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MHD Flow of Burgers' Fluid under the Effect of Pressure Gradient Through a Porous Material Pipe

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Abstract. In this research article, we will find the velocity for time dependent pressure gradient which may increases, decreases or pulsate with respect to time. The MHD flow of a Burgers' fluid with porous medium in circular channel is take into account for the study. We derived the governing equation for the analytical solutions of this problem. The solution for the velocity field are give in the form of Bessel and modified Bessel function of zero order by using the modified Darcy's law and the resistance of the porous medium. The behaviour of other physical, magnetic and permeability parameters is observed by assuming constant value of pressure gradient. The graphical depict and possible special cases are also discussed.

AMS (MOS) Subject Classification Codes: (2010), 76A05; 76S05; 33C10
Key Words: Burgers' fluids; Porous medium; Velocity field; Pressure gradient; Bessel function.

1. Introduction

The Magnetohydrodynamics(MHD) is the combination of two major fields i.e. fluid mechanics and electrodynamics. The MHD flow has considerable interest in different field of sciences. In 19^{th} centaury it becomes more important for the researchers because of the fabrication of electromagnetic pump by Hartman [16] use to make the connection between jet-wave therapeutic. In electromagnetic pump "Lorentz force" arises while shifting between the AC current into the magnetic field varying with respect to time.

The feasible application of MHD flow is that its allow to convert the heat into electronic energy for which MHD power generator is used. The MHD flow has wide application in form of rotating cylinders in the field of physical sciences such as astrophysics and geophysics. Hayat et al. [17] considered the two problems of an Oldroyd-B fluid model over an infinite oscillating plate in the presence of conducting heat and electricity when the whole system is rotating normal to the oscillatory plate. The modified Darcy's law is used to find out the behaviour of oscillatory flow of Burger's fluid in porous medium by Hayat et al. [18] and Khan et al. [20]. The limiting cases and effect of physical parameter by graphical illustrations are also discussed. Ellahi [9] derived the analytical solution for velocity field, temperature and nano concentration with the help of dimensionless technique and Homotopy analysis method (HAM). He took the viscosity model for MHD flow of nano fluid in pipe depending on two different temperature i.e. temperature of pipe is greater than temperature of pipe. The MHD flow of third grade fluid with variable viscosity is used to obtained the analytical solution of velocity and temperature by Riaz et al. [10]. Again the Homotopy analysis method (HAM) is used for this purpose and the impact of parameters shown by graphs. Haq [15] analyzed the MHD squeezed nano fluid flowing over a sensor surface. They embedded the three different type of nano particles in the base fluid and concluded that the water fractionalized copper nano particle provides better heat convection than other mixtures, under consideration, through numerical simulation.

Safdar et al. [27] derived the expressions for velocity and stress for unsteady flow of incompressible Burgers' fluid in rotating circular domain with fractional derivatives. The finite Laplace and Hankel transformations with generalized Lorenzo Hartley function are used for this purpose. Park [23] considered the circular cylinder with partially porous wall to find out the wave forces and run up wave along with wave conditions that all depending on the hydrodynamic property. Darcy's resistance law is used to depict the porosity on the surface. To analyze the hydrodynamic control, they divided the surface into three different portions and used eigenvalue expansion method for results. Awan et al. [4] discussed about the chemical reaction of incompressible fluid over an infinite vertical plate with oscillation. They found the expression for velocity depending upon temperature and also discussed the behaviour of different physical parameters in results. Pop [24] wrote a report on the flow passing through a circular cylinder and saturated in a porous medium with Brinkman model. They find out the exact solution from governing equations leading to the result of velocity behaviour. The importance of circular domain in MHD flow, MHD flow in curved channel and accelerated periodic body can be seen in [12, 11, 31, 32]. Furthermore, the interested reader can consult the references [1, 2, 3, 5, 6, 7, 8, 13, 19, 21, 25, 26, 28, 29, 30].

In medical sciences the pulsating flow going through the arteries has great importance therefore it attracted the researcher towards them for a log time. In normal conditions the working of heart is to circulate blood in arteries through its natural pumping system due to this pressure gradient arises in arteries. These results motivated us to work on the analytical solution of velocity filed for MHD flow of Buergers' fluid passing through the porous medium under the influence of pressure gradient including pulsating case. A circular magnetic field is applied perpendicular to the flow direction. The obtained velocity profile is used to predict the behaviour of different physical parameters and it concluded that the velocity and permeability parameter of porous medium has direct relation where as magnetic parameter behaved inversely to the velocity. Also, the frequency parameter have oscillatory behaviour with respect to time and other physical parameters. The obtained results are appreciably influenced on these parameters so discussion of their behaviour is more important in the problem.

2. Notations and Preliminaries

ρ	density
$\mathbf{v}$	velocity
$\mathbf{T}$	extra stress tensor
p	pressure
$\mathbf{R}$	Darcy's resistance
$\mathbf{J}$	current density
$\mathbf{B}$	magnetic field
$\mathbf{J}\times\mathbf{B}$	Lorentz force due to magnetic field
μ	dynamic viscosity
λ_1, λ_2	relaxation parameters
λ_3	retardation parameter

3. Formulation of basic equations

The linear momentum equation and equation of continuity for porous medium of incompressible fluid and unsteady flow are,

