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On the Concept of Off-Diagonal Generalized Sensitivity Functions and Their Relations to the Parameter Estimates and Correlation

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Abstract. Generalized sensitivity functions describe the effects of the changes in the true values of a parameter over its estimates. In this paper, we define the off-diagonal generalized sensitivity functions which describe the effects of the changes in the true values of one parameter over the estimates of the other parameter. Their relations with the estimates of the parameters and the correlation amongst the parameters have been developed theoretically and verified with the help of the logistic growth population model. Changes in the parameter estimates are reflected in the changes in their generalized and off-diagonal generalized sensitivity functions.

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Key Words: Off-diagonal Generalized Sensitivity Functions.Generalized Sensitivity Matrix. Parameter Estimates. Correlation Coefficients

1. Introduction

The idea of the Generalized Sensitivity Functions (GSFs) was given by Carl Cobelli and Karl Thomaseth [13] in 1999. The GSFs describe the effects of the changes in the parameter estimates due to the changes in the true values of the parameters. F. Kappel et al., used the GSFs in studying the cardiovascular model [4]. The GSFs essentially provide two types of information. First, they show the correlation between parameters with respect to the measurements. Oscillatory and monotonic behavior of the GSFs indicates a strong correlation between the parameters. A more or less monotonic increase of the GSFs from 0 to 1 indicate little correlation between parameters. Second, the GSFs provide information carried by the measurements at different times for the parameters. If the GSFs increase monotonically, then those measurements taken in that time interval where they essentially increase from 0 to 1, provide all the information on the parameter whereas the measurements taken outside that interval are more or less irrelevant.

Keck et al., used the GSFs to study the size-structured population models and extended their idea to the Partial Differential Equations(PDEs) [7]. Troparevsky et al., [15] identified the regions on the head having the highest electrostatic potential during Electroencephalography (EEG). Banks et al., used them to study the nonlinear delay differential equations [3]. F. Kappel and M. Munir [6] developed the idea of the GSFs for a multiple output system. Their idea can be developed for the optimal control problems like the one as in [1].

The importance of the off-diagonal elements of the Generalized Sensitivity Matrix(GSM) was realized during the studies of the GSFs. These are called the Off-Diagonal Generalized Sensitivity Functions(OD-GSFs) because of being the off-diagonal elements of the GSM; the diagonal elements of which are called the GSFs. The OD-GSFs primarily give the information of the changes of the parameters estimates due to the changes in the values of the other parameters. We develop the theory of OD-GSFs in the context of the nonlinear least square parameter estimation problem in Section 2. Moreover, we derive the relations connecting the GSFs and OD-GSFs with the parameter estimates and the correlation coefficients and then enlist some important properties of the OD-GSFs in comparison to the GSFs' using the logistic growth population model in this section. In Section 3, we describe the numerical scheme for our problem. Section 4 gives the detail of the major results of the association of the GSFs and OD-GSFs with the parameter estimates. In Section 5, we discuss the correlation amongst different parameters and the condition number of the Fisher Information Matrix (FIM). Concluding remarks are given in Section 6.

2. Generalized and Off-Diagonal Generalized Sensitivity Functions

For a single output system, the output of the model is described [4] as

$$y(t) = f(t, \theta), \quad 0 \leq t \leq T, \tag{2. 1}$$

$\theta = (\theta_1, ..., \theta_p)^{\mathsf{T}}$ is a vector of model parameters belonging to the open set $\mathcal{U} \subset \mathbb{R}^p$, called the feasible or the admissible set of parameters and f is a sufficiently smooth function.

We want to estimate θ in the context of *the non-linear least square inverse problem*. For this purpose, we can take the measurements from the system; however for our own convenience, we take the measurements as the values of the model output for a given value of θ denoted by θ_0 called the nominal or the true parameter plus some noise. The procedure is important with regards to the theoretical validation of the model to any data. We adopt this approach and assume that there exists a unique $\theta_0 \in \mathcal{U}$ such that at the times $0 \leq t_1 < ... < t_M \leq T$, the measurements corresponding to the model outputs $y(t_k), \quad k = 1, ..., M$, are assumed as $\xi_k = f(t_k, \theta_0) + \epsilon_k, \quad k = 1, ..., M$. Here θ_0 is the true or the nominal parameter vector of the system, ϵ_k is the measurement noise assumed to be a representation of some random noise process ε_k for measurement ξ_k which are themselves the realizations of the random measurement process $\Xi_k(t, \theta_0) = f_k(t, \theta_0) + \varepsilon_k$ at t_k. The errors ϵ_k are assumed to satisfy that $\mathbb{E}(\epsilon_k) = 0, k = 1, ..., M$, ϵ_k's are identically and independently distributed(i.i.d) and that Var$(\epsilon_k) = \sigma_k^2$, a constant, $k = 1, ..., M$. This is to be noted that the measurements ξ_k's have also the same distribution as that of ϵ_k's with the same variance $Var(\Xi_k) = \sigma_k^2$, but with different mean $\mathbb{E}(\Xi_k) = f(t_k, \theta_0)$, $k = 1, ..., M$.

We introduce the vector notations; $\xi = (\xi_1, ..., \xi_M)^{\mathsf{T}}$, $F(\theta) = (f(t_1, \theta), ..., f(t_M, \theta))^{\mathsf{T}}$ and $\epsilon = (\epsilon_1, ..., \epsilon_M)^{\mathsf{T}}$ in $\mathbb{R}^M$ for our convenience.

In order to measure the deviations between the model outputs and the measurements, we introduce the following *weighted non-linear least square cost functional*:

$$Q(\xi, \theta) = (\xi - F(\theta))^{\mathsf{T}} \Sigma^{-1} (\xi - F(\theta)),\ \xi \in \mathbb{R}^M\ , \theta \in \mathbb{R}^p, \tag{2. 2}$$

where $\Sigma = \operatorname{diag}\left(\sigma_1^2, ..., \sigma_M^2\right)$. The introduction of Σ in Eqn. (2. 2) signifies that the measurements with large error have less weight than the measurements with less errors in the least-square process.

We assume that we have a *unique local identifiability* [12] in $\mathcal{U}$, i.e., for any nominal parameter vector θ_0 the functional $Q(\xi, \theta)$ has a *unique local minimizer* $\hat{\theta} = \hat{\theta}(\theta_0)$. This leads to

$$\nabla_\theta Q(\xi, \theta)\big|_{\theta = \hat{\theta}(\theta_0)} = 0, \tag{2. 3}$$

$$\nabla^2_{\theta\theta} Q(\xi, \theta)\big|_{\theta = \hat{\theta}(\theta_0)} > 0. \tag{2. 4}$$

In view of our assumption on the unique local identifiability, the estimate $\hat{\theta}(\theta)$ has to satisfy the first order necessary optimal condition for any nominal parameter vector $\theta \in \mathcal{U}$ i.e.,

$$\nabla_\theta Q\left(\xi(\theta), \hat{\theta}(\theta)\right) = 0, \tag{2. 5}$$

Here as per our assumption $\xi = \xi(\theta)$ for any nominal parameter θ, so new mean of any measurement ξ is $f(., \theta)$instead of $f(., \theta_0)$. Differentiating this with respect to ξ and denoting $\hat{\theta}(\theta)$ by ϑ onward, we get

$$\nabla^2_{\theta\xi} Q\left(\xi(\theta), \hat{\theta}(\theta)\right) \frac{\partial \xi}{\partial \theta} + \nabla^2_{\vartheta\vartheta} Q\left(\xi(\theta), \hat{\theta}(\theta)\right) \frac{\partial \hat{\theta}}{\partial \theta} \equiv 0. \tag{2. 6}$$

