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Application of Bernstein Polynomials for Solving Linear Volterra Integro-Differential Equations with Convolution Kernels

Tahira Batool
Department of Mathematics and Statistics,
University of Lahore, Pakistan.
Email: tahira5914@gmail.com

Muhammad Ozair Ahmad
Department of Mathematics and Statistics,
University of Lahore, Pakistan.

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Abstract. This paper deals with a new application of Bernstein polynomials to find approximate solution of linear Volterra Integro-differential equation of a special kind. For this purpose, we first need to convert multiple integral into single integral. Since we have taken kernel of convolution type so we will use convolution product. By using properties of Bernstein polynomials integral equation is reduced into an algebraic equation. The set of algebraic equation is then solved and approximate solution is obtained. Some numerical solutions are also presented to confirm the reliability and applicability of the proposed method.

AMS (MOS) Subject Classification Codes: 35S29; 40S70; 25U09
Key Words: Linear volterra integral equations, Integro differential equations, Bernstein polynomials, Operational matrices.

1. Introduction

Integro-differential equations widely appear in the phenomena of many physical sciences, mathematical physics, electronics, biology and many engineering processes [5, 7] in which related rate of change of various quantities is involved. There are many methods to find exact and approximate solutions for Integro-differential equations such as, Laplace Transform [4] and Combined Laplace Transform-Successive Approximations Method [1]. In this paper we have discussed a new approach to find approximate solution of Linear Volterra Integro-Differential equation via Bernstein Polynomials. In this kind of equation unknown function g(u) appears under the integral sign on one side whereas its derivative of any order appears on the other side. We observe that the derivative of an unknown function can be solved by integrating the integral equation so the multiple integral appears. We investigate the result of converting a multiple integral into single integral. Bernstein polynomials have many useful properties to solve such type of integral equations to obtain algebraic equations. We have also discussed Rational Bernstein Collocation Method [8] and New Operational Matrix [9].

We consider the Volterra Integro-Differential equation
$\frac{d^n}{dt^n} f(t) = g(t) + \lambda \int_0^t k(t,w) f(w) dw.$ (1)

In this equation 'n' is the order of the derivative of an unknown function f (t), while g (t), the kernel k(t, w) and a constant parameter $'\lambda'$ are known. The kernel k(t, w) specifies the classification of integral equation. We focus on the kernel of convolution type $k(t,w) = (t-w)^z$ where $z \in Z^+$, the degree of the kernel.

The literature is very dense on the application of polynomials. In addition to their computer friendly properties, it is convenient to work on the algorithms based on Bernstein polynomials. In this paper we make an effort to resolve Volterra Integro-Differential equation into an algebraic equation to get approximate solution for (1).

Section 2 is based on the review of some basic properties of Bernstein polynomials and solution of multiple integrals. In section 3, method to solve Volterra Integro-Differential equations with Bernstein polynomials is introduced. In section 4, numerical examples are given. The graphs and the error tables are also provided. In section 5, we conclude and discuss the applicability and reliability of the proposed method.

2. Bernstein Polynomials and their properties

Definition 2.1: The Bernstein polynomials of q^{th} degree are defined as,

$$B_{p,q}(x) = \binom{p}{q} x^p (1-x)^{q-p} \qquad ; p = 0, 1, 2, \ldots, q$$

where the binomial coefficients are obtained by $\binom{p}{q} = \frac{q!}{p!(q-p)!}$.

Bernstein polynomials have many useful properties. We are going to discuss here some properties which will use in this paper.

- The Bernstein polynomials of q^{th} degree $B_{p,q}(x) = \{0 \le p \le q, q \ge 0\}$ form a complete system in $L^2[0,1]$ with inner product.

$$< g, h >= \int_0^1 g(x)h(x)dx,$$

  and the associated norm $\|g\| =< g, g >^{1/2}$.