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}.\nabla\mathbf{v}\right)=-\nabla p+\nabla.\mathbf{T}+\mathbf{J}\times\mathbf{B}+\mathbf{R}, \tag{3. 1}$$

$$\nabla.\mathbf{v}=0. \tag{3. 2}$$

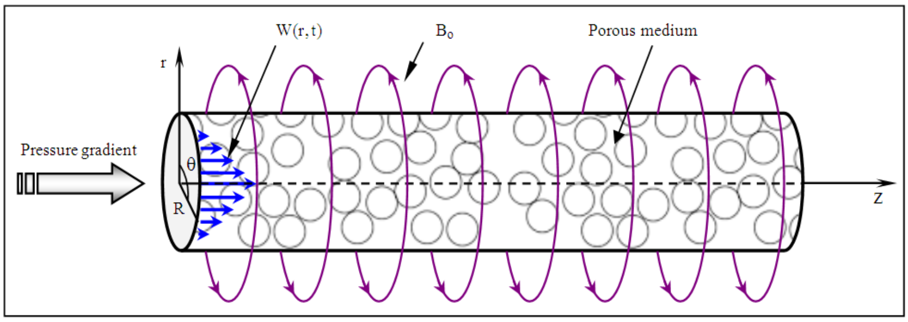

FIGURE 1. Geometry of the problem

We are dealing with Burgers' fluid model of viscoelastic fluids. The properties of Burgers' fluid can be specified by following constants μ, λ_1, λ_2 and λ_3. The constitutive equation of extra stress tensor $\mathbf{T}$ for Burgers' fluid in the absence of hydrostatic pressure is as follow

$$\mathbf{T}+(\lambda_1+\lambda_2\overset{\triangle}{\mathbf{T}})\overset{\triangle}{\mathbf{T}}=(\mathbf{A}+\lambda_3\overset{\triangle}{\mathbf{A}})\mu, \tag{3. 3}$$

$\mathbf{A}$ denotes the rate of deformation tensor and $\overset{\triangle}{\mathbf{A}}$, $\overset{\triangle}{\mathbf{T}}$ are upper convected derivative of deformation and extra stress tensor, defined as

$$\overset{\triangle}{\mathbf{T}} = \frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v}.\triangledown)\mathbf{T} - \mathbf{LT} - (\mathbf{LT})^{\top}, \tag{3. 4}$$

where $\mathbf{L}$ and $\mathbf{A}$ can be defined as;

$$\mathbf{L} = \triangledown \mathbf{v} \quad , \quad \mathbf{A} = (\mathbf{L} + \mathbf{L}^{\top}), \tag{3. 5}$$

$\top$ denotes the matrix transposition.
So equation (3.3) becomes

$$\mathbf{T} + \left(\lambda_1 + \lambda_2 \left(\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v}.\triangledown)\mathbf{T} - \mathbf{LT} - (\mathbf{LT})^{\top}\right)\right)\left(\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v}.\triangledown)\mathbf{T} - \mathbf{LT} - (\mathbf{LT})^{\top}\right)$$
$$= \left[\lambda_3 \left(\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}.\triangledown)\mathbf{A} - \mathbf{LA} - (\mathbf{LA})^{\top}\right) + \mathbf{A}\right]\mu. \tag{3. 6}$$

For $\lambda_1 = \lambda_2 = \lambda_3 = 0$ the equation (3.6) reduced to viscous Newtonian fluid , $\lambda_2 = \lambda_3 = 0$ leads to Maxwell fluid and for Oldroyd-B fluid substitute $\lambda_2 = 0;\ 0 < \lambda_3 < \lambda_1$ in equation (3.6).

The Lorentz force in terms of magnetic field defined as

$$\mathbf{J} \times \mathbf{B} = -\sigma B_o^2 \mathbf{v}, \tag{3. 7}$$

where B_o is the magnetic field and σ is the electric conductivity.

The Darcy's resistance law convinced the consequent relation for Burgers' fluid

$$\left(1 + \left(\lambda_1 + \lambda_2 \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right)\mathbf{R} = \frac{-\mu\phi}{k}\left(1 + \lambda_3 \frac{\partial}{\partial t}\right)\mathbf{v}, \tag{3. 8}$$

where ϕ is the constant of porosity and k denotes the permeability of the porous medium. For this problem we assume that

$$\mathbf{v} = [0, 0, w(r,t)] \ , \ \mathbf{B} = [0, B_o, 0] \tag{3. 9}$$

and

$$\mathbf{R} = [0, 0, R_z] \ , \ \mathbf{T} = \begin{bmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{bmatrix}. \tag{3. 10}$$

The continuity equation for velocity given in equation (3.9) will be definitely satisfied. Now, the equation (3.7) and (3.8) implies that

$$\mathbf{J} \times \mathbf{B} = [0, 0, -\sigma B_0^2 w], \tag{3. 11}$$

$$\left(1 + \left(\lambda_1 + \lambda_2 \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right)\mathbf{R}_z = \frac{-\mu\phi}{k}\left(1 + \lambda_3 \frac{\partial}{\partial t}\right) w. \tag{3. 12}$$

Therefore the equation (3.1) in component form can be written as

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\tau_{rr} = 0, \tag{3. 13}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\tau_{rz} - \frac{\partial p}{\partial z} - \sigma B_0^2 w + R_z = \rho \frac{\partial w}{\partial t}. \tag{3. 14}$$

Also, the equation (3.6) resolved into components as

$$\tau_{rr} + \left(\lambda_1 + \lambda_2 \frac{\partial}{\partial t}\right)\left(\frac{\partial \tau_{rr}}{\partial t} - 2\frac{\partial w}{\partial r}\tau_{rz}\right) = -2\mu\lambda_3 \left(\frac{\partial w}{\partial r}\right)^2, \tag{3. 15}$$

$$\tau_{r\theta} + \left(\lambda_1 + \lambda_2 \frac{\partial}{\partial t}\right)\left(\frac{\partial \tau_{r\theta}}{\partial t} - \frac{\partial w}{\partial r}\tau_{\theta z}\right) = 0, \tag{3. 16}$$

$$\tau_{rz} + \left(\lambda_1 + \lambda_2 \frac{\partial}{\partial t}\right)\left(\frac{\partial \tau_{rz}}{\partial t} - \frac{\partial w}{\partial r}\tau_{zz}\right) = \mu \left(\frac{\partial w}{\partial r} + \lambda_3 \frac{\partial^2 w}{\partial r \partial t}\right), \tag{3. 17}$$

$$\tau_{\theta\theta} + \left(\lambda_1 + \lambda_2 \frac{\partial}{\partial t}\right)\left(\frac{\partial \tau_{\theta\theta}}{\partial t}\right) = 0, \tag{3. 18}$$

$$\tau_{\theta z} + \left(\lambda_1 + \lambda_2 \frac{\partial}{\partial t}\right)\left(\frac{\partial \tau_{\theta z}}{\partial t}\right) = 0, \tag{3. 19}$$

$$\tau_{zz} + \left(\lambda_1 + \lambda_2 \frac{\partial}{\partial t}\right)\left(\frac{\partial \tau_{zz}}{\partial t}\right) = 0. \tag{3. 20}$$