We get from Eqn.(2. 2)

$$\nabla_\vartheta Q\left(\xi(\theta), \vartheta\right)\big|_{\vartheta = \hat{\theta}(\theta)} = -2\left(\xi(\theta) - F\left(\hat{\theta}(\theta)\right)\right)^{\mathsf{T}} \Sigma^{-1} \nabla_\vartheta F\left(\hat{\theta}(\theta)\right). \tag{2. 7}$$

and

$$\nabla^2_{\xi\vartheta} Q\left(\xi(\theta), \vartheta\right)\big|_{\vartheta = \hat{\theta}(\theta)} = -2\left(\nabla_\vartheta F\left(\hat{\theta}(\theta)\right)\right)^{\mathsf{T}} \Sigma^{-1}. \tag{2. 8}$$

Moreover,

$$\frac{\partial \xi}{\partial \theta} = \frac{\partial}{\partial \theta}\left(F(\theta) + \epsilon\right) = \nabla_\theta F(\theta). \tag{2. 9}$$

We again get from Eqn. (2. 7)

$$\begin{aligned} &\nabla^2_{\vartheta\vartheta} Q\left(\xi(\theta), \vartheta\right)\big|_{\vartheta = \hat{\theta}(\theta)} \\ &= -2\left[\nabla^2_{\vartheta\vartheta} F\left(\hat{\theta}(\theta)\right)^{\mathsf{T}} \Sigma^{-1}\left(\xi(\theta) - F\left(\hat{\theta}(\theta)\right)\right) - \left(\nabla_\vartheta F \hat{\theta}(\theta)\right)^{\mathsf{T}} \Sigma^{-1} \nabla_\vartheta F\left(\hat{\theta}(\theta)\right)\right] \end{aligned} \tag{2. 10}$$

Here we make an important assumption frequently used in the treatment of the non-linear least square parameter estimation that $\nabla^2_{\vartheta\vartheta} F\left(\hat{\theta}(\theta)\right) \approx 0$. This makes the first term on the right side in the above equation zero as non-linear model can be transformed into linear one [9]. So we get,

$$\nabla^2_{\vartheta\vartheta} Q\left(\xi(\theta), \vartheta\right)\big|_{\vartheta=\hat{\theta}(\theta)} \approx 2\left(\nabla_\vartheta F\left(\hat{\theta}(\theta)\right)\right)^{\mathsf{T}} \Sigma^{-1} \nabla_\vartheta F\left(\hat{\theta}(\theta)\right) \tag{2. 11}$$

Using the values from the Equations (2. 8), (2. 9) and (2. 11) in Equation(2. 6), we obtain

$$-\left(\nabla_\vartheta F\left(\hat{\theta}(\theta)\right)\right)^{\mathsf{T}} \Sigma^{-1} \nabla_\theta F(\theta) + \left(\nabla_\vartheta F\left(\hat{\theta}(\theta)\right)\right)^{\mathsf{T}} \Sigma^{-1} \nabla_\vartheta F\left(\hat{\theta}(\theta)\right)\left(\frac{\partial\hat{\theta}}{\partial\theta}\right) \approx 0 \tag{2. 12}$$

Now we assume that parameter estimates are unbiased i.e. $\mathbb{E}(\hat{\Theta}(\theta)) = 0$ and that $\left(\nabla_\vartheta F\left(\hat{\Theta}(\theta)\right)\right)^{\mathsf{T}} \Sigma^{-1} \nabla_\vartheta F\left(\hat{\Theta}(\theta)\right)$ and $\frac{\partial\hat{\Theta}}{\partial\theta}$ are approximately independent. It is to be noted that $\hat{\theta}(\theta)$ is a realization of the random process $\hat{\Theta}(\theta)$. Since

$$\mathbb{E}\left(\nabla_\vartheta F\left(\hat{\Theta}(\theta)\right)\right)^{\mathsf{T}} \Sigma^{-1} \nabla_\theta F(\Theta) \approx \left(\nabla_\vartheta F\left(\hat{\theta}(\theta)\right)\right)^{\mathsf{T}} \Sigma^{-1} \nabla_\theta F(\theta)$$

and as

$$\left(\nabla_\vartheta F\left(\hat{\theta}(\theta)\right)\right)^{\mathsf{T}} \Sigma^{-1} \nabla_\theta F(\theta) = \sum_{k=1}^{M} \frac{1}{\sigma_k^2} \nabla_\theta f(t_k, \theta)^{\mathsf{T}} \nabla_\theta f(t_k, \theta),$$

therefore by taking

$$\mathcal{F}(\theta) = \sum_{k=1}^{M} \frac{1}{\sigma_k^2} \nabla_\theta f(t_k, \theta)^{\mathsf{T}} \nabla_\theta f(t_k, \theta), \tag{2. 13}$$

being invertible and called the Fisher Information Matrix for the parameter estimation problem and taking the expected values of Equation (2) and using the above assumption, we get

$$\mathbb{E}\left(\frac{\partial\hat{\Theta}}{\partial\theta}\right) \approx (\mathcal{F}(\theta))^{-1} \mathcal{F}(\theta) \tag{2. 14}$$

or

$$\mathbb{E}\left(\frac{\partial\hat{\Theta}(\theta)}{\partial\theta}\right) \approx I_{p\times p}. \tag{2. 15}$$

Now if we assume that only measurements up to a sample times t_k vary with θ and the measurements taken at later times are fixed to their values corresponding to the nominal parameter vector θ_0, we obtain the generalized sensitivity matrix(GSM) as

$$\mathcal{G}(t_k, \theta) = \left(\frac{\partial\hat{\theta}(\theta)}{\partial\theta}\right)_k = (\mathcal{F}(\theta))^{-1} \mathcal{F}_k(\theta), k = 1, ..., M. \tag{2. 16}$$

where

$$\mathcal{F}_k(\theta) = \sum_{j=1}^{k} \frac{1}{\sigma_j^2} \nabla_\theta f(t_j, \theta)^\mathsf{T} \nabla_\theta f(t_j, \theta). \tag{2. 17}$$

The generalized sensitivity functions for the parameter component θ_i, $i = 1, ..., p$, is the i-th element in the main diagonal of the matrix $\mathcal{G}$ as function of t_k,

$$g_{\theta_i}(t_k, \theta) = (\mathcal{G}(t_k, \theta))_{i,i}, \quad k = 1, ..., M. \tag{2. 18}$$

The off-diagonal elements (i, j)-th of this matrix $\mathcal{G}$, are defined as the *off-diagonal generalized sensitivity functions* of the parameter component θ_i with respect to the component θ_j [10] as

$$g_{\theta_i|\theta_j}(t_k, \theta) = \left(\mathcal{G}(t_k, \theta)\right)_{i,j}, k = 1, ..., M, i, j = 1, ..., p. \tag{2. 19}$$

2.1. **Parameter Estimates.** We have initially developed the idea of the OD-GSFs in the previous section. Next, we want to explore the relation between the OD-GSFs and the parameter estimates of a parameter, so by using Taylor's series approximation up to second term in the neighbourhood of θ_0, we get the approximation of the parameter estimate $\hat{\theta}(\theta)$

$$\hat{\theta}(\theta) = \hat{\theta}(\theta_0) + \frac{\partial \hat{\theta}(\theta)}{\partial \theta}(\theta - \theta_0)$$

