9252], [0.7367, 0.9269]$,
$[0.7439, 0.9289], [0.7609, 1], [0.7647, 1], [0.7711, 1], [0.8422, 1], [0.8448, 1], [0.8490, 1]\}$,

$\{0.3721, 0.3860, 0.3923, 0.4933, 0.5117, 0.5201, 0.4742, 0.4919, 0.4999, 0.6287, 0.6521, 0.6628, 0.5509, 0.5714, 0.5808, 0.7303, 0.7576, 0.7700, 0.5901, 0.6121, 0.6221, 0.7823, 0.8115, 0.8247\} >\},$

$GCHFHA_1(ch_{41}, ch_{42}, ch_{43}, ch_{44}) = \psi_4^1 = ((\oplus)_{l=1}^4)_{k=4}(\phi_l ch_{kl}^{o\sim}) = \{< \{[1 - \Pi_{l=1}^4(1-\mu_{i_{4l}}^{-o\sim})^{\phi_l}, 1 - \Pi_{l=1}^4(1-\mu_{i_{4l}}^{+o\sim})^{\phi_l}]\}, \{\Pi_{l=1}^4(h_{i_{4l}}^{o\sim})^{\phi_l}\} >\},$

$GCHFHA_1(ch_{41}, ch_{42}, ch_{43}, ch_{44}) = \psi_4^1 = \{< \{[0.2934, 0.4588], [0.3074, 0.4711], [0.3378, 0.5156], [0.3393, 0.5010], [0.3525, 0.5123], [0.3808, 0.5534], [0.4035, 0.5656], [0.4154, 0.5754], [0.4410, 0.6112], [0.5212, 1], [0.5307, 1], [0.5512, 1], [0.4351, 0.7158], [0.4463, 0.7222], [0.4706, 0.7456], [0.4718, 0.7379], [0.4823, 0.7439], [0.5050, 0.7654], [0.5231, 0.7718], [0.5326, 0.7770], [0.5531, 0.7958], [0.6172, 1], [0.6248, 1], [0.6413, 1]\}, \{0.4403, 0.4712, 0.4981, 0.4820, 0.5158, 0.5452, 0.5128, 0.5488, 0.5801, 0.5377, 0.5754, 0.6082, 0.5508, 0.5894, 0.6230, 0.6029, 0.6452, 0.6820, 0.6415, 0.6670, 0.7256, 0.6726, 0.7197, 0.7608\} >\}.$

By Threome 3.19 and $\sigma = 2$ we get,

$GCHFHA_2(ch_{kl})_{k,l=1}^n = \psi_k^2 = (((\oplus)_{l=1}^n)_{k=1}^n(\phi_l(ch_{kl}^{o\sim})^2))^{\frac{1}{2}} = \{< \{[(1-\Pi_{k,l=1}^n(1-(\mu_{i_{kl}}^{-o\sim})^2)^{\phi_l})^{\frac{1}{2}}, (1-\Pi_{k,l=1}^n(1-(\mu_{i_{kl}}^{+o\sim})^2)^{\phi_l})^{\frac{1}{2}}]\}, \{1-(1-\Pi_{k,l=1}^n(1-(1-h_{i_{kl}}^{o\sim})^2)^{\phi_l})^{\frac{1}{2}}\} >\},$

$GCHFHA_2(ch_{11}, ch_{12}, ch_{13}, ch_{14}) = \psi_1^2 = ((\oplus)_{l=1}^4)_{k=1}(\phi_l(ch_{kl}^{o\sim})^2))^{\frac{1}{2}} = \{< \{[(1-\Pi_{l=1}^4(1-(\mu_{i_{1l}}^{-o\sim})^2)^{\phi_l})^{\frac{1}{2}}, (1-\Pi_{l=1}^4(1-(\mu_{i_{1l}}^{+o\sim})^2)^{\phi_l})^{\frac{1}{2}}]\}, \{1-(1-\Pi_{l=1}^4(1-(1-h_{i_{1l}}^{o\sim})^2)^{\phi_l})^{\frac{1}{2}}\} >\},$