As we observed, by solving the equation (3.13) gives

$$\tau_{rr} = \frac{f(t)}{r},$$

where $f(t)$ is an arbitrary function of time. Where as solving equations (3.16), (3.18), (3.19) and (3.20) gives output

$$\tau_{r\theta} = \tau_{\theta\theta} = \tau_{\theta z} = \tau_{zz} = \frac{-g(r)(1+\lambda_1)e^t}{\lambda_2},$$

here g is arbitrary function with respect to radius of cylinder "r''. It can be notice that these stress tensor components are showing negative behaviour while; $r, t, \lambda_1, \lambda_2 \geq 0$. So, the reduced system of equations is as follows:

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\tau_{rz} - \frac{\partial p}{\partial z} - \sigma B_0^2 w + R_z = \rho \frac{\partial w}{\partial t}, \tag{3. 21}$$

$$\tau_{rz} + \left(\lambda_1 + \lambda_2 \frac{\partial}{\partial t}\right)\frac{\partial \tau_{rz}}{\partial t} = \mu \left[\frac{\partial w}{\partial r} + \lambda_3 \frac{\partial^2 w}{\partial r \partial t}\right]. \tag{3. 22}$$

4. Formulation of the problem

Here, we study the electrically conducting Burgers' fluid flowing through the circular cylindrical domain with porous medium. We consider that the radius of the cylinder is R and we are taking the flow in the Z-direction only where magnetic field is applied perpendicular to z-axis. We also assuming that the motion is due to pressure gradient. By neglecting the induced magnetic field and eliminating the τ_{rz} in equation (3.21) and (3.22),

$$\left(1+\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\right)\frac{\partial w}{\partial t}-\frac{\mu}{\rho}\left(1+\lambda_3\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)$$
$$+\frac{\sigma B_0^2}{\rho}\left(1+\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right)w-\frac{1}{\rho}\left(1+\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right)R_z$$
$$=\frac{-1}{\rho}\left(1+\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial z}. \tag{4. 23}$$

Using equation (3.12) into equation (4.23)

$$\left(1+\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\right)\frac{\partial w}{\partial t}-\frac{\mu}{\rho}\left(1+\lambda_3\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)$$
$$+\frac{\sigma B_0^2}{\rho}\left(1+\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right)w+\frac{\mu\phi}{\rho k}\left(1+\lambda_3\frac{\partial}{\partial t}\right)w$$
$$=\frac{-1}{\rho}\left(1+\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial z}, \tag{4. 24}$$

and appropriate boundary conditions are

$$w(R,t)=0 \quad , \quad \frac{\partial w(0,t)}{\partial r}=0. \tag{4. 25}$$

We defined the following dimensionless variables

$$\overset{\star}{r}=\frac{r}{R} \quad , \quad \overset{\star}{t}=\frac{\mu t}{\rho R^2} \quad , \quad \overset{\star}{w}=\left(\frac{\mu L}{\triangle p R^2}\right)w,$$
$$\overset{\star}{\lambda}_1=\frac{\lambda_1\mu}{\rho R^2} \ , \ \overset{\star}{\lambda}_2=\left(\frac{\mu}{\rho R^2}\right)^2\lambda_2 \ \ and \ \ \overset{\star}{\lambda}_3=\frac{\mu\lambda_3}{\rho R^2}, \tag{4. 26}$$

putting equation (4.26) into governing equation (3.2) and boundary conditions. For the sake of simplicity we drop the sign " $\star$ " here

$$\left(1+\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right)\left(\frac{\partial w}{\partial t}+Mw\right)-\left(1+\lambda_3\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)$$
$$+\frac{1}{K}\left(1+\lambda_3\frac{\partial}{\partial t}\right)w=\left(1+\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right)\psi(t), \tag{4. 27}$$

where

$$M=\frac{\sigma B_0^2 R^2}{\mu} \ , \quad \frac{1}{K}=\frac{\psi R^2}{k} \ , \quad \psi(t)=\frac{-L}{\triangle\, p}\frac{\partial p}{\partial z}, \tag{4. 28}$$

also, the boundary conditions becomes as

$$w(1,t)=0 \quad and \quad \frac{\partial w(0,t)}{\partial r}=0. \tag{4. 29}$$

In equation (4.28) co-efficient M is the magnetic parameter and K is the permeability parameter. When $M=0$, magnetic forces are absent there. When M increases the magnetic forces increases. Similarly, for $K=0$ we have solid cylinder and for $K\to\infty$ we get hollow cylinder. Equation (4.27) has different behaviour with respect to the types of pressure gradient and boundary conditions.

5. Solution of the problem

Case 1: Increasing pressure gradient:-

Let

$$\psi(t) = P_0 e^{\alpha t} \tag{5. 30}$$

and

$$w(r,t) = F(r)e^{\alpha t}. \tag{5. 31}$$

In equation (5.30) and (5.31) P_0 and α are constants. Equation (4.27) in terms of equation (5.30) and (5.31) is

$$\begin{aligned}&\left(1+\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right)\left(\frac{\partial F(r)e^{\alpha t}}{\partial t}+MF(r)e^{\alpha t}\right)\\ &-\left(1+\lambda_3\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 F(r)e^{\alpha t}}{\partial r^2}+\frac{1}{r}\frac{\partial F(r)e^{\alpha t}}{\partial r}\right)+\frac{1}{K}\left(1+\lambda_3\frac{\partial}{\partial t}\right)F(r)e^{\alpha t}\\ &=\left(1+\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right)P_0e^{\alpha t}.\end{aligned} \tag{5. 32}$$