This gives

$$\hat{\theta}(\theta) - \hat{\theta}(\theta_0) = \frac{\partial \hat{\theta}(\theta)}{\partial \theta}(\theta - \theta_0)$$

Replacing $\theta - \theta_0$ by $\Delta\theta$ and dividing both sides by the Euclidean norm-$\|\Delta\theta\|_2$ (however, we can take any norm). Taking limit as $\|\Delta(\theta)\|$ tends to 0 on both sides, we get

$$\lim_{\|\Delta\theta\|_2 \to 0} \frac{\hat{\theta}(\theta) - \hat{\theta}(\theta_0)}{\|\Delta\theta\|_2} = \lim_{\|\Delta\theta\|_2 \to 0} \frac{\partial \hat{\theta}(\theta)}{\partial \theta} \frac{\Delta\theta}{\|\Delta\theta\|_2}$$

or

$$\frac{d}{d\tau}\left(\hat{\theta}(\theta_0; \Delta\theta)\right) = \frac{\partial \hat{\theta}(\theta)}{\partial \theta}\psi \tag{2. 20}$$

where

$$\frac{d}{d\tau}\left(\hat{\theta}(\theta_0; \Delta\theta)\right) = \lim_{\|\Delta\theta\|_2 \to 0} \frac{\hat{\theta}(\theta) - \hat{\theta}(\theta_0)}{\|\Delta\theta\|_2},$$

$\tau = \|\Delta\theta\|_2$ and $\psi = \lim_{\|\Delta\theta\|_2 \to 0} \frac{\Delta\theta}{\|\Delta\theta\|_2}$. Here ψ is of unit magnitude. The term on the left of the above Equation (2. 20) quantifies the changes in the parameter estimate $\hat{\theta}$ due to the changes in the value of the parameter θ, so does the term on the right-side. But this term contains $\frac{\partial \hat{\theta}(\theta)}{\partial \theta}$ - a realization of the random variable $\frac{\partial \hat{\Theta}(\theta)}{\partial \theta}$; the expected value of which is GSM. Since the diagonal elements of this matrix are the GSFs and the off-diagonal elements are OD-GSFs, so any change in the parameter estimates $\hat{\theta}$ will bring a change in the GSFs and OD-GSFs.

2.2. **Correlation Matrix.** In order to explore the relation of the OD-GSFs with the correlation, we again use the Taylor's series expansion for $F(\theta)$ in the neighborhood of θ_0 as follows [10].

$$F(\theta) = F(\theta_0) + \frac{\partial F(\theta_0)}{\partial \theta}(\theta - \theta_0) + (\theta-\theta_0)^{\mathsf{T}}\frac{\partial^2 F(\theta_0)}{\partial \theta^2}(\theta - \theta_0) + \text{h.o.t.}$$

Considering only the first order *linear approximation* for $F(\theta)$, we get

$$F(\theta) \approx F(\theta_0) + \frac{\partial F(\theta_0)}{\partial \theta}(\theta - \theta_0).$$

Putting this approximation for $F(\theta)$ in Equation (2. 2) (We take $Q = Q(\theta)$ as $Q = Q(\xi, \theta)$ and $\xi = \xi(\theta)$ and therefore $Q = Q(\theta)$), we have

$$Q(\theta) \approx \left(\underbrace{\xi - F(\theta_0)}_{\epsilon} - \frac{\partial F(\theta_0)}{\partial \theta}(\theta - \theta_0)\right)^{\mathsf{T}} \Sigma^{-1}\left(\underbrace{\xi - F(\theta_0)}_{\epsilon} - \frac{\partial F(\theta_0)}{\partial \theta}(\theta - \theta_0)\right),$$
$$= \left(\epsilon - \frac{\partial F(\theta_0)}{\partial \theta}(\theta - \theta_0)\right)^{\mathsf{T}} \Sigma^{-1}\left(\epsilon - \frac{\partial F(\theta_0)}{\partial \theta}(\theta - \theta_0)\right).$$

Differentiating with respect to θ, we get

$$\frac{\partial Q(\theta)}{\partial \theta} \approx \frac{\partial}{\partial \theta}\left[\left(\epsilon - \frac{\partial F(\theta_0)}{\partial \theta}(\theta - \theta_0)\right)^{\mathsf{T}} \Sigma^{-1}\left(\epsilon - \frac{\partial F(\theta_0)}{\partial \theta}(\theta - \theta_0)\right)\right].$$

After successive differentiation, we get

$$0 = \frac{\partial Q(\theta)}{\partial \theta}|_{\theta=\hat{\theta}(\theta_0)} \approx -2\left(\frac{\partial F(\theta_0)}{\partial \theta}\right)^{\mathsf{T}} \Sigma^{-1}\left(\epsilon - \frac{\partial F(\theta_0)}{\partial \theta}\left(\hat{\theta}(\theta_0) - \theta_0\right)\right),$$

or

$$\left(\frac{\partial F(\theta_0)}{\partial \theta}\right)^{\mathsf{T}} \Sigma^{-1}\epsilon - \left(\frac{\partial F(\theta_0)}{\partial \theta}\right)^{\mathsf{T}} \Sigma^{-1}\left(\frac{\partial F(\theta_0)}{\partial \theta}\left(\hat{\theta}(\theta_0) - \theta_0\right)\right) \approx 0.$$

or

$$\hat{\theta}(\theta_0) - \theta_0 \approx \left(\left(\frac{\partial F(\theta_0)}{\partial \theta}\right)^{\mathsf{T}} \Sigma^{-1}\frac{\partial F(\theta_0)}{\partial \theta}\right)^{-1}\left(\frac{\partial F(\theta_0)}{\partial \theta}\right)^{\mathsf{T}} \Sigma^{-1}\epsilon, \tag{2. 21}$$

Now since

$$\text{Var}(\hat{\Theta}(\theta_0)) = \mathbb{E}\left(\left(\hat{\Theta}(\theta_0) - \mathbb{E}\left(\hat{\Theta}(\theta_0)\right)\right)\left(\hat{\Theta}(\theta_0) - \mathbb{E}\left(\hat{\Theta}(\theta_0)\right)\right)^{\mathsf{T}}\right), \tag{2. 22}$$

so taking the Equation (2. 21) for the random variable and placing it in Equation (2. 22), we get after making the use of the assumption that the parameter estimates are unbiased, after a little simplification:

$$\text{Var}\left(\hat{\Theta}(\theta_0)\right) \approx \left(\left(\frac{\partial F(\theta_0)}{\partial \theta}\right)^{\mathsf{T}} \Sigma^{-1}\frac{\partial F(\theta_0)}{\partial \theta}\right)^{-1}, \tag{2. 23}$$

or

$$\text{Var}\left(\hat{\Theta}(\theta_0)\right) \approx (\mathcal{F}(\theta_0))^{-1}$$

called the *dispersion* or the *variance-covariance* matrix of the estimates $\hat{\Theta}(\theta_0)$. The right hand side of this matrix is the inverse of the *Fisher Information Matrix* given by Equation (2. 13). From here, it is evident that the GSFs and OD-GSFs give us the information about the correlation between the parameters. This relation is depicted by the oscillations amongst them. The *correlation matrix* is obtained by dividing the $(i, j)th$ element of the Covariance Matrix (2. 23) by the square root of the product of the ith and jth elements of its main diagonal (*standard deviations*) [12]. As Σ is unknown in Equation (2. 23), it is usually estimated, as in [12], by

$$\Sigma \approx \frac{1}{M-p} Q\left(\hat{\theta}(\theta_0)\right). \tag{2. 24}$$

The *Standard Errors of Estimates* (SEEs) of $\hat{\theta}(\theta_0)$ are then given by

$$SEEs\left(\hat{\theta}(\theta_0)\right) = \text{diag}\left(\sqrt{\Sigma}\right) \tag{2. 25}$$

2.3. **Properties of the OD-GSFs in comparison to the GSFs.** In contrast to the properties of the generalized sensitivity functions [13], we here describe the main properties of the OD-GSFs.