- They form a partition of unity that is :

$$\sum_{p=0}^{q} B_{p,q}(x) = \sum_{p=0}^{q-1} B_{p,q-1}(x).$$

- They can be written in terms of power basis. $\{1, x, x^2, \ldots, x^q\}$, also

$$B_{p,q}(x) = \sum_{r=p}^{q} (-1)^{r-p} \binom{p}{r} \binom{r}{p} x^r.$$

- Power basis function can be written in terms of Bernstein polynomials

$$x^p = \sum_{r=p}^{q} \frac{\binom{r}{p}}{\binom{q}{p}} B_{r,q}(x).$$

3. APPROXIMATION

Theorem 3.1: If S is a closed subspace of a Hilbert space H, then for any $h \in H$, the unique approximation exists.
Proof: The proof can be found in [2].
As, H is a Hilbert space , assume that

$H = L^2[0,1]$ and spanning set $S = \{B_{0,q}, B_{0,1}, B_{0,2}, \ldots, B_{q,q}\}$ then for any $h \in H$,

$$h(x) \approx h_0(x) = \sum_{l=0}^{q} u_l B_{l,q}(x) = u_q^T(x),$$

$u = [u_1, u_2, \ldots, u_q]^T$, we need to calculate u

$$< h, \phi >= \int_0^1 h(x)\phi^T(x)dx,$$
$$< \phi, \phi >= \int_0^1 \phi_q(x)\phi_q^T(x)dx = Q(say), \qquad (2)$$

where, 'Q' is a square matrix of order $(q+1) \times (q+1)$.

Lemma 3.2: Suppose that $h \in C^{m+1}[0,1]$ and $S = span\{B_{0,q}, B_{1,q}, \ldots, B_{q,q}\}$. If $u^T B$ is the best approximation of h in S. then,

$$\|h - u^T B\|_{L^2[0,1]} \leq \frac{max|h^{(m+1)}(x)|}{(q+1)!\sqrt{2q+3}}.$$

Proof: The proof can be found in [3].
Which shows that the error reduces as $q \to \infty$ (q is the degree of Bernstein Polynomial).

Lemma 3.3: [6] Formula for converting a multiple integral into a single ordinary integral:
We have a formula for converting a multiple integral into single integral,

$$\int_a^t y(x)dx^n = \int_a^t \frac{(t-x)^{(n-1)}}{(n-1)!} y(x)dx, \qquad (3)$$

where 'n' is the order of the integral.

4. SOLUTION OF LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATION

We let
$\phi_q(x) = [B_{0,q}, B_{1,q}, \ldots, B_{q,q}]^T$ and $T_q(x) = [1, x, x^2, \ldots, x^q]^T \qquad (4)$

so, we can write $\phi_q(x) = AT_q(x)$, where A is an upper triangular matrix obtained by taking coefficients of Bernstein basis.

We consider the linear Volterra integro-differential equation of the type

$$\frac{d^n}{dt^n} f(t) = g(t) + \lambda \int_0^t k(t,w) f(w) dw.$$

with kernel $K(t,w) = (t-w)^z$ thus we have,

$$\frac{d^n}{dt^n} f(t) = g(t) + \lambda \int_0^t (t-w)^z f(w) dw. \qquad (5)$$

where 'n' is the order of the derivative, f (t) is an unknown function, while g (t) and constant parameter $'\lambda'$ and 'z' the degree of kernel are known.
Since we have the formula to convert multiple integral into single integral,

$$f(t) = G(t) + \lambda \int_0^t (t-w)^{n'} f(w) dw, \qquad (6)$$

where $n' = z + n - 1$.
Now let $f(t) \approx u^T \phi_q(t)$ and $G(t) \approx v^T \phi_q(t)$. (7)
By substituting (7) in (6), we get

$$u^T \phi_q(t) = v^T \phi_q(t) + \lambda \int_0^t (t-w)^{n'} u^T \phi_q(w) dw. \qquad (8)$$

To transform (8) into algebraic equation, we consider

$$\int_0^t (t-w)^{n'} \phi_q(w) dw \approx F \phi_q(t),$$

where, 'F' is the operational matrix of order $(q+1) \times (q+1)$.
Since, the kernel is of convolution type so, we will use convolution product and write it as

$$\int_0^t (t-w)^{n'} \phi_q(w) dw \approx t^{n'} * \phi_q(t),$$

where, $'*'$ is the convolution product. And

$$\begin{aligned} t^{n'} * \phi_q(t) &= t^{n'} * [B_{0,q}, B_{1,q}, \ldots, B_{q,q}]^T \\ &= [t^{n'} * B_{0,q}, t^{n'} * B_{1,q}, \ldots, t^{n'} * B_{q,q}]^T \\ &= t^{n'} * (AT_q(t)) \\ &= A(t^{n'} * T_q(t)), \qquad (9) \end{aligned}$$

where, T_q and A are defined.