$GCHFHA_2(ch_{11}, ch_{12}, ch_{13}, ch_{14}) = \psi_1^2 = \{< \{[0.1400, 0.3356], [0.2190, 0.4069], [0.3018, 0.5594], [0.3480, 0.5950], [0.2114, 0.3718], [0.2689, 0.4355], [0.3437, 0.5769], [0.3790, 0.6105], [0.2902, 0.4380], [0.3328, 0.4900], [0.3932, 0.6118], [0.4232, 0.6416]\}, \{0.2695, 0.2900, 0.4402, 0.4806, 0.2856, 0.3078, 0.4718, 0.5171, 0.2915, 0.4703, 0.4836, 0.5309\} >\},$

$GCHFHA_2(ch_{21}, ch_{22}, ch_{23}, ch_{24}) = \psi_2^2 = ((\oplus)_{l=1}^4)_{k=2}(\phi_l(ch_{kl}^{o\sim})^2))^{\frac{1}{2}} = \{< \{[(1-\Pi_{l=1}^4(1-(\mu_{i_{2l}}^{-o\sim})^2)^{\phi_l})^{\frac{1}{2}}, (1-\Pi_{l=1}^4(1-(\mu_{i_{2l}}^{+o\sim})^2)^{\phi_l})^{\frac{1}{2}}]\}, \{1-(1-\Pi_{l=1}^4(1-(1-h_{i_{2l}}^{o\sim})^2)^{\phi_l})^{\frac{1}{2}}\} >\},$

$GCHFHA_2(ch_{21}, ch_{22}, ch_{23}, ch_{24}) = \psi_2^2 = \{< \{[0.2768, 0.4149], [0.2917, 0.4389], [0.3206, 0.4870], [0.3332, 0.5061], [0.4473, 0.5707], [0.4555, 0.5851], [0.5767, 0.7096], [0.5820, 0.7182], [0.3539, 0.4566], [0.3651, 0.4777], [0.3874, 0.5203], [0.3973, 0.5374], [0.4922, 0.5961], [0.4992, 0.6093], [0.6065, 0.7248], [0.6112, 0.7329], [0.4443, 0.6505], [0.4526, 0.6614], [0.4692, 0.6844], [0.4768, 0.6940], [0.5521, 0.7281], [0.5578, 0.7360], [0.6480, 0.8087], [0.6521.0.8140]\}, \{0.3246, 0.3411, 0.4584, 0.4855, 0.5343, 0.5697, 0.5666, 0.6064, 0.3407, 0.3582, 0.4848, 0.5145, 0.5687, 0.6087, 0.6054, 0.6513, 0.3664, 0.3858, 0.5287, 0.5634, 0.6285, 0.6789, 0.6746, 0.7358\} >\},$

$GCHFHA_2(ch_{31}, ch_{32}, ch_{33}, ch_{34}) = \psi_3^2 = ((\oplus)_{l=1}^4)_{k=3}(\phi_l(ch_{kl}^{o\sim})^2))^{\frac{1}{2}} = \{< \{[(1-\Pi_{l=1}^4(1-(\mu_{i_{3l}}^{-o\sim})^2)^{\phi_l})^{\frac{1}{2}}, (1-\Pi_{l=1}^4(1-(\mu_{i_{3l}}^{+o\sim})^2)^{\phi_l})^{\frac{1}{2}}]\}, \{1-(1-\Pi_{l=1}^4(1-(1-h_{i_{3l}}^{o\sim})^2)^{\phi_l})^{\frac{1}{2}}\} >\},$

$GCHFHA_2(ch_{31}, ch_{32}, ch_{33}, ch_{34}) = \psi_3^2 = \{< \{[0.4149, 0.6609], [0.4181, 0.6644], [0.4300, 0.6709], [0.6318, 0.8578], [0.6334, 0.8590], [0.6392, 0.8614], [0.5193, 0.7112], [0.5216, 0.7140], [0.5301, 0.7193], [0.6856, 0.8764], [0.6869, 0.8775], [0.6916, 0.8795], [0.6236, 0.8416], [0.6252, 0.8430], [0.6311, 0.8456], [0.7460, 0.9290], [0.7470, 0.9296], [0.7506, 0.9308], [0.7960, 1], [0.7968, 1], [0.7996, 1], [0.8568, 1], [0.8573, 1], [0.8592, 1]\}, \{0.3502, 0.3589, 0.3612, 0.4346, 0.4466, 0.4496, 0.4626, 0.4758, 0.4791, 0.5970, 0.6180, 0.6236, 0.5336, 0.5504, 0.5548, 0.7202, 0.7546, 0.7642, 0.5528, 0.5707, 0.5754, 0.7596, 0.8020, 0.8140\} >\},$

