Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 48(2)(2016) pp. 81-90

A New and Reliable Modification of Homotopy Perturbation Method

Mubashir Qayyum*, Hamid Khan, Omar Khan
National University of Computer and Emerging Sciences-FAST, Peshawar, Pakistan
Email: {mubashir.qayyum, hamid.khan, omar.khan}@nu.edu.pk
* Corresponding Author

Received: 04 April, 2016 / Accepted: 21 June, 2016 / Published online 24 August, 2016

Abstract. This manuscript introduces a new alteration to the Homotopy Perturbation Method by coupling it with the Laplace Transform. The corresponding Homotopy Perturbation Laplace Method (HPLM) promises better results in terms of accuracy, efficiency, and easy-of-use when compared to other semi-numerical schemes, and is therefore conveniently poised to be used for various problems in science and engineering. The method is tested against standard fifth and sixth order linear and non-linear ordinary differential equations. For validity, the obtained results are compared with well known analytical and numerical schemes.

AMS (MOS) Subject Classification Codes: 3404; 65H20; 65L10; 35A25
Key Words: Ordinary Differential Equations; Homotopy Perturbation Laplace method; analytical methods.

1. Introduction

Various real-world physical problems can be modeled as ordinary differential equations (ODE). Naturally, their accurate solution is of great interest to the scientific community. For finding the solution, two approaches in the area of numerical analysis can be used; one is the pure numerical, while the other is semi-numerical (analytical). Both approaches are commonly used to solve and study mathematical models accurately in order to have a better understanding of the application at hand. However, they do have their own advantages and disadvantages. In the case of numerical schemes, accuracy can be affected due to discretization and round-off errors. Similarly, they may incur a large print on memory, and may also be computationally expensive. Examples of well-known pure numerical schemes are the family of Runge Kutta, Finite Difference, Wavelet, Finite Element and shooting methods.

On the other hand, analytical methods produce solutions that are differential over the domain of a problem and is thus amenable for mathematical treatments. These methods develop series solutions. In certain cases, the developed series may lead to the closed form solution. In other words, the analytical approaches are restrictive and may be used in special cases [8, 9]. This restriction can removed by the usage of generalized analytic methods [1, 2, 14–16, 19, 21, 22, 36].

This paper pertains to obtain approximate solutions to fifth and sixth order linear and non-linear boundary value problems (BVP). Fifth order BVP's occur frequently in areas such as fluid dynamics [10, 11], while sixth order BVP's are observed in astrophysics [13]. In this context, the ADM has been used for obtaining solutions of fifth & sixth order problems in [33–35]. The same using Variational Iteration Decomposition Method (VIDM), VIM, HPM, and Modified Variational Iteration Method (MVIM) are solved in [25–29, 31]. Similarly, higher order BVPs have also been solved using HAM in [20], non-polynomial spline approach in [17, 18], Iterative Method (ITM) in [24], OHAM and DTM in [3, 4, 6, 30], and Quintic B-Spline Galerkin scheme in [32].

In this manuscript, a novel alteration of HPM is proposed which is then used to approximate the solution of fifth & sixth order linear and non-linear differential equations with boundary conditions. The alteration extends the Homotopy Perturbation with a Laplace transform. The Laplace transform is an integral transform and is known for its utility to solve various problems in science and engineering. In the rest of the manuscript, this new alteration is referred to as the Homotopy Perturbation Laplace Method (HPLM).

It is pertinent to mention that attempts to couple the Laplace transform with HPM have also been attempted before. Homotopy perturbation method for non-linearities distribution along with Laplace transform is given in [12]. Their method is relevant to equations involving non-homogeneous and non-polynomial terms. Another variant known as the Laplace Transform - Homotopy Perturbation Method (LT-HPM) has been reported in [7] which is used to find solutions to stiff systems of ODE's with initial conditions. A Laplace homotopy perturbation method (LHPM) has been used to solve partial differential equations with variable coefficients in [23]. In contrast, HPLM proposed in this manuscript is more efficient as can be seen from the comparison of errors with other analytical and numerical schemes.

In the rest of the manuscript, the proposed approach is presented in section 2, which is then applied to various fifth and sixth order BVPs in section 3 while the conclusion is in section 4.