After solving (5.32) we get the following differential equation

$$F^{''}(r)+\frac{1}{r}F^{'}(r)-\left[\frac{(\alpha+M)(1+\alpha(\lambda_1+\alpha\lambda_2))}{(1+\alpha\lambda_3)}+\frac{1}{K}\right]F(r)=\frac{-(1+\alpha(\lambda_1+\alpha\lambda_2))P_0}{(1+\alpha\lambda_3)}, \tag{5. 33}$$

and boundary conditions are

$$F(1)=F^{'}(0)=0. \tag{5. 34}$$

Therefore

$$F(r)=\frac{(1+\alpha(\lambda_1+\alpha\lambda_2))P_0}{(1+\alpha\lambda_3)\beta^2}\left[1-\frac{I_0(\beta r)}{I_0(\beta)}\right]. \tag{5. 35}$$

In above equation $I_0(.)$ represents the zero order modified Bessel function and

$$\beta=\sqrt{\frac{(\alpha+M)(1+\alpha(\lambda_1+\alpha\lambda_2))}{(1+\alpha\lambda_3)}+\frac{1}{K}}. \tag{5. 36}$$

So the velocity field is given by

$$w(r,t)=\frac{(1+\alpha(\lambda_1+\alpha\lambda_2))P_0}{(1+\alpha\lambda_3)\beta^2}\left[1-\frac{I_0(\beta r)}{I_0(\beta)}\right]e^{\alpha t}. \tag{5. 37}$$

Case 2: Decreasing pressure gradient:-

In this case, let

$$\psi(t) = P_0 e^{-\alpha t} \tag{5. 38}$$

and also assume that

$$w(r,t) = G(r)e^{-\alpha t}. \tag{5. 39}$$

In equation (5.38) and (5.39) P_0 and α are constants. Equation (4.27) in terms of equation (5.38) and (5.39) is

$$\left(1+\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right)\left(\frac{\partial G(r)e^{-\alpha t}}{\partial t}+MG(r)e^{-\alpha t}\right)$$
$$-\left(1+\lambda_3\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 G(r)e^{-\alpha t}}{\partial r^2}+\frac{1}{r}\frac{\partial G(r)e^{-\alpha t}}{\partial r}\right)$$
$$+\frac{1}{K}\left(1+\lambda_3\frac{\partial}{\partial t}\right)G(r)e^{-\alpha t}=\left(1+\left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\right)P_0e^{-\alpha t}. \quad (5.\ 40)$$

Equation (5.40) gives

$$G^{''}(r)+\frac{1}{r}G^{'}(r)-\left[\frac{(\alpha-M)(1-\alpha(\lambda_1-\alpha\lambda_2))}{(1-\alpha\lambda_3)}-\frac{1}{K}\right]G(r$$
$$=-\frac{(1-\alpha(\lambda_1-\alpha\lambda_2))P_0}{(1-\alpha\lambda_3)}, \quad (5.\ 41)$$

with boundary conditions

$$G(1)=G^{'}(0)=0. \quad (5.\ 42)$$

Therefore

$$G(r)=\frac{-(1-\alpha(\lambda_1-\alpha\lambda_2))P_0}{(1-\alpha\lambda_3)\beta_1^2}\left[1-\frac{J_0(\beta_1 r)}{J_0(\beta_1)}\right], \quad (5.\ 43)$$

where $J_0(.)$ is the zero order modified Bessel function and

$$\beta_1=\sqrt{\frac{(\alpha-M)(1-\alpha(\lambda_1-\alpha\lambda_2))}{(1-\alpha\lambda_3)}-\frac{1}{K}}. \quad (5.\ 44)$$

So the associated velocity field is

$$w(r,t)=\frac{-(1-\alpha(\lambda_1-\alpha\lambda_2))P_0}{(1-\alpha\lambda_3)\beta_1^2}\left[1-\frac{J_0(\beta_1 r)}{J_0(\beta_1)}\right]e^{-\alpha t}. \quad (5.\ 45)$$

Case 3: Pulsating pressure gradient:-

We calculate the periodic pressure gradient due to cosine pulses in order to solve Eq. (4.27) subject to the boundary condition (4.29), we assume that the function $\psi(t)$ has oscillation with frequency ω and amplitude P_0 i.e.

$$\psi(t)=P_0 cos\omega t=\Re(P_0e^{\iota\omega t}). \quad (5.\ 46)$$

For this let the velocity function is of the form;

$$w(r,t)=\Re(H(r)e^{\iota\omega t}), \quad (5.\ 47)$$

boundary conditions

$$H(1)=H^{'}(0)=0. \quad (5.\ 48)$$

By putting (5.46), (5.47) into equation (4.27) and using (5.48),

$$H(r)=\frac{(1+\iota\omega(\lambda_1+\iota\omega\lambda_2))P_0}{(1+\iota\omega\lambda_3)\beta_2^2}\left[1-\frac{I_0(\beta_2 r)}{I_0(\beta_2)}\right], \quad (5.\ 49)$$

where

$$\beta_2 = \pm\sqrt{\frac{(\iota\omega + M)(1 + \iota\omega(\lambda_1 + \iota\omega\lambda_2))}{(1 + \iota\omega\lambda_3)} + \frac{1}{K}}. \tag{5. 50}$$

Hence the velocity field is

$$w(r,t) = \Re\left(\frac{(1 + \iota\omega(\lambda_1 + \iota\omega\lambda_2))P_0}{(1 + \iota\omega\lambda_3)\beta_2^2}\left[1 - \frac{I_0(\beta_2 r)}{I_0(\beta_2)}\right]e^{\iota\omega t}\right). \tag{5. 51}$$

6. Results and discussions

The analytical solution of unsteady MHD flow of Burgers' fluid for velocity field is available in this research work. We assumed that the Burgers' fluid is passing through the circular domain which have porous walls. For different values of physical parameters, we calculated the velocity profile having variations with respect to r and t in Eqs. (37), (45) and (51). These variations can be seen from Figs. 2 to 19 for $P_0 = 1$ and for fixed values of parameters $\lambda_1, \lambda_2, \lambda_3, M, K, \alpha$ and ω. The effects of relaxation parameters λ_1