$GCHFHA_2((ch_{41}, ch_{42}, ch_{43}, ch_{44}) = \psi_4^2 = ((\oplus)_{l=1}^4)_{k=4}(\phi_l(ch_{kl}^{o\sim})^2))^{\frac{1}{2}} = \{< \{[(1-\Pi_{l=1}^4(1-(\mu_{i_{4l}}^{-o\sim})^2)^{\phi_l})^{\frac{1}{2}}, (1-\Pi_{l=1}^4(1-(\mu_{i_{4l}}^{+o\sim})^2)^{\phi_l})^{\frac{1}{2}}]\}, \{1-(1-\Pi_{l=1}^4(1-(1-h_{i_{4l}}^{o\sim})^2)^{\phi_l})^{\frac{1}{2}}\} >\}$,

$GCHFHA_2((ch_{41}, ch_{42}, ch_{43}, ch_{44}) = \psi_4^2 = \{< \{[0.3357, 0.5096], [0.3432, 0.5162], [0.3737, 0.5575], [0.3625, 0.5320], [0.3692, 0.5382], [0.3972, 0.5767], [0.4156, 0.5805], [0.4212, 0.5857], [0.4448, 0.6187], [0.5339, 1], [0.5377, 1], [0.5539, 1], [0.5026, 0.7711], [0.5068, 0.7735], [0.5246, 0.7890], [0.5180, 0.7793], [0.5220, 0.7816], [0.5390, 0.7965], [0.5506, 0.7980], [0.5541, 0.8001], [0.5694, 0.8136], [0.6307, 1], [0.6334, 1], [0.6450, 1]\}, \{0.4166, 0.4496, 0.4710, 0.4399, 0.4757, 0.4992, 0.4506, 0.4778, 0.5124, 0.4543, 0.4920, 0.5196, 0.5344, 0.5847, 0.6192, 0.5694, 0.6269, 0.6673, 0.5862, 0.6474, 0.6915, 0.5921, 0.6547, 0.7000\} >\}$.

Now by using Theorem 3.22 we have,

$GCHFHG_\sigma(ch_{kl})_{k,l=1}^n = \chi_k^\sigma = \frac{1}{\sigma}(((\otimes)_{l=1}^n)_{k=1}^n(\sigma ch_l^{o\sim})^{\phi_l}) = \{< \{[1-(1-\Pi_{k,l=1}^n(1-(1-\mu_{i_{kl}}^{-o\sim})^\sigma)^{\phi_l})^{\frac{1}{\sigma}}, 1-(1-\Pi_{k,l=1}^n(1-(1-\mu_{i_{kl}}^{+o\sim})^\sigma)^{\phi_l})^{\frac{1}{\sigma}}]\}, \{(1-\Pi_{k,l=1}^n(1-(h_{i_{kl}}^{o\sim})^\sigma)^{\phi_l})^{\frac{1}{\sigma}}\} >\}$,

for $\sigma = 1$ we have,

$GCHFHG_1(ch_{11}, ch_{12}, ch_{13}, ch_{14}) = \chi_1^1$
$= ((\oplus)_{l=1}^4)_{k=1}((ch_l^{o\sim})^{\phi_l}) = \{< \{[\Pi_{l=1}^4(\mu_{i_{1l}}^{-o\sim})^{\phi_l}, \Pi_{l=1}^4(\mu_{i_{1l}}^{+o\sim})^{\phi_l}]\}, \{1-\Pi_{l=1}^4(1-h_{i_{1l}}^{o\sim})^{\phi_l}\} >\}$,