2. Analysis of HPLM for Differential Equations

To understand the fundamentals of HPLM, it is applied to the following differential equation:

$$\Im\left[w\left(x\right)\right]+\aleph\left[w\left(x\right)\right]-g\left(x\right)\ =0 \tag{2. 1}$$

where x represents an independent variable, $\Im$ is the linear operator, $\aleph$ is the non-linear operator, $w\left(x\right)$ is an unknown, and $g\left(x\right)$ is a known function. According to HPLM, Homotopy can be constructed as $y\left(x,q\right):\mathbb{R}\times\left[0,1\right]\rightarrow\mathbb{R}$ such that it satisfies:

$$\left(1-q\right)\left[\Im\left(y\left(x,q\right)\right)-g\left(x\right)\right]+q\left[\Im\left(y\left(x,q\right)\right)+\aleph\left(y\left(x,q\right)-g\left(x\right)\right)\right]\ =0 \tag{2. 2}$$

where $x\epsilon\mathbb{R}$, $q\epsilon\left[0,1\right]$ and $y\left(x,q\right)$ is an unknown function. Clearly, $y\left(x,0\right)\ =\ y_0\left(x\right)$ for $q=0$ and $y\left(x,1\right)=\tilde{y}\left(x\right)$ for $q=1$.

To obtain an approximate solution, expanding $y\left(x,q\right)$ about q gives:

$$y\left(x,q\right)\ =y_o\left(x\right)+\sum_{k=1}^{m}y_kq^k \tag{2. 3}$$

Substituting (2. 3) into (2. 2) and equating the coefficients of like powers of q, different order problems can be obtained. The zero-th order problem is:

$$\Im\left[y_0\left(x\right)\right]-g\left(x\right)\ =0 \tag{2. 4}$$

Using differential property of Laplace transform to (2. 4) results in:

$$s^n Ly_0(x) - s^{n-1}y_0(\alpha) - s^{n-2}y_0'(\alpha) - \ldots - y_0^{n-1}(\alpha) - L\{g(x)\} = 0 \quad (2.\ 5)$$

Applying the inverse Laplace Transform gives:

$$y_o(x) = L^{-1}\left\{\frac{1}{s^n}\left[s^{n-1}y_0(\alpha) + s^{n-2}y_0'(\alpha) + \ldots + y_0^{n-1}(\alpha) + L\{g(x)\}\right]\right\} \quad (2.\ 6)$$

The general k^{th} order problem is:

$$\Im[y_k(x)] - \aleph_{k-1}[y_0, y_1, \ldots, y_{k-1}] = 0 \quad k = 1, 2, 3, \ldots \quad (2.\ 7)$$

Application of the Laplace transform to (2. 7) gives:

$$\begin{aligned} s^n Ly_k(x) - &s^{n-1}y_k(\alpha) - s^{n-2}y_k'(\alpha) - \ldots \\ &\ldots - y_k^{n-1}(\alpha) - L\{\aleph_{k-1}[y_0, y_1, \ldots, y_{k-1}]\} = 0 \end{aligned} \quad (2.\ 8)$$

and then, the application of the inverse Laplace Transform on both sides gives:

$$y_k(x) = L^{-1}\left\{\frac{1}{s^n}\left[\begin{aligned} &s^{n-1}y_k(\alpha) + s^{n-2}y_k'(\alpha)\ldots \\ &\qquad \ldots + y_k^{n-1}(\alpha) + L\{\aleph_{k-1}[y_0, y_1, \ldots, y_{k-1}]\}\end{aligned}\right]\right\} \quad (2.\ 9)$$

Let the initial approximation be of the form $y_k(\alpha) = a_0$, $y_k'(\alpha) = a_1$, and likewise until $y_k^{n-1}(\alpha) = a_{n-1}$, the approximate solution is:

$$\tilde{y} = \lim_{q \to 1} y = y_0 + y_1 + y_2 + \ldots \quad (2.\ 10)$$

Substituting (2. 10) in (2. 2), the expression for residual is:

$$R(x) = \Im[\tilde{y}(x)] + \aleph[\tilde{y}(x)] - g(x) \quad (2.\ 11)$$

If $R = 0$, then $\tilde{y}$ is the exact solution. However, this usually does not happen in most of the problems. The method described in this section minimizes the limitations of the ordinary perturbation techniques.