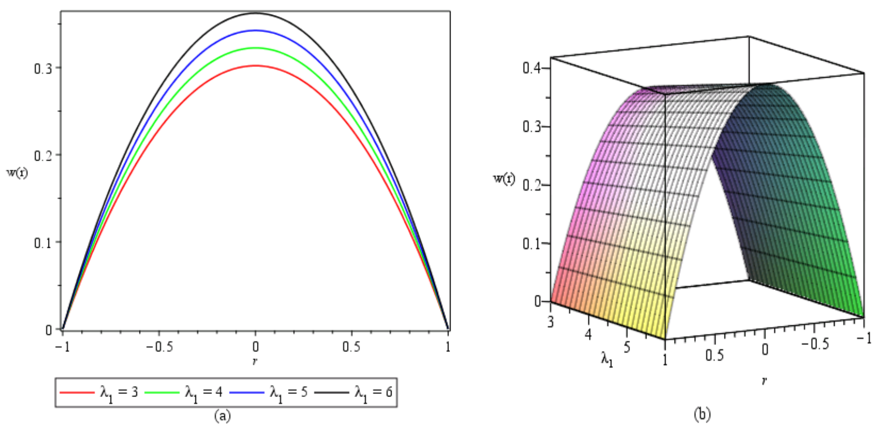

FIGURE 2. Velocity profile of increasing pressure gradient for different values of λ_1 and fixed values for $t = 3, \lambda_2 = 2, \lambda_3 = 1, M = 0.5, K = 10$ and $\alpha = 0.1$

is illustrated in Figs. 2, 8 and 14. We observed that the velocity is increasing with respect to λ_1 in case(1) (Increasing pressure gradient) and case(2) (Decreasing pressure gradient) where as velocity is decreasing with respect to λ_1 in case(3) (Periodic pressure gradient).
Similarly for non-Newtonian parameter λ_2, the velocity increases in case(1) and case(2) but decreases in case(3) as shown in Figs. 3, 9 and 15 for fix values of $\lambda_1, \lambda_3, M, K, \alpha$ and ω.
The influence of retardation parameters λ_3 is presented in Figs. 4, 10 and 16. The velocity field decreases in case(1), (2) and it increases in case(3) for fixed values of $\lambda_1, \lambda_2, M, K, \alpha$ and ω.

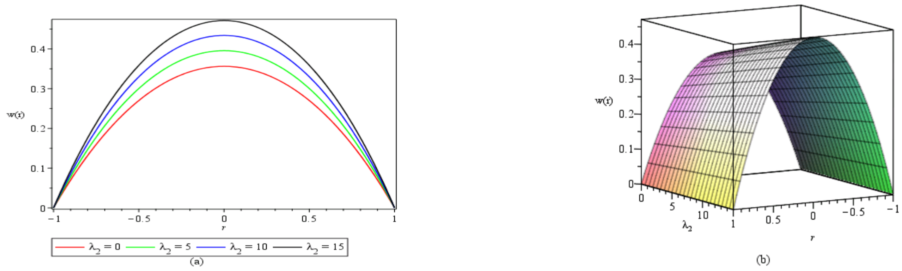

FIGURE 3. Velocity profile of increasing pressure gradient for different values of λ_2 and fixed values for $t = 3, \lambda_1 = 3, \lambda_3 = 4, M = 0.5, K = 10$ and $\alpha = 0.2$

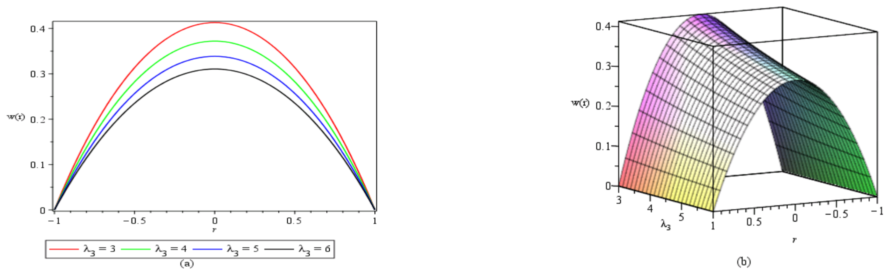

FIGURE 4. Velocity profile of increasing pressure gradient for different values of λ_3 and fixed values for $t = 3, \lambda_1 = 3, \lambda_2 = 2, M = 0.5, K = 10$ and $\alpha = 0.2$

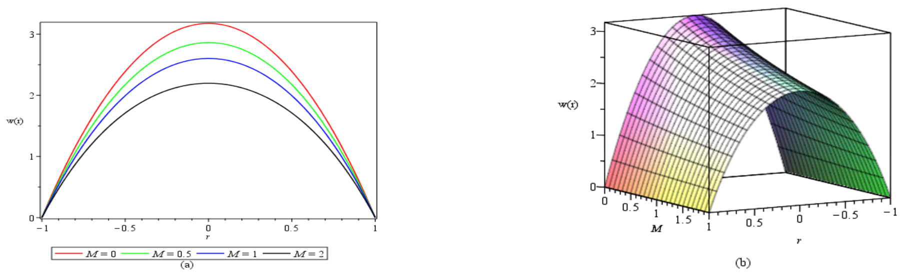

FIGURE 5. Velocity profile of increasing pressure gradient for different values of M and fixed values for $t = 24, \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1, K = 10$ and $\alpha = 0.1$

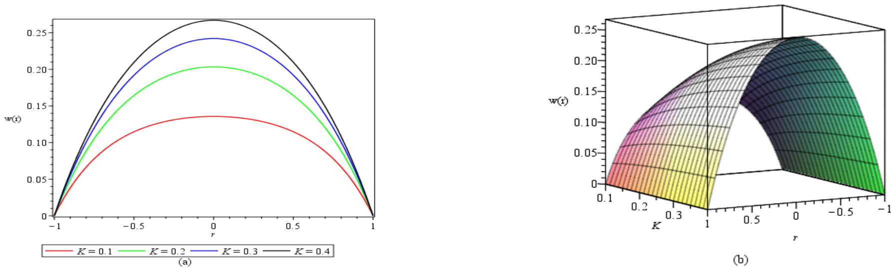

FIGURE 6. Velocity profile of increasing pressure gradient for different values of K and fixed values for $t = 3, \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 0.2, M = 0.5$ and $\alpha = 0.1$