$$\begin{aligned} t^{n'} * T_q(t) &= [t^{n'} * 1, t^{n'} * t, \ldots, t^{n'} * t^n]^T \\ &= D\overline{T}, \qquad (10) \end{aligned}$$

where 'D' is the square matrix of order $(q+1) \times (q+1)$ with entries,

$$D = \begin{cases} \frac{q!}{(l+1)(l+2)\cdots(l+q+1)} & l = m \\ 0 & otherwise \end{cases} \qquad (11)$$

for l=m=0, 1, 2,..., q.
Now,
We need to approximate $T \approx E\phi_q(t)$, where 'E' is the square matrix of order $(q+1) \times$

$(q+1)$ and $\overline{T} = [t^{n'+1}, t^{n'+2}, \ldots, t^{n'+q+1}]$.
It can be calculated using Bernstein polynomials as,

$$\begin{aligned} t^{n'+1} &= E_1^T \phi_q(t) \\ t^{n'+2} &= E_2^T \phi_q(t) \\ &. \\ &. \\ &. \\ t^{n'+q+1} &= E_{q+1}^T \phi_q(t). \end{aligned}$$

So,

$$\begin{gathered} t^{n'+i} = E_i^T \phi_q(t)\ ; i = 1, 2, 3, \ldots, q+1, \\ E^T \phi_i = < t^{n'+1}.\phi_q(t) > < \phi_q(t).\phi_q(t) >^{-1} \\ E^T \phi_i = (\int_0^1 t^{n'+i}.\phi_q(t)^T dt).Q^{-1}, \qquad (12) \end{gathered}$$

and we have, $E = [E_1, E_2, E_3, \ldots, E_{q+1}]^T$.
So, operational matrix 'F' can be calculated in this manner.
F = ADE. (13)
Now, at the end, we are ready to transform integral equation into algebraic equation. Combining (3) to (13)

$$\begin{gathered} u^T \phi_q(t) = v^T \phi_q(t) + \lambda F u^T \phi_q(t) \\ u^T \phi_q(t)(I - \lambda F) = v^T \phi_q(t) \\ u^T = v^T (I - \lambda F)^{-1}. \qquad (14) \end{gathered}$$

And the algebraic equation is evaluated from $f(t) \approx u^T \phi_q(t)$.
This is the approximated solution for the assumed integral equation.

5. Numerical Examples

Example 1: Consider the following linear Volterra Integro differential equation.

$$f(t)' = e^t - \int_0^t (t-w) f(w) dw; \qquad f(0) = 1.$$

Integrating both sides from 0 to t, and using formula to convert multiple integral into single integral, we have

$$f(t) = e^t - \int_0^t (t-w)^2 f(w) dw.$$

Here $G(t) =, e^t \lambda = 1, k(t,w) = (t-w)\ and\ n' = 2$.
We first take $q = 3$, so we use Bernstein polynomials of degree 3 to approximate function. Let $f(t) \approx u^T \phi_q(t)$ and $G(t) \approx v^T \phi_q(t) = e^t$.

$$A = \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad Q = \begin{bmatrix} 1/7 & 1/14 & 1/35 & 1/140 \\ 1/14 & 3/35 & 9/140 & 1/35 \\ 1/35 & 9/140 & 3/35 & 1/14 \\ 1/140 & 1/35 & 1/14 & 1/7 \end{bmatrix}$$

$$D = \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/12 & 0 & 0 \\ 0 & 0 & 1/30 & 0 \\ 0 & 0 & 0 & 1/60 \end{bmatrix}$$

$$F = \begin{bmatrix} 1/840 & -1/2520 & 37/1260 & 211/1260 \\ 1/2520 & -1/840 & -1/280 & 127/1260 \\ -1/1260 & 1/210 & -13/840 & 25/504 \\ -1/1260 & 1/252 & -23/2520 & 13/840 \end{bmatrix}$$

$u^T = [0.9991, 1.3385, 1.8183, 2.7172]$ and $v^T = [0.9969, 1.3504, 1.7847, 3.1577]$.

Thus, the approximated solution for the assumed integral equation is:
$f(t) = 0.9969 + 1.0605t + 0.2424t^2 + 0.8579t^3$.