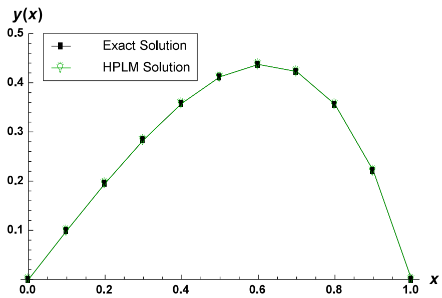

FIGURE 1. Comparison of solutions in case of fifth order linear BVP

3. NUMERICAL ILLUSTRATIONS WITH HPLM

In this section, four standard examples are presented to illustrate both HPLM as well as its ability to work with BVPs. For each example, the second order approximation is:

$$\tilde{w}(x) = w_0(x) + w_1(x) + w_2(x) \quad (3.\ 12)$$

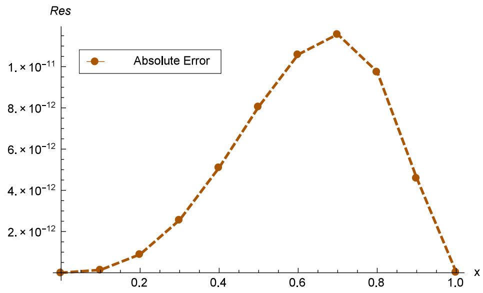

FIGURE 2. Plot of HPLM residual error in case of fifth order linear BVP

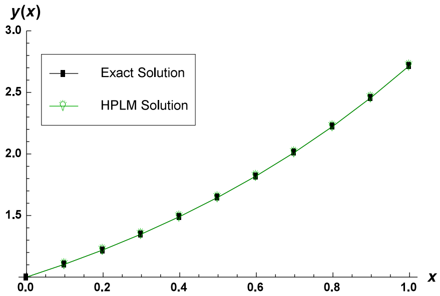

FIGURE 3. Comparison of solutions in case of fifth order non-linear BVP

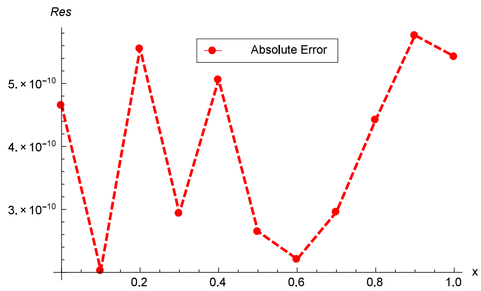

FIGURE 4. Plot of HPLM residual error in case of fifth order non-linear BVP

3.1. **Example 1: Fifth order linear BVP.**

$$\begin{array}{c} w^{v}\left(x\right)=w-15\exp(x)-10x\exp(x), \qquad x\epsilon[0,1] \\ w\left(0\right)=0, w'\left(0\right)=1, w''\left(0\right)=0, w\left(1\right)=0, w'\left(1\right)=-e \end{array} \tag{3. 13}$$

having exact solution $w(x) = \left(x - x^2\right)\exp(x)$. Using HPLM, we obtain $A = -2.9999$ and $B = -8.0000$ in the given domain. Therefore, the second order solution is:

$$\begin{aligned} \tilde{w}\left(x\right)= \quad & -120+120e^{x}-99x-20xe^{x}-40x^{2}-10.5x3-2.x^{4}-7x^{5}/24 \\ & -x^{6}/30-x^{7}/336-0.000198413x^{8}-8.2672\times10^{-6}x^{9} \\ & +x^{11}/39916800-4.81771\times10^{-10}x^{13}+O\left(x^{14}\right) \end{aligned} \tag{3. 14}$$

The numerical results for the solution (3. 14) using HPLM are shown in Table 1.