$GCHFHG_1(ch_{11}, ch_{12}, ch_{13}, ch_{14}) = \chi_1^1 = \{< \{[0.0866, 0.2759], [0.2096, 0.3967], [0.1082, 0.3157], [0.2620, 0.4540], [0.0999, 0.2946], [0.2420, 0.4236], [0.1249, 0.3371], [0.3024, 0.4848], [0.1057, 0.3096], [0.2559, 0.4453], [0.1321, 0.3543], [0.3198, 0.5096]\}, \{0.4448, 0.6140, 0.5105, 0.6596, 0.4839, 0.6411, 0.5449, 0.6836, 0.5293, 0.6727, 0.5850, 0.7114\} >\}$,

$GCHFHG_1(ch_{21}, ch_{22}, ch_{23}, ch_{24}) = \chi_2^1$
$= ((\oplus)_{l=1}^4)_{k=2}((ch_l^{o\sim})^{\phi_l}) = \{< \{[\Pi_{l=1}^4(\mu_{i_{2l}}^{-o\sim})^{\phi_l}, \Pi_{l=1}^4(\mu_{i_{2l}}^{+o\sim})^{\phi_l}]\}, \{1-\Pi_{l=1}^4(1-h_{i_{2l}}^{o\sim})^{\phi_l}\} >\}$,

$GCHFHG_1(ch_{21}, ch_{22}, ch_{23}, ch_{24}) = \chi_2^1 = \{< \{[0.1930, 0.3526], [0.2073, 0.3796], [0.2005, 0.3600], [0.2154, 0.3876], [0.2057, 0.3730], [0.2210, 0.4016], [0.2525, 0.4390], [0.2713, 0.4727], [0.2623, 0.4482], [0.2819, 0.4826], [0.2691, 0.4644], [0.2891, 0.5000], [0.2970, 0.4735], [0.3191, 0.5098], [0.3086, 0.4834], [0.3316, 0.5205], [0.3166, 0.5009], [0.3402, 0.5393], [0.3160, 0.4983], [0.3396, 0.5365], [0.3284, 0.5087], [0.3529, 0.5477], [0.3368, 0.5271], [0.3620, 0.5675]\}, \{0.5619, 0.5813, 0.5722, 0.5911, 0.6159, 0.6329, 0.5932, 0.6113, 0.6028, 0.6204, 0.6433, 0.6592, 0.6412, 0.6571, 0.6496, 0.6652, 0.6854, 0.6592, 0.7080, 0.7209, 0.7148, 0.7275, 0.7439, 0.7553\} >\}$,

$GCHFHG_1(ch_{31}, ch_{32}, ch_{33}, ch_{34}) = \chi_3^1$
$= ((\oplus)_{l=1}^4)_{k=3}((ch_l^{o\sim})^{\phi_l}) = \{< \{[\Pi_{l=1}^4(\mu_{i_{3l}}^{-o\sim})^{\phi_l}, \Pi_{l=1}^4(\mu_{i_{3l}}^{+o\sim})^{\phi_l}]\}, \{1-\Pi_{l=1}^4(1-h_{i_{3l}}^{o\sim})^{\phi_l}\} >\}$,

$GCHFHG_1(ch_{31}, ch_{32}, ch_{33}, ch_{34}) = \chi_3^1 = \{< \{[0.3303, 0.5623], [0.3837, 0.6113], [0.4126, 0.6386], [0.4265, 0.6664], [0.4954, 0.7245], [0.5328, 0.7568], [0.4037, 0.6091], [0.4690, 0.6623], [0.5043, 0.6918], [0.5213, 0.7219], [0.6056, 0.7849], [0.6512, 0.8199], [0.4290, 0.6463], [0.4983, 0.7027], [0.5359, 0.7340], [0.5539, 0.7660], [0.6435, 0.8328], [0.6920, 0.8699], [0.4442, 0.6504], [0.5160, 0.7072], [0.5549, 0.7387], [0.5736, 0.7709], [0.6663, 0.8381], [0.7166, 0.8755]\}, \{0.4889, 0.5222, 0.5590, 0.6704, 0.6918, 0.7156, 0.5291, 0.5598, 0.5936, 0.6963, 0.7161, 0.7379, 0.6065, 0.6321, 0.6604, 0.7462, 0.7627, 0.7810, 0.7183, 0.7367, 0.7570, 0.8183, 0.8302, 0.8432\} >\}$,