Now, we take $q = 5$, the 5^{th} order Bernstein polynomials to find approximated solution,

$$A = \begin{bmatrix} 1 & -5 & 10 & -10 & 5 & -1 \\ 0 & 5 & -20 & -20 & -20 & 5 \\ 0 & 0 & 10 & 10 & 30 & -10 \\ 0 & 0 & 0 & 0 & -20 & 10 \\ 0 & 0 & 0 & 0 & 5 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/105 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/168 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/11 & 1/22 & 2/99 & 1/132 & 1/462 & 1/2772 \\ 1/22 & 5/99 & 5/132 & 5/231 & 25/2772 & 1/462 \\ 2/99 & 5/132 & 10/231 & 25/693 & 5/231 & 1/22 \\ 1/132 & 5/231 & 5/132 & 10/231 & 5/132 & 2/99 \\ 1/462 & 25/2772 & 5/99 & 5/132 & 5/99 & 1/22 \\ 1/2772 & 1/462 & 1/22 & 2/99 & 1/22 & 1/11 \end{bmatrix}$$

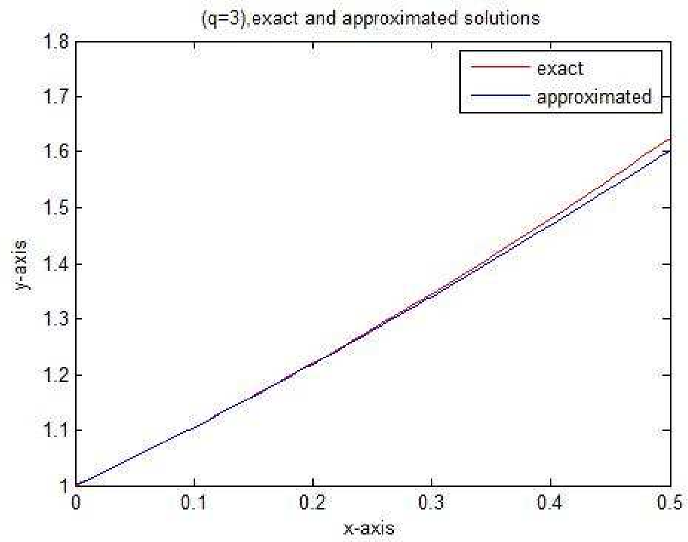
$$F = \begin{bmatrix} -145/3171 & 781/6850 & -921/6103 & 771/5401 & 65/2983 & 534/4025 \\ 281/6151 & -1975/17371 & 491/3283 & -29/278 & 219/2498 & 433/5305 \\ -2/188249 & 64/564929 & -15/91202 & -48/23509 & 52/2361 & 424/7133 \\ 7/143714 & -36/81995 & 64/36851 & -61/14251 & 95/13861 & 128/3579 \\ 4/85485 & -1/3509 & 19/26820 & -17/25406 & -16/8571 & 146/8139 \\ -1/24024 & 1/3432 & -67/72072 & 43/22754 & -29/9240 & 117/19841 \end{bmatrix}$$

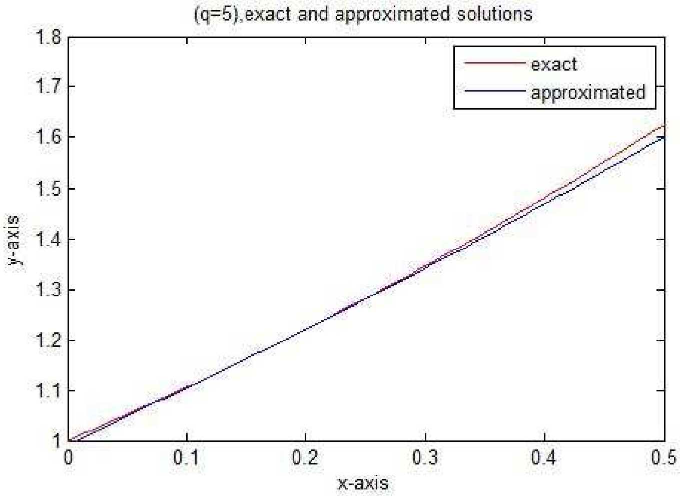
$u^T = [1, 1.2, 1.4499, 1.7668, 2.1746, 2.7183]$
and $v^T = [1.0009, 1.1804, 1.4759, 1.7813, 2.3305, 3.1602]$.