(1) The OD-GSFs denoted by $g_{\theta_i|\theta_j}(t_k, \theta)$ of the parameter component θ_i with respect to θ_j, are defined at the time points $t_k, k = 1, ..., M$, $i = 1, ..., p$.
(2) The OD-GSFs are defined for a model to have minimum two parameters.
(3) There is a transition of the OD-GSFs; $g_{\theta_i|\theta_j}(t_k, \theta)$ from 0 to 0 as is shown in the Figure(1) unlike the GSFs which have transition from 0 to 1 as is evident in the Figure(2). So we define $g_{\theta_i|\theta_j}(t_k, \theta) = 0$ for $t < t_1$ and $g_{\theta_i|\theta_j}(t_k, \theta) = 0$ for $t \geq t_M$,
(4) The identification procedure is not only unbiased, but also efficient i.e., the covariance of the estimates $Cov(\hat{\theta})$ equals the $Cramer - Rao(Lower)bound$ [4]:

$$Cov(\hat{\theta}) = \mathcal{J}(\hat{\theta})^{-1} \tag{2. 26}$$

(5) The OG-GSFs are, in general, either convex or concave for $k = 1, ..., M$. In the intervals where the OD-GSFs of θ_i with respect to θ_j viz., $g_{\theta_i|\theta_j}(t_k, \theta)$ have higher slope, the measurements have more information about $\hat{\theta}_i$ with respect to θ_j, though the magnitude of the change is not so large. However, the OD-GSFs first increase/decrease from 0, then attain their maximum/minimum values and then again decrease/increase and at the end they attain value 0. This is called forced to 0 artifact as is visible in Figures (1, 3, 4 and 5). Off-course, the second increase or decrease is due to the correlation amongst the parameters. In the case where the parameters are strongly correlated, there are considerable oscillations in the OD-GSFs. On the other hand, the GSFs when drawn for a single parameter are monotonically increasing as is shown in the first three panels of the Figure(2). These show oscillations due to the correlation when drawn combined for two or

[scale=.38]

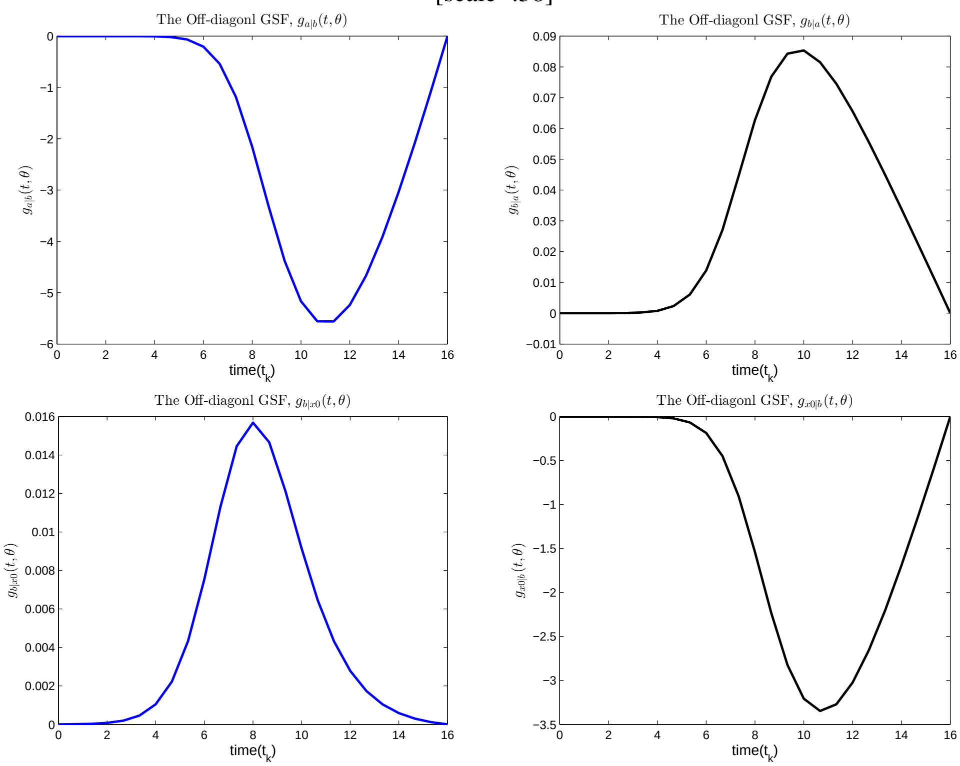

FIGURE 1. Off-diagonal Generalized Sensitivity Functions of a with respect to b (upper-left), b with respect to a (upper-right), b with respect to x_0 (lower-left) and x_0 with respect to b(lower-right) for the Logistic Growth Population Model

more than two parameters as is depicted from the last panel of the Figure(2). However, this figure shows that the information contents given by the measurements for the parameter a are spread over whole the interval $[0, 16]$ i.e., the data taken from this interval contains all the information about a. The data taken from the beginning of the interval to almost time instant 5 contains no information about the parameter b, whereas the data taken from the later part of $[0, 16]$, loosely from $[12, 16]$ contains no information about x_0. The high degree of correlation is reflected by the oscillations amongst them. The GSFs are also bound to force to 1 artifact. See Figures(2, 3, 4 and 5).

3. NUMERICAL IMPLEMENTATION

We explain the above results with the help of the logistic growth population model described by the first order non-linear ordinary differential equation [8]:

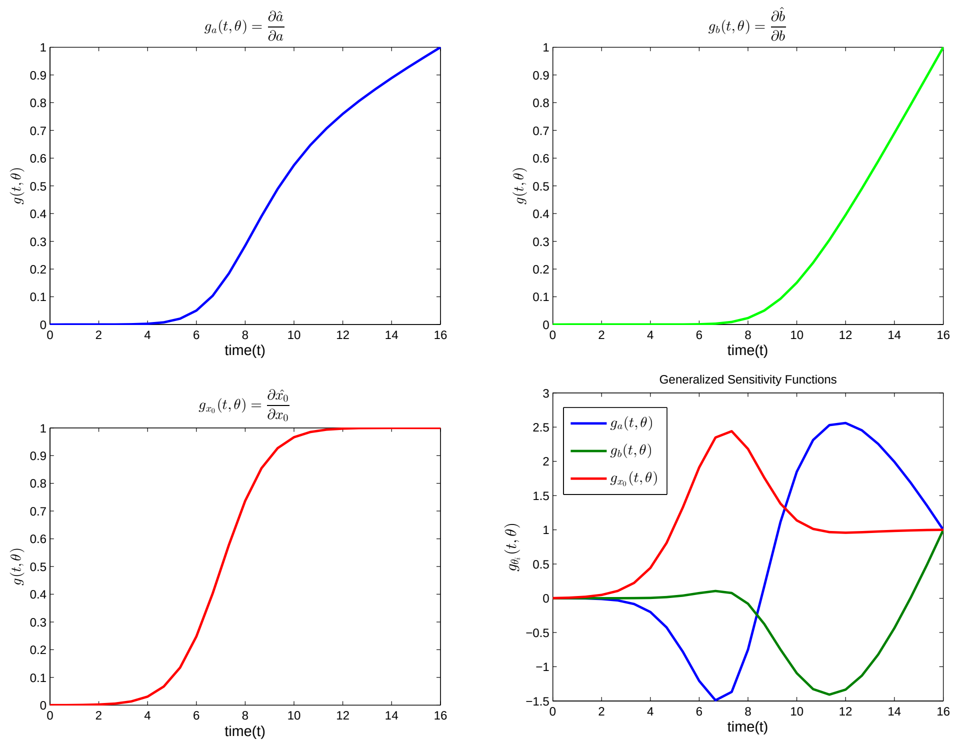

FIGURE 2. Generalized Generalized Sensitivity functions of a (upper-left), b's(upper-right), x_0's(lower-left) and combined for three parameters viz., a,b and x_0(lower-right) for the Logistic Growth Population Model

$$\frac{dx}{dt} = ax - bx^2, \qquad (3.\ 27)$$
$$x(0) = x_0.$$

The parameter vector is $\theta = (a, b, x_0)$. In accordance with our theoretical development of the GSFs and OD-GSFs, we use the *Matlab* solver *ode*45 to solve the Equation (3. 27) with nominal parameter $\theta_0 = [0.7, 0.04, 0.1]$ [2] in the time interval $[0, 16]$ dividing it into $n = 25$ equal sub-intervals. We use the numerical solution as model output $\{y_i, i = 1, ..., n\}$. We created the simulated data set $\{\xi_i, i = 1, ..., n\}$, by adding a noise having normal distribution with *mean* 0 and various *standard deviations* σ to the numerical solution. We use three sets of data for our results; first with a small standard deviations $\sigma = 0.0005$, second with medium standard $\sigma = 0.005$ and the third with a reasonable large standard deviations $\sigma = 0.05$. The optimization routine *solvopt* [5] is used to find the parameter estimates $\hat{\theta}$ from the cost functional $Q(\theta)$ as given in Equation (2. 2) with various *standard deviations*. We find the *standard error of estimates* by Equation(2. 25).

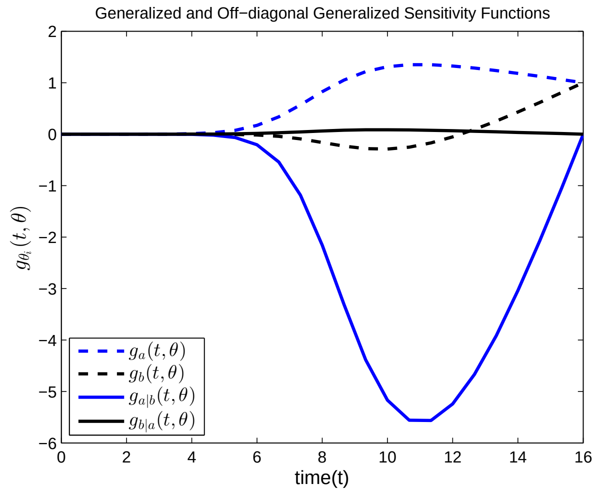

FIGURE 3. Generalized and the off-diagonal generalized sensitivity functions for the selected parameters of the logistic growth population model.

4. SIMULATION RESULTS

4.1. **Generalized versus Off-diagonal Generalized Sensitivity Functions.** We draw the GSFs and the OD-GSFs in comparison for the numerical scheme given above using Equation (2. 18) and Equation (2. 19) respectively by taking two parameters at a time. The comparison also indicates the amplitude of the effect of the changes in true value of one parameter over its estimates and over the estimates of other parameter. The results are shown in Figures(3), (4) and (5). We are primarily interested in exploring the relation between the GSFs and OD-GSFs and the estimates of the parameters. Since the GSFs respectively the OD-GSFs quantify the effects of the changes in the true value of one parameter over its estimates and the estimates of the others, therefore, in order to get some insight in this regard, we estimate the parameters by making changes (perturbation) in the true value θ_{0_i}, $i = 1, 2, 3$, of one of them and keeping the other two fixed; we give perturbations in two different ways- first by changing the initial guess in a fixed proportion and second by changing the initial guess randomly and then observing their effects over the estimates $\hat{\theta}_i(\theta_0)$, $i = 1, 2, 3,$.

4.2. **Perturbing a, keeping b and x_0 fixed.** We change the initial value of the parameter a in two different ways; first by changing the value in a fixed proportion and second randomly. Estimates of parameters, in this case using three data with different standard deviations, are given in Tables(1, 2). When we perturb initial value of a in a fixed proportion, we see that the estimate of b as compared to estimates of a and x_0 are more accurate; see Table(1). Changing a randomly, we see the same trend in the parameters; see Table(2). The standard errors of b too in both the cases are less and more stable. These facts indicates that the changes in estimates of a, b and x_0 are not due to errors. On the other hand, we see that

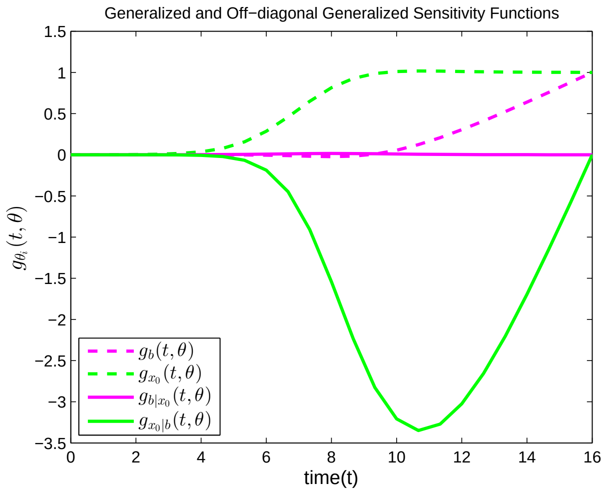

FIGURE 4. Generalized and the off-diagonal generalized sensitivity functions for the selected parameters of the logistic growth population model.