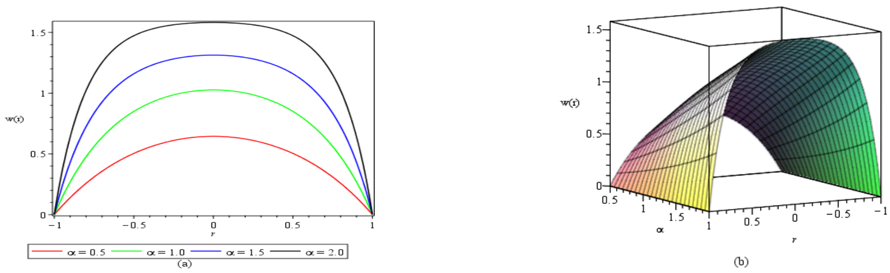

FIGURE 7. Velocity profile of increasing pressure gradient for different values of α and fixed values for $t = 0.7, \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 0.1, M = 0.5$ and $K = 10$

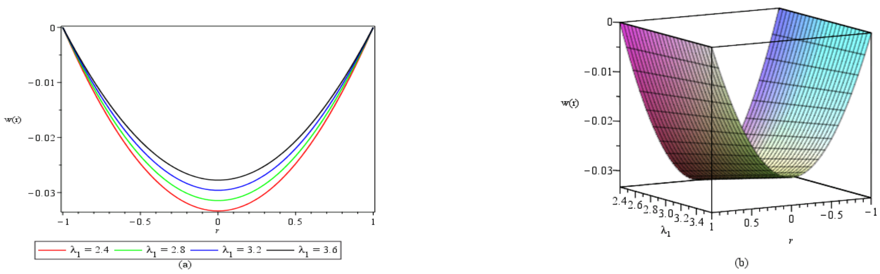

FIGURE 8. Velocity profile of decreasing pressure gradient for different values of λ_1 and fixed values for $t = 24, \lambda_2 = 2, \lambda_3 = 4, M = 0.5, K = 10$ and $\alpha = 0.1$

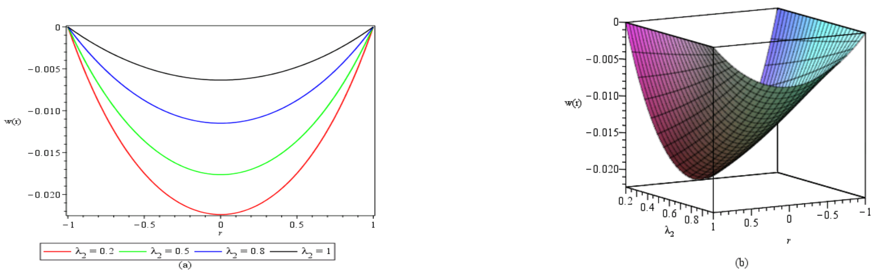

FIGURE 9. Velocity profile of decreasing pressure gradient for different values of λ_2 and fixed values for $t = 1, \lambda_1 = 3.3, \lambda_3 = 2, M = 0.5, K = 10$ and $\alpha = 2.4$

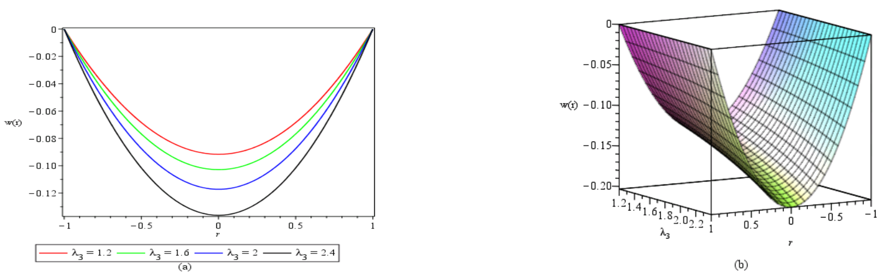

FIGURE 10. Velocity profile of decreasing pressure gradient for different values of λ_3 and fixed values for $t = 3, \lambda_1 = 3, \lambda_2 = 2, M = 0.5, K = 10$ and $\alpha = 0.2$

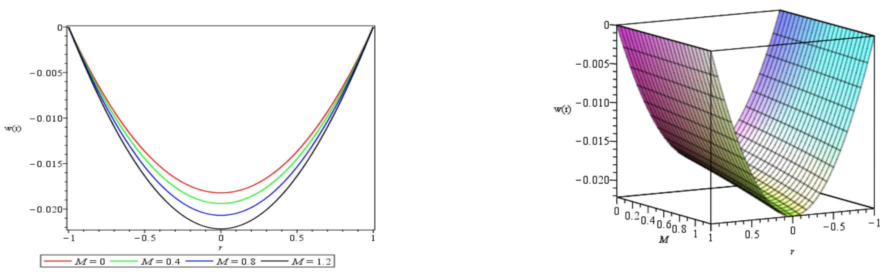

FIGURE 11. Velocity profile of decreasing pressure gradient for different values of M and fixed values for $t = 24, \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1, K = 10$ and $\alpha = 0.1$

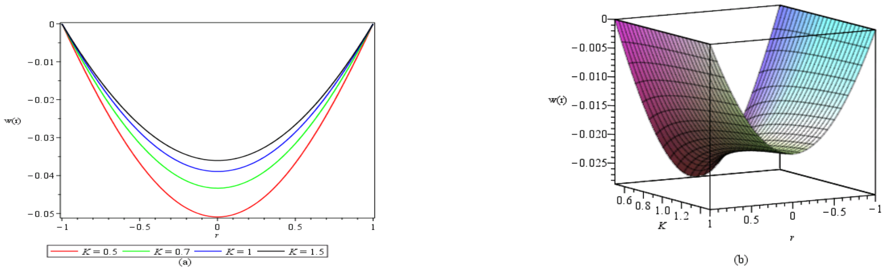

FIGURE 12. Velocity profile of decreasing pressure gradient for different values of K and fixed values for $t = 24, \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 0.2, M = 0.5$ and $\alpha = 0.1$