Thus, the approximated solution for the assumed integral equation is:
$f(t) = 0.2359t^5 - 0.8815t^4 + 0.8870t^3 - 0.0760t^2 + 1.1305t + 1.0009.$

TABLE 1. Comparison of exact and approximated solutions.

t	Exact Solution	Approximated sol for (q=3)	Approximated sol for (q=5)	Error for (q=3)	Error for (q=5)
0	1	0.9969000	1.0009	0.0031000	0.000900
0.1	1.1050000	1.1062319	1.1054871	0.0012319	0.000487
0.2	1.2200001	1.2255592	1.2231539	0.0055591	0.003154
0.3	1.3450011	1.3600293	1.3505441	0.0150282	0.005543
0.4	1.4800060	1.5147896	1.4961590	0.0347836	0.016153
0.5	1.6250233	1.6949876	1.6345594	0.0699643	0.009536





Example2: Consider the following linear Volterra Integro differential equation.

$$f(t)'' = t + \lambda \int_0^t (t-w)f(w)dw; \qquad f(0) = 0, f'(0) = 1$$

Integrating both sides from 0 to t, and using formula to convert multiple integral into single integral, we have

$$f(t) = (t + \frac{t^3}{6}) + \frac{1}{2}\int_0^t (t-w)^3 f(w)dw.$$

Here $G(t) = t + t^3/6, \lambda = 1/2, k(t,w) = (t-w)\ and\ n' = 3.$

We first take $q = 3$, so we use Bernstein polynomials of degree 3 to approximate function.

Let $f(t) \approx u^T\phi_q(t)$ and $G(t) \approx v^T\phi_q(t) = t + \frac{t^3}{6}$

$$A = \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad Q = \begin{bmatrix} 1/7 & 1/14 & 1/35 & 1/140 \\ 1/14 & 3/35 & 9/140 & 1/35 \\ 1/35 & 9/140 & 3/35 & 1/14 \\ 1/140 & 1/35 & 1/14 & 1/7 \end{bmatrix}$$

$$D = \begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/20 & 0 & 0 \\ 0 & 0 & 1/60 & 0 \\ 0 & 0 & 0 & 1/140 \end{bmatrix}$$

$$F = \begin{bmatrix} -1/1320 & 1/198 & -43/1980 & 47/330 \\ -1/770 & 23/3080 & -53/2310 & 65/924 \\ -1/924 & 13/2310 & -43/3080 & 3/110 \\ -1/2310 & 29/13860 & -31/13860 & 59/9240 \end{bmatrix}$$

$u^T = [-0.0008, 0.3377, 0.6556, 1.1912]\ and\ v^T = [0, 1/3, 2/3, 7/6].$

Thus, the approximated solution for the assumed integral equation is:
$f(t) = -0.0008 + 1.0155t - 0.0618t^2 + 0.2383t^3$.

Now, we take $q = 5$, the 5^{th} order Bernstein polynomials to find approximated solution,

$$A = \begin{bmatrix} 1 & -5 & 10 & -10 & 5 & -1 \\ 0 & 5 & -20 & -20 & -20 & 5 \\ 0 & 0 & 10 & 10 & 30 & -10 \\ 0 & 0 & 0 & 0 & -20 & 10 \\ 0 & 0 & 0 & 0 & 5 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/60 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/140 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/280 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/504 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/11 & 1/22 & 2/99 & 1/132 & 1/462 & 1/2772 \\ 1/22 & 5/99 & 5/132 & 5/231 & 25/2772 & 1/462 \\ 2/99 & 5/132 & 10/231 & 25/693 & 5/231 & 1/22 \\ 1/132 & 5/231 & 5/132 & 10/231 & 5/132 & 2/99 \\ 1/462 & 25/2772 & 5/99 & 5/132 & 5/99 & 1/22 \\ 1/2772 & 1/462 & 1/22 & 2/99 & 1/22 & 1/11 \end{bmatrix}$$