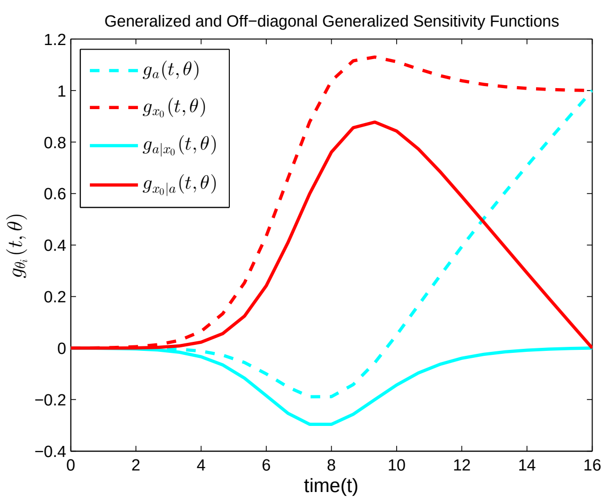

FIGURE 5. Generalized and the off-diagonal generalized sensitivity functions for the selected parameters of the logistic growth population model.

the OD-GSFs of b with respect to a denoted by $g_{b/a}(t_k, \theta_0)$ in Figure(3) have less slope or are more close to a straight line indicating that the estimates of b do not change much with changes in true value of a, whereas the effect of the changes of the true value of a on the estimates of a and x_0 is more enormous. This change is reflected more in the GSFs of a as

θ^0	$\hat{\theta}$	$SEEs$
Panel A: $\sigma = 0.0005$		
(0.84, 0.04, 0.1)	(0.70004060, 0.04000338, 0.09996607)	(6.9689e-5, 4.5643e-6, 4.9612e-5)
(0.77, 0.04, 0.1)	(0.70004060, 0.04000338, 0.09996607)	(6.9682e-5, 4.5638e-6, 4.9607e-5)
(0.7, 0.04, 0.1)	(0.70004061, 0.04000338, 0.09996607)	(6.9678e-5, 4.5636e-6, 4.9604e-5)
(0.63, 0.04, 0.1)	(0.70004062, 0.040003383, 0.09996606)	(6.9687e-5, 4.5642e-6, 4.9610e-5)
(0.56, 0.04, 0.1)	(0.70004063, 0.04000338, 0.09996605)	(6.9681e-5, 4.5638e-6, 4.9606e-5)
Panel B: $\sigma = 0.005$		
(0.84, 0.04, 0.1)	(0.70040631, 0.04003384, 0.09966097)	(3.0203e-4, 1.9784e-5, 2.1437e-4)
(0.77, 0.04, 0.1)	(0.70040621, 0.04003384, 0.09966123)	(3.0199e-4, 1.9782e-5, 2.1435e-4)
(0.7, 0.04, 0.1)	(0.70040669, 0.04003387, 0.09966079)	(3.0202e-4, 1.9783e-5, 2.1437e-4)
(0.63, 0.04, 0.1)	(0.70040678, 0.04003387, 0.09966070)	(3.0202e-4, 1.9784e-5, 2.1437e-4)
(0.56, 0.04, 0.1)	(0.70040675, 0.04003387, 0.09966077)	(3.0201e-4, 1.9783e-5, 2.1436e-4)
Panel C: $\sigma = 0.05$		
(0.84, 0.04, 0.1)	(0.70410159, 0.04034150, 0.09663266)	(3.0156e-3, 1.9783e-4, 2.0772e-3)
(0.77, 0.04, 0.1)	(0.70410226, 0.04034153, 0.09663244)	(3.0156e-3, 1.9783e-4, 2.0772e-3)
(0.7, 0.04, 0.1)	(0.70410640, 0.04034182, 0.09662866)	(3.0156e-3, 1.9784e-4, 2.0771e-3)
(0.63, 0.04, 0.1)	(0.70410683, 0.04034185, 0.09662962)	(3.0156e-3, 1.9783e-4, 2.0771e-3)
(0.56, 0.04, 0.1)	(0.70410057, 0.04034145, 0.09663333)	(3.0156e-3, 1.9783e-4, 2.0772e-3)

TABLE 1. Initial and optimized parameters together with their respective SEEs with noise having normal distribution with zero mean and different standard deviations; changing a in a fixed proportion and keeping other fixed.

in Figure(3) and the OD-GSFs of x with respect to a as in Figure (5). So, we can say that the linear nature of the OD-GSFs of a parameter indicates that its estimates do not change with the changes in true value of the corresponding parameters.

4.3. **Perturbing b, keeping a and x_0 fixed.** We give perturbation to b; first changing its value in a fixed proportion and then randomly. From the three data set with different standard deviations, we find all the three estimates given in Tables(3 and 4). When we perturb initial value of b in a fixed proportion, we see that the estimate of b as compared to estimates of a and x_0 are more accurate i.e., correct to more decimal places. These results are given in Table(3). Changing b randomly, we see the same trend in the estimates of parameters in Table(4). The standard errors of b in both the cases are less and more stable. These facts indicates that the changes in estimates of a, b and x_0 are not dependent on the errors. In this case, the OD-GSFs of a with respect to b denoted by $g_{a/b}(t_k, \theta_0)$ in Figure(3) has more slope than the OD-GSF of x_0 with respect to b denoted by $g_{x_0/b}(t_k, \theta_0)$ shown in Figure(4) indicates that the changes in b more affect the estimates of a than estimates of x_0. However, changes in b least affect the estimates of b.

4.4. **Perturbing x_0, keeping a and b fixed.** Proceeding analogously, we perturb the initial condition x_0 and keep a and b constant; our results are given in Tables(5 and 6). Changing x_0's value in a fixed proportion for three different data sets, estimates along with their standard errors are given in Table(5). We see almost no change in the estimates of b indicating that changes in x_0 do not affect the estimates of b. Changing x_0 randomly, we see the same trend in the parameters estimates in Table(6). The standard errors of b in both the cases

θ^0	$\hat{\theta}$	$SEEs$
Panel A: $\sigma = 0.0005$		
(1.9, 0.04, 0.1)	(0.70004063, 0.04000338, 0.09996604)	(6.9682e-5, 4.5639e-6, 4.9607e-5)
(1.3, 0.04, 0.1)	(0.70004063, 0.04000338, 0.09996605)	(6.9683e-5, 4.5639e-6, 4.9607e-5)
(0.7, 0.04, 0.1)	(0.70004061, 0.04000338, 0.09996607)	(6.9678e-5, 4.5636e-6, 4.9604e-5)
(0.3, 0.04, 0.1)	(0.70004064, 0.04000338, 0.09996604)	(6.9681e-5, 4.5638e-6, 4.9606e-5)
(0.09, 0.04, 1.1)	(0.70004062, 0.04000338, 0.09996606)	(6.9684e-5, 4.5640e-6, 4.9608e-5)
Panel B: $\sigma = 0.005$		
(1.9, 0.04, 0.1)	(0.70040689, 0.04003388, 0.09966068)	(3.0200e-4, 1.9783e-5, 2.1436e-4)
(1.3, 0.04, 0.1)	(0.70040671, 0.04003387, 0.09966075)	(3.0201e-4, 1.9783e-5, 2.1436e-4)
(0.7, 0.04, 0.1)	(0.70040669, 0.04003387, 0.09966079)	(3.0202e-4, 1.9783e-5, 2.1437e-4)
(0.3, 0.04, 0.1)	(0.70040647, 0.04003385, 0.09966090)	(3.0202e-4, 1.9784e-5, 2.1437e-4)
(0.09, 0.04, 1.1)	(0.70040669, 0.04003387, 0.09966086)	(3.0200e-4, 1.9782e-5, 2.1435e-4)
Panel C: $\sigma = 0.05$		
(1.9, 0.04, 0.1)	(0.70410367, 0.04034166, 0.09663123)	(3.0156e-3, 1.9783e-4, 2.0772e-3)
(1.3, 0.04, 0.1)	(0.70410489, 0.04034169, 0.09663010)	(3.0156e-3, 1.9783e-4, 2.0771e-3)
(0.7, 0.04, 0.1)	(0.70410640, 0.04034182, 0.09662866)	(3.0156e-3, 1.9784e-4, 2.0771e-3)
(0.3, 0.04, 0.1)	(0.70409962, 0.04034137, 0.09663355)	(3.0156e-3, 1.9783e-4, 2.0772e-3)
(0.09, 0.04, 1.1)	(0.70410242, 0.04034154, 0.09663164)	(3.0156e-3, 1.9783e-4, 2.0772e-3)

TABLE 2. Initial and optimized parameters together with their respective SEEs with noise having normal distribution with zero mean and different standard deviations; changing a randomly and keeping other fixed.