$GCHFHG_1(ch_{41}, ch_{42}, ch_{43}, ch_{44}) = \chi_4^1$

$= ((\oplus)_{l=1}^4)_{k=4}((ch_l^{o\sim})^{\phi_l}) = \{< \{[\Pi_{l=1}^4(\mu_{i_{4l}}^{-o\sim})^{\phi_l}, \Pi_{l=1}^4(\mu_{i_{4l}}^{+o\sim})^{\phi_l}]\}, \{1 - \Pi_{l=1}^4(1 - h_{i_{4l}}^{o\sim})^{\phi_l}\} >\},$

$GCHFHG_1(ch_{41}, ch_{42}, ch_{43}, ch_{44}) = \chi_4^1 = \{< \{[0.2114, 0.3117], [0.2365, 0.3349], [0.2546, 0.3617], [0.2852, 0.4017], [0.3190, 0.4316], [0.3435, 0.4661], [0.3583, 0.4995], [0.4009, 0.5368], [0.4316, 0.5797], [0.4496, 0.7250], [0.5031, 0.7791], [0.5416, 0.8414], [0.2438, 0.3423], [0.2727, 0.3678], [0.2936, 0.3972], [0.3289, 0.4411], [0.3679, 0.4740], [0.3961, 0.5119], [0.4132, 0.5486], [0.4623, 0.5895], [0.4977, 0.6367], [0.5185, 0.7963], [0.5801, 0.8556], [0.6246, 0.9241]\}, \{0.5241, 0.5390, 0.5756, 0.6252, 0.6369, 0.6658, 0.7321, 0.7404, 0.7611, 0.8842, 0.8878, 0.8967, 0.5873, 0.6002, 0.6320, 0.6750, 0.6851, 0.7102, 0.7677, 0.7280, 0.7928, 0.8996, 0.9027, 0.9105\} >\}.$

By Threome 3.22 and $\sigma = 2$ we get,

$GCHFHG_2(ch_{kl})_{k,l=1}^n = \chi_k^2 = \frac{1}{2}(((\otimes)_{l=1}^n)_{k=1}^n(2ch_l^{o\sim})^{\phi_l}) = \{< \{[1-(1-\Pi_{k,l=1}^n(1-(1-\mu_{i_{kl}}^{-o\sim})^2)^{\phi_l})^{\frac{1}{2}}, 1-(1-\Pi_{k,l=1}^n(1-(1-\mu_{i_{kl}}^{+o\sim})^2)^{\phi_l})^{\frac{1}{2}}]\}, \{(1-\Pi_{k,l=1}^n(1-(h_{i_{kl}}^{o\sim})^2)^{\phi_l})^{\frac{1}{2}}\} > \},$

$GCHFHG_2(ch_{11}, ch_{12}, ch_{13}, ch_{14}) = \chi_1^2 = \frac{1}{2}(((\oplus)_{l=1}^4)_{k=1}(2ch_l^{o\sim})^{\phi_l}) = \{< \{[1 - (1 - \Pi_{l=1}^4(1 - (1 - \mu_{i_{1l}}^{-o\sim})^2)^{\phi_l})^{\frac{1}{2}}, 1 - (1 - \Pi_{l=1}^4(1 - (1 - \mu_{i_{1l}}^{+o\sim})^2)^{\phi_l})^{\frac{1}{2}}]\}, \{(1 - \Pi_{l=1}^4(1 - (h_{i_{1l}}^{o\sim})^2)^{\phi_l})^{\frac{1}{2}}\} >\},$

$GCHFHG_2(ch_{11}, ch_{12}, ch_{13}, ch_{14}) = \chi_1^2 = \{< \{[0.0852, 0.2699], [0.2085, 0.3950], [0.1020, 0.2902], [0.2544, 0.4326], [0.0967, 0.5100], [0.2397, 0.4214], [0.1159, 0.3101], [0.2936, 0.4629], [0.1004, 0.2954], [0.2501, 0.4377], [0.1205, 0.3207], [0.3068, 0.4816]\}, \{0.4845, 0.6606, 0.5235, 0.6822, 0.5210, 0.6807, 0.5558, 0.7007, 0.5704, 0.7093, 0.6000, 0.7271\} >\},$