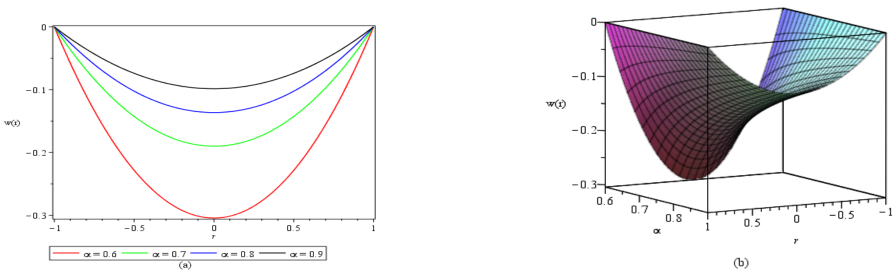

FIGURE 13. Velocity profile of decreasing pressure gradient for different values of α and fixed values for $t = 0.1, \lambda_1 = 3.3, \lambda_2 = 2, \lambda_3 = 2, M = 0.5$ and $K = 10$

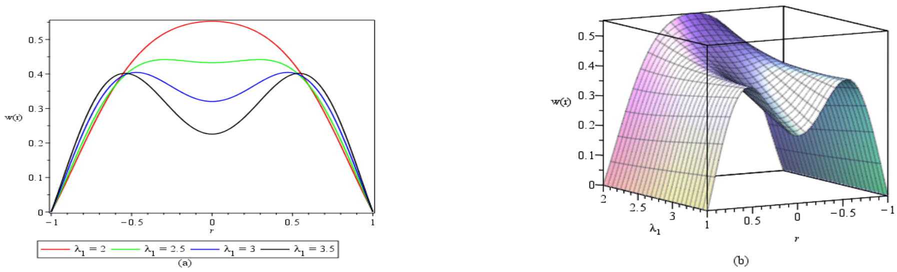

FIGURE 14. Velocity profile of pulsating pressure gradient for different values of λ_1 and fixed values for $t = 1/4, \lambda_2 = 0.15, \lambda_3 = 0.25, M = 0.5, K = 8$ and $\omega = \pi$

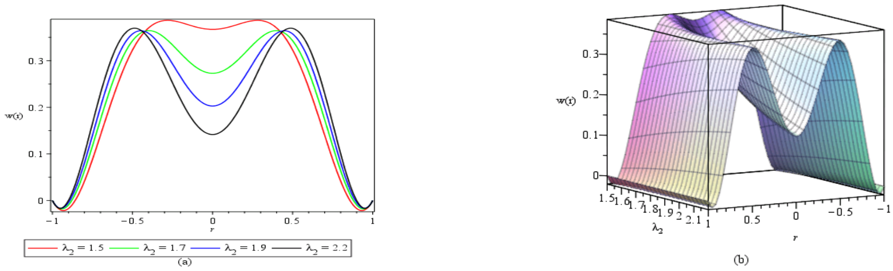

FIGURE 15. Velocity profile of pulsating pressure gradient for different values of λ_2 and fixed values for $t = 1/4, \lambda_1 = 0.15, \lambda_3 = 0.25, M = 0.5, K = 8$ and $\omega = \pi$

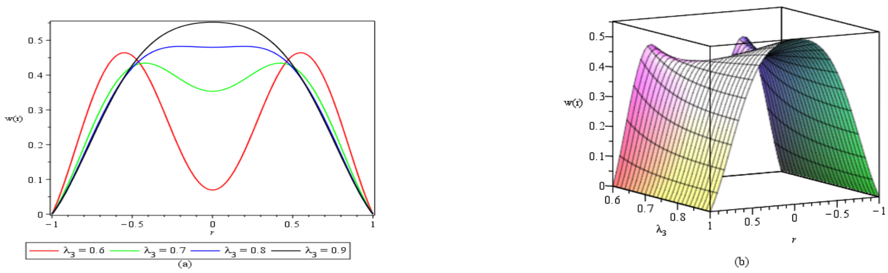

FIGURE 16. Velocity profile of pulsating pressure gradient for different values of λ_3 and fixed values for $t = 1/4, \lambda_1 = 3.5, \lambda_2 = 1.6, M = 0.5, K = 8$ and $\omega = \pi$

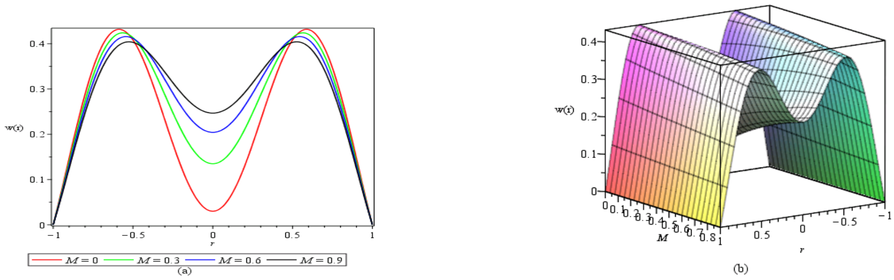

FIGURE 17. Velocity profile of pulsating pressure gradient for different values of M and fixed values for $t = 1/4, \lambda_1 = 3.3, \lambda_2 = 0.15, \lambda_3 = 0.2, K = 8$ and $\omega = \pi$

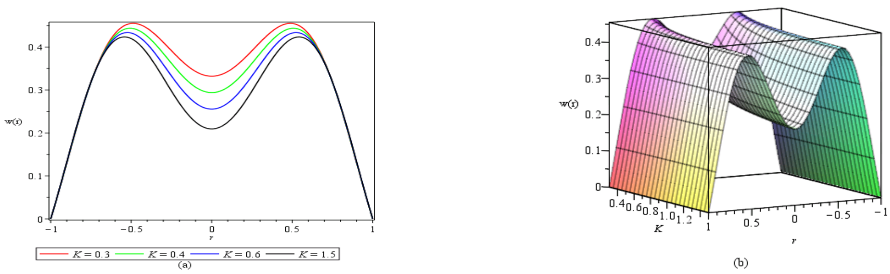

FIGURE 18. Velocity profile of pulsating pressure gradient for different values of K and fixed values for $t = 1/4, \lambda_1 = 3.3, \lambda_2 = 0.15, \lambda_3 = 0.2, M = 0.5$ and $\omega = \pi$