θ^0	$\hat{\theta}$	$SEEs$
Panel A: $\sigma = 0.0005$		
(0.7, 0.048, 0.1)	(0.70004056, 0.04000337, 0.09996610)	(6.9683e-5, 4.5639e-6, 4.9607e-5)
(0.7, 0.044, 0.1)	(0.70004062, 0.04000338, 0.09996606)	(6.9684e-5, 4.5640e-6, 4.9608e-5)
(0.7, 0.04, 0.1)	(0.70004064, 0.04000338, 0.09996605)	(6.9675e-5, 4.5634e-6, 4.9601e-5)
(0.7, 0.036, 0.1)	(0.70004067, 0.04000338, 0.09996603)	(6.9672e-5, 4.5632e-6, 4.9600e-5)
(0.7, 0.032, 0.1)	(0.70004057, 0.04000338, 0.09996610)	(6.9685e-5, 4.5640e-6, 4.9609e-5)
Panel B: $\sigma = 0.005$		
(0.7, 0.048, 0.1)	(0.70040657, 0.04003386, 0.09966091)	(3.0199e-4, 1.9782e-5, 2.1435e-4)
(0.7, 0.044, 0.1)	(0.70040678, 0.04003387, 0.09966074)	(3.0201e-4, 1.9783e-5, 2.1436e-4)
(0.7, 0.04, 0.1)	(0.70040669, 0.04003387, 0.09966079)	(3.0202e-4, 1.9783e-5, 2.1437e-4)
(0.7, 0.036, 0.1)	(0.70040663, 0.04003386, 0.09966080)	(3.0201e-4, 1.9783e-5, 2.1436e-4)
(0.7, 0.028, 0.1)	(0.70040724, 0.04003391, 0.09966036)	(3.0204e-4, 1.9785e-5, 2.1438e-4)
Panel C: $\sigma = 0.05$		
(0.7, 0.048, 0.1)	(0.70409, 0.040341, 0.096634)	(3.0156e-3, 1.9783e-4, 2.0772e-3)
(0.7, 0.044, 0.1)	(0.70410, 0.040341, 0.096631)	(3.0156e-3, 1.9783e-4, 2.0772e-3)
(0.7, 0.04, 0.1)	(0.70410, 0.040341, 0.096628)	(3.0156e-3, 1.9784e-4, 2.0771e-3)
(0.7, 0.036, 0.1)	(0.70410, 0.040341, 0.096631)	(3.0156e-3, 1.9783e-4, 2.0772e-3)
(0.7, 0.032, 0.1)	(0.70410, 0.040341, 0.096633)	(3.0156e-3, 1.9783e-4, 2.0772e-3)

TABLE 3. Initial and optimized parameters together with their respective SEEs with noise having normal distribution with zero mean and different standard deviations; changing b in a fixed proportion and keeping other fixed.

θ^0	$\hat{\theta}$	$SEEs$
Panel A: $\sigma = 0.0005$		
(0.7, 0.1, 0.1)	(0.70004057, 0.04000338, 0.09996609)	(6.9686e-5, 4.5641e-6, 4.9610e-5)
(0.7, 0.09, 0.1)	(0.70004061, 0.04000338, 0.09996606)	(6.9683e-5, 4.5639e-6, 4.9607e-5)
(0.7, 0.04, 0.1)	(0.70004064, 0.04000338, 0.09996605)	(6.9675e-5, 4.5634e-6, 4.9601e-5)
(0.7, 0.009, 0.1)	(0.70004063, 0.04000338, 0.09996605)	(6.9683e-5, 4.5639e-6, 4.9608e-5)
(0.7, 0.004, 0.1)	(0.70004061, 0.04000338, 0.09996606)	(6.9690e-5, 4.5644e-6, 4.9612e-5)
Panel B: $\sigma = 0.005$		
(0.7, 0.1, 0.1)	(0.70040663, 0.04003386, 0.09966076)	(3.0203e-4, 1.9784e-5, 2.1438e-4)
(0.7, 0.09, 0.1)	(0.70040678, 0.04003387, 0.09966071)	(3.0201e-4, 1.9783e-5, 2.1436e-4)
(0.7, 0.04, 0.1)	(0.70040669, 0.04003387, 0.09966079)	(3.0202e-4, 1.9783e-5, 2.1437e-4)
(0.7, 0.009, 0.1)	(0.70040658, 0.04003386, 0.09966087)	(3.0201e-4, 1.9783e-5, 2.1436e-4)
(0.7, 0.004, 0.1)	(0.70040699, 0.04003389, 0.09966052)	(3.0203e-4, 1.9785e-5, 2.1438e-4)
Panel C: $\sigma = 0.05$		
(0.7, 0.1, 0.1)	(0.704105,0.040341, 0.096629)	(3.0156e-3, 1.9783e-4, 2.0771e-3)
(0.7, 0.09, 0.1)	(0.704106,0.040341, 0.096629)	(3.0156e-3, 1.9783e-4, 2.0771e-3)
(0.7, 0.04, 0.1)	(0.704106, 0.040341, 0.096628)	(3.0156e-3, 1.9784e-4, 2.0771e-3)
(0.7, 0.009, 0.1)	(0.704102, 0.040341, 0.096632)	(3.0156e-3, 1.9783e-4, 2.0772e-3)
(0.7, 0.004, 0.1)	(0.704102, 0.040341, 0.096632)	(3.0156e-3, 1.9783e-4, 2.0772e-3)

TABLE 4. Initial and optimized parameters together with their respective SEEs with noise having normal distribution with zero mean and different standard deviations; changing b randomly and keeping other fixed.

are less and more stable as compared to others. These facts indicate that the changes in estimates of a, b and x_0 are not due to errors.

In this case, the OD-GSF of b with respect to x_0 denoted by $g_{b/x_0}(t_k, \theta_0)$ in Figure(4) is almost straight line signifying that changes in true value of x_0 do not change the parameter estimates of b. That is, x_0 least affects the estimates of b as compared to a's and x_0's which is very common to be found in many other models.

5. DISCUSSION

The importance of the GSFs being the diagonal elements of the GSM given by Equation (2. 16) in relation to the estimates of the parameters has now been fully realized. The off-diagonal elements of this matrix have been defined as the OD-GSFs. An effort has been made to explore the relation of these functions with the parameter estimates first time. However, some important points are needed to be discussed and clarified here which supplement our earlier results.

5.1. **Condition Number of the FIM.** The Condition number of the Fisher Information Matrix given by Eqn.(2. 13) is the measure to know how much the results given by the generalized and the off-diagonal sensitivity functions of the parameters are reliable. Large condition numbers indicate that the matrix is close to singularity or is ill-conditioned, whereas the small and moderate condition numbers indicate that the matrix is well-conditioned and that the results are reliable. Condition number of the FIM namely $\mathcal{F}(\theta_0)$ in the region $[0, 16]$ is 13485 which is considered moderate one in the context of the inverse problem.