$GCHFHG_2(ch_{21}, ch_{22}, ch_{23}, ch_{24}) = \chi_2^2 = \frac{1}{2}(((\oplus)_{l=1}^4)_{k=2}(2ch_l^{o\sim})^{\phi_l}) = \{< \{[1 - (1 - \Pi_{l=1}^4(1 - (1 - \mu_{i_{2l}}^{-o\sim})^2)^{\phi_l})^{\frac{1}{2}}, 1 - (1 - \Pi_{l=1}^4(1 - (1 - \mu_{i_{2l}}^{+o\sim})^2)^{\phi_l})^{\frac{1}{2}}]\}, \{(1 - \Pi_{l=1}^4(1 - (h_{i_{2l}}^{o\sim})^2)^{\phi_l})^{\frac{1}{2}}\} >\},$

$GCHFHG_2(ch_{21}, ch_{22}, ch_{23}, ch_{24}) = \chi_2^2 = \{< \{[0.1887, 0.3447], [0.2023, 0.3708], [0.2469, 0.4296], [0.2656, 0.4655], [0.2808, 0.4532], [0.3026, 0.4923], [0.2890, 0.4619], [0.3116, 0.5021], [0.1930, 0.3487], [0.2070, 0.3752], [0.2529, 0.4351], [0.2721, 0.4716], [0.2877, 0.4591], [0.3102, 0.4989], [0.2962, 0.4679], [0.3196, 0.5091], [0.1950, 0.3523], [0.2091, 0.3792], [0.2556, 0.4400], [0.2749, 0.4772], [0.2908, 0.4645], [0.3136, 0.5050], [0.2994, 0.4734], [0.3231, 0.5153]\}, \{0.6038, 0.6184, 0.6175, 0.6314, 0.6551, 0.6672, 0.7209, 0.7302, 0.6099, 0.6242, 0.6234, 0.6370, 0.6602, 0.6721, 0.7248, 0.7339, 0.6470, 0.6595, 0.6587, 0.6707, 0.6911, 0.7017, 0.7487, 0.7569\} >\},$

$GCHFHG_2(ch_{31}, ch_{32}, ch_{33}, ch_{34}) = \chi_3^2 = \frac{1}{2}(((\oplus)_{l=1}^4)_{k=3}(2ch_l^{o\sim})^{\phi_l}) = \{< \{[1 - (1 - \Pi_{l=1}^4(1 - (1 - \mu_{i_{3l}}^{-o\sim})^2)^{\phi_l})^{\frac{1}{2}}, 1 - (1 - \Pi_{l=1}^4(1 - (1 - \mu_{i_{3l}}^{+o\sim})^2)^{\phi_l})^{\frac{1}{2}}]\}, \{(1 - \Pi_{l=1}^4(1 - (h_{i_{3l}}^{o\sim})^2)^{\phi_l})^{\frac{1}{2}}\} >\},$

$GCHFHG_2(ch_{31}, ch_{32}, ch_{33}, ch_{34}) = \chi_3^2 = \{< \{[0.3153, 0.5246], [0.3773, 0.5918], [0.4092, 0.6278], [0.3788, 0.5749], [0.4595, 0.6565], [0.5026, 0.7030], [0.3760, 0.5597], [0.4558, 0.6365], [0.4983, 0.6792], [0.4577, 0.6169], [0.5685, 0.7145], [0.6328, 0.7753], [0.3877, 0.5723], [0.4714, 0.6532], [0.5164, 0.6988], [0.4734, 0.6323], [0.5915, 0.7375], [0.6619, 0.8064], [0.3904, 1], [0.4751, 1], [0.5205, 1], [0.4770, 1], [0.5970, 1], [0.6689, 1]\}, \{0.5255, 0.5600, 0.6015, 0.6945, 0.7135, 0.7372, 0.5484, 0.5804, 0.6192, 0.7070, 0.7251, 0.7477, 0.6204, 0.6456, 0.6765, 0.7484, 0.7634, 0.7824, 0.7390, 0.7547, 0.7744, 0.8218, 0.8319, 0.8448\} >\},$