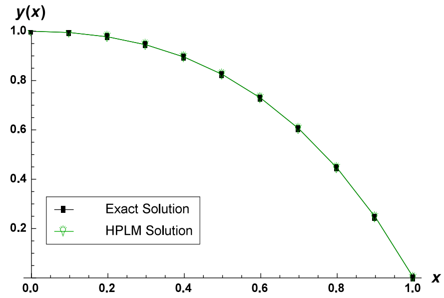

FIGURE 5. Comparison of solutions in case of sixth order linear BVP

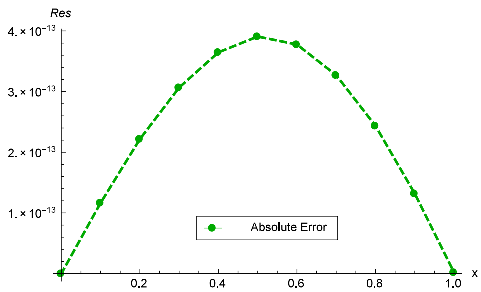

FIGURE 6. Plot of HPLM residual error in case of sixth order linear BVP

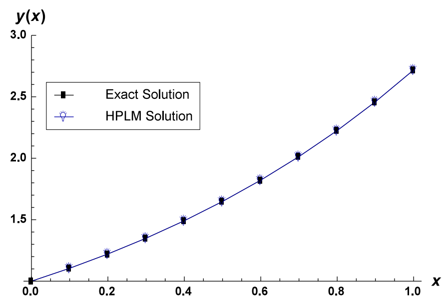

FIGURE 7. Comparison of solutions in case of sixth order non-linear BVP

3.2. **Example 2: Fifth order non-linear BVP.**

$$\begin{array}{c} w^{v}(x) = w^{2}(x)\exp(-x), \qquad x\epsilon[0,1] \\ w(0) = 1, w'(0) = 1, w''(0) = 1, w(1) = e, w'(1) = e \end{array} \tag{3. 15}$$

having exact solution $w(x) = \exp(x)$. In this case, $A = 0.9999$ and $B = 1.0000$ in the given domain using HPLM. Knowing these values, the approximate second order solution

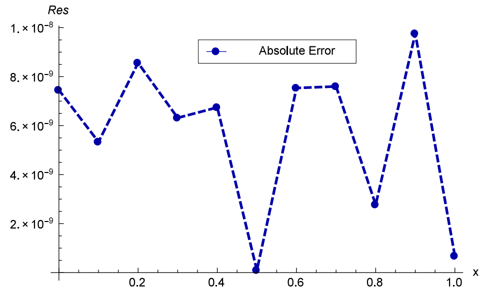

FIGURE 8. Plot of HPLM residual error in case of sixth order non-linear BVP

x	Exact Solution	HPLM Solution	E* HPLM	E* HPM [25]	E* OHAM [5]	E* VIMHP [29]	E* ADM [34]	E* B-Spline [34]
0.0	0.	0.	0.	0.	0.	0.	0.	0.
0.1	0.099465	0.0994654	-1E-13	-3E-11	-9E-11	-3E-11	-3E-11	-8E-03
0.2	0.195424	0.195424	-8E-13	-2E-10	-4E-10	-2E-10	-2E-10	-1E-03
0.3	0.28347	0.28347	-2E-12	-4E-10	-5E-10	-4E-10	-4E-10	-5E-03
0.4	0.358038	0.358038	-5E-12	-8E-10	-2E-11	-8E-10	-8E-10	3E-03
0.5	0.41218	0.41218	-8E-12	-1E-09	1E-09	-1E-09	-1E-09	8E-03
0.6	0.437309	0.437309	-1E-11	-2E-09	2E-09	-2E-09	-2E-09	6E-03
0.7	0.422888	0.422888	-1E-11	-2E-09	2E-09	-2E-09	-2E-09	-0.000
0.8	0.356087	0.356087	-9E-12	-1E-09	1E-09	-2E-09	-1E-09	9E-03
0.9	0.221364	0.221364	-4E-12	-1E-09	4E-10	-1E-09	-1E-09	-9E-03
1.0	0.	2.93E-14	-2E-14	0.	0.	0.	0.	0.

E* = Exact - Approx.

TABLE 1. Comparison of second order error of HPLM with various schemes for fifth order linear BVP (Section 3. 13).

is:

$$\begin{aligned}\tilde{w}\left(x\right) = \quad & 2.51072\times 10^{6} + 989285.e^{-2x} - 3.5\times 10^{6} e^{-x} - 741435.x \\ & +1.11051\times 10^{6} x e^{-x} - 1.8905110^{6} x e^{-x} + 88660.5x^{2} \\ & +590663.e^{-2x}x^{2} - 577381.e^{-x}x^{2} - 5135.82x^{3} + 196535.e^{-2x}x^{3} \\ & -127504.e^{-x}x^{3} + 123.469x^{4} + 45492.7e^{-2x}x^{4} - 20783.3e^{-x}x^{4} \\ & +7712.51e^{-2x}x^{5} - 2432.e^{-x}x^{5} + 983.339e^{-2x}x^{6} - 200.486e^{-x}x^{6} \\ & +95.3254e^{-2x}x^{6} - 12.75e^{-x}x^{7} + 7.00659e^{-2x}x^{8} \\ & -0.873264e^{-x}x^{8} + 0.38298e^{-2x}x^{9} + 0.01487e^{-2x}x^{10} \\ & +0.000370732e^{-2x}x^{11} + 4.52112\times 10^{-6} e^{-2x}x^{12}\end{aligned} \tag{3. 16}$$

The numerical results for the solution (3. 16) using HPLM are shown in Table 2.