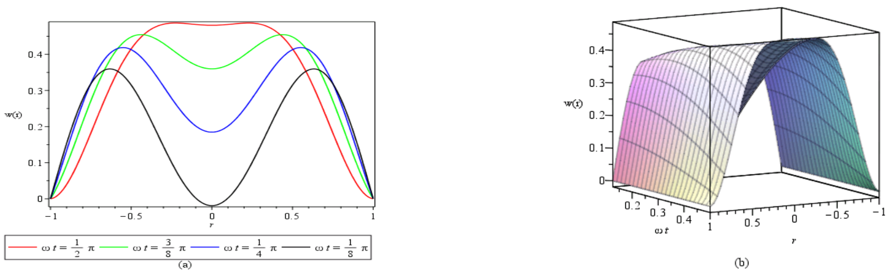

FIGURE 19. Velocity profile of pulsating pressure gradient for different values of ωt and fixed values for $\lambda_1 = 3.3, \lambda_2 = 0.15, \lambda_3 = 0.2, M = 0.5$ and $K = 8$

We display the influence of magnetic parameter M on velocity field in Figs. $5, 11$ and 17 for fixed values of other parameters. It can be noticed that the velocity profile is decreasing in case(1) and (2) while it is increasing in case(3). Which shows that the magnetic force and effect of porous medium is inversely proportional to velocity. The permeability parameter K has direct proportion with velocity for fixed value of other parameters as shown in Figs. $6, 12$ and 18.

For the known constant α velocity profile is given in Figs. 7 and 13. The velocity filed has directly proportional behaviour with respect to this constant for fixed values of $\lambda_1, \lambda_2, \lambda_3, M$ and K. In Fig. 19, velocity showing the oscillatory behaviour with respect to the oscillation parameter ω for fixed values of $\lambda_1, \lambda_2, \lambda_3, M$ and K.
Finally, the illustration of non-Newtonian parameters i.e. relaxation parameters λ_1, λ_2 and retardation parameters λ_3 is given in Figs. 20 and 21. These figures showing that we have more retardation factor in our porous cylindrical domain than the relaxation factor.

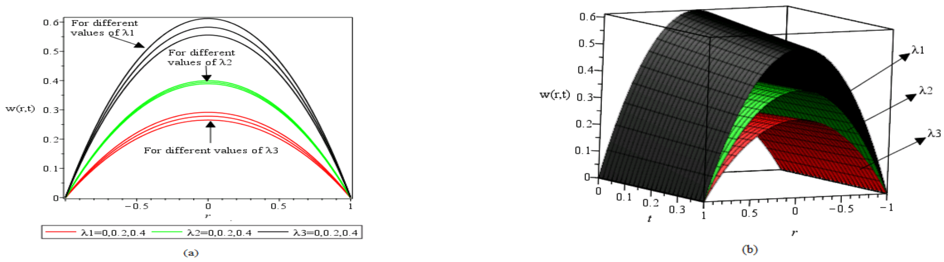

FIGURE 20. Comparison velocity profile of increasing pressure gradient for different values of $\lambda_1, \lambda_2, \lambda_3$ respectively and taking fixed values as $\lambda_1 = 0.5, \lambda_2 = 0.5, \lambda_3 = 0.5, M = 0.5, K = 10$ and $\alpha = 0.3$

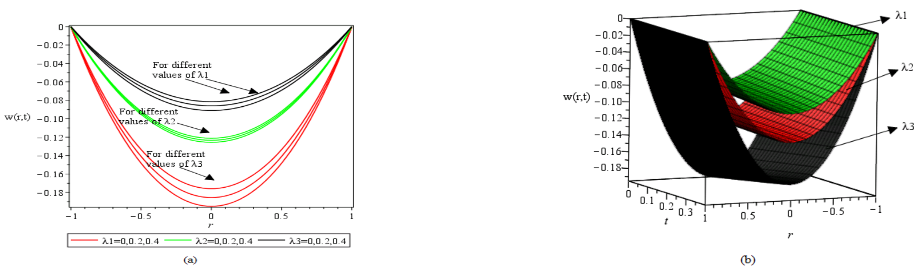

FIGURE 21. Comparison velocity profile of decreasing pressure gradient for different values of $\lambda_1, \lambda_2, \lambda_3$ respectively and taking fixed values as $\lambda_1 = 0.5, \lambda_2 = 0.5, \lambda_3 = 0.5, M = 0.5, K = 10$ and $\alpha = 0.3$

7. CONCLUSION

We have following concluding remarks:

(1) : The behaviour of solution is depending on β, where $\beta = \sqrt{\frac{(\alpha \pm M)(1 \pm \alpha(\lambda_1 \pm \alpha\lambda_2))}{(1 \pm \alpha\lambda_3)} \pm \frac{1}{K}}$. In this formula the positive sign leads to increasing pressure gradient(case(1)) and negative sign leads to decreasing pressure gradient (case(2)).

(2) : As solution is depending upon the parameter β where as β is depending on parameters $\lambda_1, \lambda_2, \lambda_3, M, K$ and α. So these physical parameters are also important for solution.

(3) : When $M = 0$ and $1/K \to 0$, we have the same solution for MHD Burgers' fluid with $\beta = \sqrt{\frac{\alpha(1 \pm \alpha(\lambda_1 \pm \alpha\lambda_2))}{(1 \pm \alpha\lambda_3)}}$.

(4) : For $\lambda_2 = 0$ we have coincide results for MHD Oldroyd-B fluid as given in Eq. (33), (40) and (47) in [14].

(5) : For $\lambda_2 = 0$ and $\lambda_3 = 0$, the resulting solution will be for MHD Maxwell fluid [14]. Which showing the consistency in results.

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