$GCHFHG_2(ch_{41}, ch_{42}, ch_{43}, ch_{44}) = \chi_4^2 = \frac{1}{2}(((\oplus)_{l=1}^4)_{k=4}(2ch_l^{o\sim})^{\phi_l}) = \{< \{[1 - (1 - \Pi_{l=1}^4(1-(1-\mu_{i_{4l}}^{-o\sim})^2)^{\phi_l})^{\frac{1}{2}}, 1-(1-\Pi_{l=1}^4(1-(1-\mu_{i_{4l}}^{+o\sim})^2)^{\phi_l})^{\frac{1}{2}}]\}, \{(1-\Pi_{l=1}^4(1-(h_{i_{4l}}^{o\sim})^2)^{\phi_l})^{\frac{1}{2}}\} >\}$,

$GCHFHG_2(ch_{41}, ch_{42}, ch_{43}, ch_{44}) = \chi_4^2 = \{< \{[0.2025, 0.2892], [0.2277, 0.3117], [0.2432, 0.3318], [0.2765, 0.3801], [0.3132, 0.4127], [0.3361, 0.4422], [0.3476, 0.4773], [0.3973, 0.5239], [0.4292, 0.5678], [0.4256, 1], [0.4929, 1], [0.5379, 1], [0.2224, 0.2994], [0.2505, 0.3229], [0.2679, 0.3440], [0.3054, 0.3947], [0.3471, 0.4291], [0.3734, 0.4605], [0.3866, 0.4979], [0.4446, 0.5481], [0.4824, 0.5961], [0.4782, 1], [0.5603, 1], [0.6180, 1]\}, \{0.5456, 0.5555, 0.5921, 0.6523, 0.6591, 0.6847, 0.7600, 0.7643, 0.7806, 0.9018, 0.9034, 0.9096, 0.5944, 0.6028, 0.6340, 0.6863, 0.6923, 0.7148, 0.7817, 0.7855, 0.8002, 0.9099, 0.9114, 0.9170\} >\}$.

**Step 4**

$T(\psi_1^1) = 0.1108, T(\psi_2^1) = 0.2808, T(\psi_3^1) = 0.5484, T(\psi_4^1) = 0.3940,$
$T(\psi_1^2) = 0.1071, T(\psi_2^2) = 0.3061, T(\psi_3^2) = 0.5453, T(\psi_4^2) = 0.3967,$
$T(\chi_1^1) = 0.0805, T(\chi_2^1) = 0.2022, T(\chi_3^1) = 0.4649, T(\chi_4^1) = 0.3302,$
$T(\chi_1^2) = 0.0924, T(\chi_2^2) = 0.1905, T(\chi_3^2) = 0.4622, T(\chi_4^2) = 0.3403,$
$T(\psi_3^1) > T(\psi_4^1) > T(\psi_2^1) > T(\psi_1^1),$
$T(\psi_3^2) > T(\psi_4^2) > T(\psi_2^2) > T(\psi_1^2),$
$T(\chi_3^1) > T(\chi_4^1) > T(\chi_2^1) > T(\chi_1^1),$
$T(\chi_3^2) > T(\chi_4^2) > T(\chi_2^2) > T(\chi_1^2),$
so we conclude that $c_3 > c_4 > c_2 > c_1$ and hence $c_3$ is the best choice.

## 6. Conclusion

In our work we defined aggregation operators for cubic hesitant fuzzy sets (CHFSs) which includes generalized cubic hesitant fuzzy averaging (geometric) operator, cubic hesitant fuzzy ordered weighted averaging (geometric) operator, generalized cubic hesitant fuzzy ordered weighted averaging (geometric) operator, cubic hesitant fuzzy hybrid averaging (geometric) operator, cubic hesitant fuzzy arithmetical averaging (geometric) operator, generalized cubic hesitant fuzzy hybrid averaging (geometric) operator. We also solved a multi criteria decision making argument by applying generalized cubic hesitant fuzzy hybrid averaging (geometric) operator. In future we shall apply CHFSs in pattern recognition, medical diagnosis and also define distance measures for CHFSs.

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