θ^0	$\hat{\theta}$	$SEEs$
Panel A: $\sigma = 0.0005$		
(0.7, 0.04, 0.12)	(0.70004062, 0.04000338, 0.09996606)	(6.9679e-5, 4.5637e-6, 4.9605e-5)
(0.7, 0.04, 0.11)	(0.70004064, 0.04000338, 0.09996605)	(6.9685e-5, 4.5641e-6, 4.9609e-5)
(0.7, 0.04, 0.1)	(0.70004061, 0.04000338, 0.09996607)	(6.9678e-5, 4.5636e-6, 4.9604e-5)
(0.7, 0.04, 0.09)	(0.70004064, 0.04000338, 0.09996604)	(6.9680e-5, 4.5637e-6, 4.9605e-5)
(0.7, 0.04, 0.08)	(0.70004062, 0.04000338, 0.09996605)	(6.9686e-5, 4.5641e-6, 4.9610e-5)
Panel B: $\sigma = 0.005$		
(0.7, 0.04, 0.12)	(0.7004067, 0.0400338, 0.0996607)	(3.0202e-4, 1.9784e-5, 2.1437e-4)
(0.7, 0.04, 0.11)	(0.7004067, 0.0400338, 0.0996607)	(3.0202e-4, 1.9784e-5, 2.1437e-4)
(0.7, 0.04, 0.1)	(0.7004066, 0.0400338, 0.0996607)	(3.0202e-4, 1.9783e-5, 2.1437e-4)
(0.7, 0.04, 0.09)	(0.7004065, 0.0400338, 0.0996609)	(3.0200e-4, 1.9783e-5, 2.1436e-4)
(0.7, 0.04, 0.08)	(0.7004069, 0.0400338, 0.0996606)	(3.0200e-4, 1.9783e-5, 2.1436e-4)
Panel C: $\sigma = 0.05$		
(0.7, 0.04, 0.12)	(0.704102, 0.040341, 0.096631)	(0.00301564, 0.00019783, 0.00207721)
(0.7, 0.04, 0.11)	(0.704102, 0.040341, 0.096631)	(0.00301564, 0.00019783, 0.00207721)
(0.7, 0.04, 0.1)	(0.704106, 0.040341, 0.096628)	(0.00301569, 0.00019784, 0.00207717)
(0.7, 0.04, 0.09)	(0.704102, 0.040341, 0.096631)	(0.00301566, 0.00019783, 0.00207721)
(0.7, 0.04, 0.08)	(0.704101, 0.040341, 0.096633)	(0.00301562, 0.00019783, 0.00207722)

TABLE 5. Initial and optimized parameters together with their respective SEEs with noise having normal distribution with zero mean and different standard deviations; changing x_0 in a fixed proportion and keeping other fixed.

θ^0	$\hat{\theta}$	$SEEs$
Panel A: $\sigma = 0.0005$		
(0.7, 0.04, 0.006)	(0.70004059, 0.04000338, 0.09996608)	(6.9692e-5, 4.5645e-6, 4.9614e-5)
(0.7, 0.04, 0.07)	(0.70004064, 0.04000338, 0.09996605)	(6.9675e-5, 4.5634e-6, 4.9601e-5)
(0.7, 0.04, 0.1)	(0.70004061, 0.04000338, 0.09996607)	(6.9678e-5, 4.5636e-6, 4.9604e-5)
(0.7, 0.04, 0.9)	(0.70004062, 0.04000338, 0.09996606)	(6.9685e-5, 4.5641e-6, 4.9609e-5)
(0.7, 0.04, 1.5)	(0.70004065, 0.04000338, 0.09996604)	(6.9680e-5, 4.5637e-6, 4.9606e-5)
Panel B: $\sigma = 0.005$		
(0.7, 0.04, 0.006)	(0.7004068, 0.0400338, 0.9966068)	(3.0200e-4, 1.9783e-5, 2.1436e-4)
(0.7, 0.04, 0.07)	(0.7004067, 0.0400338, 0.0996607)	(3.0202e-4, 1.9784e-5, 2.1437e-4)
(0.7, 0.04, 0.1)	(0.7004066, 0.0400338, 0.0996607)	(3.0202e-4, 1.9783e-5, 2.1437e-4)
(0.7, 0.04, 0.9)	(0.7004064, 0.0400338, 0.0996610)	(3.0200e-4, 1.9782e-5, 2.1436e-4)
(0.7, 0.04, 1.5)	(0.7004065, 0.0400338, 0.0996608)	(3.0202e-4, 1.9784e-5, 2.1437e-4)
Panel C: $\sigma = 0.05$		
(0.7, 0.04, 0.006)	(0.707691, 0.040518, 0.095220)	(4.8420e-3, 3.6395e-4, 3.0837e-3)
(0.7, 0.04, 0.07)	(0.707686, 0.040518, 0.095223)	(4.8419e-3, 3.6395e-4, 3.0838e-3)
(0.7, 0.04, 0.1)	(0.707686, 0.040518, 0.095223)	(4.8419e-3, 3.6395e-4, 3.0838e-3)
(0.7, 0.04, 0.9)	(0.707687, 0.040518, 0.095222)	(4.8419e-3, 3.6395e-4, 3.0838e-3)
(0.7, 0.04, 1.5)	(0.707688, 0.040518, 0.095221)	(4.8420e-3, 3.6395e-4, 3.0838e-3)

TABLE 6. Initial and optimized parameters together with their respective SEEs with noise having normal distribution with zero mean and different standard deviations; changing x_0 randomly and keeping other fixed.

5.2. **Correlation.** Correlations amongst the parameters is reflected by the oscillations in the GSFs and OD-GSFs of the corresponding parameters [14]. The information contents given by the measurements for three parameters are highly correlated in the case of the logistic growth population model as $corr\left(\hat{a},\hat{b}\right) = 0.9977$, $corr\left(\hat{a},\hat{x}_0\right) = 0.9909$ and $corr\left(\hat{b},\hat{x}_0\right) = 0.9977$, which are reflected only by the oscillations in their GSFs and OD-GSFs. However, there can be a good-fit to the data in the presence of high correlation [11].

6. CONCLUSION

The generalized and off-diagonal generalized sensitivity functions have a well-defined relation with the parameter estimates and the correlation coefficients. The OD-GSFs describe the effects of the changes in the true values of one parameter over the estimates of other parameters whereas the GSFs quantify the effects of the changes in true value of the parameters over its estimate. This relation have been exhibited with the help of an example. High correlation amongst the parameters is shown in the oscillations of the GSFs and OD-GSFs.

The numerical results of parameter estimates given in Tables(1) to Table(6) show that the OD-GSFs of a parameter with respect to another parameter are close to straight line when the changes in one parameter do not effect the estimates of other parameters. This change is reflected more when the change in one parameter affects its estimates or other parameters' estimates more. We displayed the GSFs and OD-GSFs by taking two parameters at a time; however we also got the similar results by taking all the three parameters together which are not given here.

Change in one parameter brings enormous change in the other parameters in several dynamical systems; in order to study these systems, the study of the off-diagonal generalized sensitivity functions is as important as that of the generalized sensitivity functions.

We have used the *non-linear Least Square* method to develop the idea of the OD-GSFs, but the *maximum-likelihood estimation* method may be used to develop it. However, both the methods coincide as the assumed distribution of the measurement process is normal.

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