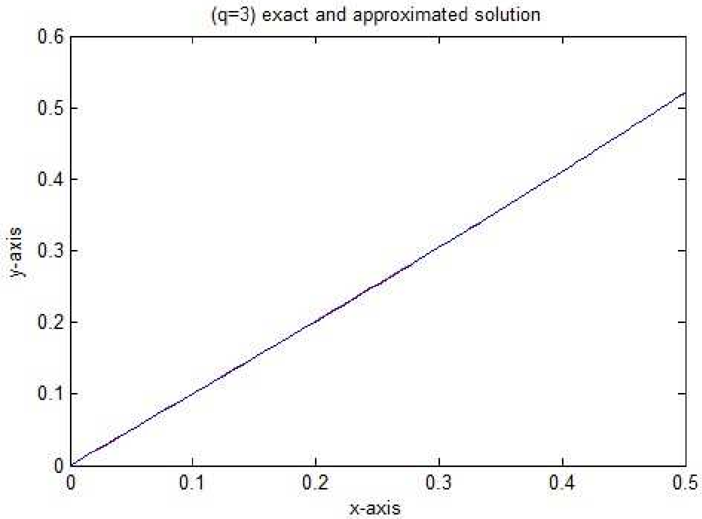
$$F = \begin{bmatrix} 1230/44839 & -8/1243 & 387/4279 & -379/4279 & 219/3437 & 574/5389 \\ 3/129029 & -17/131373 & 70/134679 & -46/18105 & 167/10413 & 541/7793 \\ 9/489811 & -50/227017 & 31/31406 & -53/20616 & 24/6427 & 17/428 \\ 6/169013 & -9/42371 & 25/47308 & -47/96127 & -7/3712 & 43/2163 \\ -1/721072 & 1/9009 & -31/72072 & 19/16380 & -34/12095 & 1/126 \\ -1/45045 & 1/6552 & -17/36036 & 5/5544 & -17/12870 & 63/3229 \end{bmatrix}$$

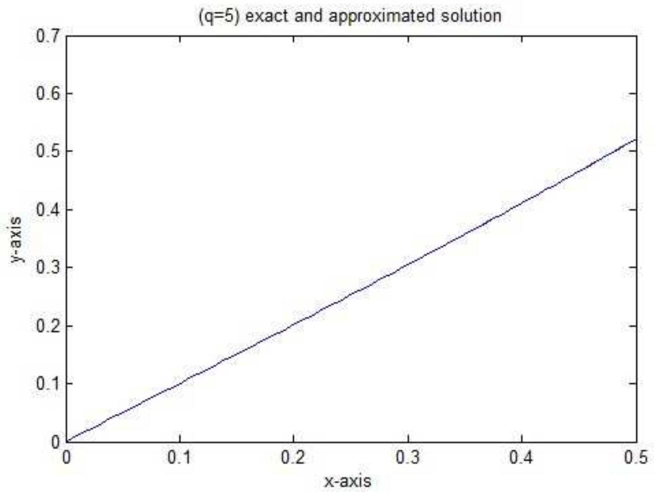
$u^T = [0, 0.2, 0.399, 0.6168, 0.8664, 1.1923]$
and $v^T = [0, 0.2, 0.4, 0.6167, 0.8667, 1.1667]$.

Thus, the approximated solution for the assumed integral equation is:
$f(t) = 0.0293t^5 - 0.007t^4 + 0.171t^3 - 0.001t^2 + t$.

TABLE 2. Comparison of exact and approximated solutions.

t	Exact Solution	Approximated sol for (q=3)	Approximated sol for (q=5)	Error for (q=3)	Error for (q=5)
0	0	-0.0008000	0	0.0008	0
0.1	0.1001668	0.1003703	0.1001606	0.0002036	0.0000616
0.2	0.2013360	0.2017344	0.2013269	0.0003984	0.00009827
0.3	0.3045203	0.3047221	0.3045415	0.0002018	0.00002121
0.4	0.4107523	0.4107632	0.4109048	1.0000109	0.00015231
0.5	0.5210953	0.5212875	0.5216031	0.0001912	0.0000508





6. Conclusion

In this paper, we have discussed a new technique to find approximate solution for linear Volterra Integro-differential equation using Bernstein polynomials. In this method we get good approximation which depends on the order of Bernstein polynomials and the error reduces as $q \to \infty$. We make use the formula for converting multiple integral into single ordinary integral. Two examples are considered with exact solutions which are obtained by Laplace transformation and the approximate solutions are obtained by using Bernstein polynomials. Comparison of exact and approximated solutions are given and error columns are also provided for better understanding and result evaluation.

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