3.3. **Example 3: Sixth order linear BVP.**

$$\begin{gathered} w^{vi}\left(x\right) = w\left(x\right) - 6e^{x} \qquad x\epsilon[0,1] \\ w\left(0\right) = 1, w''\left(0\right) = -1, w^{vi}\left(0\right) = -3, w\left(1\right) = 0, w''\left(1\right) = -2e, w^{vi}\left(1\right) = -4e \end{gathered} \tag{3. 17}$$

x	Exact Solution	HPLM Solution	E* HPLM	E* HPM [25]	E* OHAM [5]	E* VIMHP [29]	E* ADM [34]	E* B-Spline [34]
0.0	1.	1.	4E-10	0.	0.0000	0.	0.	0.
0.1	1.105170918	1.10517	2E-10	1E-09	1E-10	1E-09	1E-09	-7E-04
0.2	1.221402758	1.2214	5E-10	2E-09	1E-09	2E-09	2E-09	-7E-04
0.3	1.349858808	1.34986	2E-10	1E-08	3E-09	1E-08	1E-08	4E-04
0.4	1.491824698	1.49182	5E-10	2E-08	6E-09	2E-08	2E-08	4E-04
0.5	1.648721271	1.64872	2E-10	3E-08	9E-09	3E-08	3E-08	4E-04
0.6	1.822118800	1.82212	2E-10	3E-08	1E-08	3E-08	3E-08	4E-04
0.7	2.013752707	2.01375	2E-10	4E-08	1E-08	4E-08	4E-08	3E-04
0.8	2.225540928	2.22554	4E-10	3E-08	8E-09	3E-08	3E-08	3E-04
0.9	2.459603111	2.4596	5E-10	1E-08	1E-09	1E-08	1E-08	1E-04
1.0	2.718281828	2.71828	5E-10	0.	0.	0.	0.	0.

E* = Exact - Approx.

TABLE 2. Comparison of second order error of HPLM with various schemes for fifth order non-linear BVP (Section 3. 15).

having exact solution $w(x) = \exp(x) - x\exp(x)$. Considering HPLM, the second order approximation can be obtained by determining $A = -1.1692075593864748 \times 10^{-12}$, $B = -1.9999999999898874$, and $C = -4.000000000062179$:

$$\begin{aligned}\tilde{w}(x) = \; & 13 - 12e^{x} + 12.x + 11x^{2}/2 + 1.66667x^{3} + 3x^{4}/8 + 0.06666x^{5} \\ & +7x^{6}/720 + 0.00119x^{7} + x^{8}/8064 + 1.1022 \times 10^{-5}x^{9} + x^{10}/1209600 \\ & +5.01042 \times 10^{-8}x^{11} + x^{12}/479001600 - 1.87764 \times 10^{-22}x^{13} + O\left(x^{14}\right)\end{aligned} \tag{3. 18}$$

The numerical results for the solution (3. 18) using HPLM are mentioned in Table 3.

x	Exact Solution	HPLM Solution	E* HPLM	E* HPM [28]	E* OHAM [5]	E* VIM [27]	E* ADM [35]
0.0	1.	1.	0.	0.	0.	0.	0.
0.1	0.994654	0.994654	1.1E-13	-4.0E-4	2.0E-8	-4.0E-4	-4.0E-4
0.2	0. 977122	0. 977122	2.2E-13	-7.7E-4	4.0E-8	-7.7E-4	-7.7E-4
0.3	0.944901	0.944901	3.0E-13	-1.0E-3	5.6E-8	-1.0E-3	-1.0E-3
0.4	0.895095	0.895095	3.6E-13	-1.2E-3	6.9E-8	-1.2E-3	-1.2E-3
0.5	0.824361	0.824361	3.9E-13	-1.3E-3	7.5E-8	-1.3E-3	-1.3E-3
0.6	0.728848	0.728848	3.7E-13	-1.2E-3	7.4E-8	-1.2E-3	-1.2E-3
0.7	0.604126	0.604126	3.2E-13	-1.0E-3	6.5E-8	-1.0E-3	-1.0E-3
0.8	0.445108	0.445108	2.4E-13	-4.0E-4	4.8E-8	-4.0E-4	-4.0E-4
0.9	0.24596	0.24596	1.3E-13	-7.7E-4	2.5E-8	-7.7E-4	-7.7E-4
1.0	0.	1.97E-15	-1.9E-15	0.	-2.0E-9	0.	0.

E* = Exact - Approx.

TABLE 3. Comparison of second order error of HPLM with various schemes for sixth order linear BVP (Section 3. 17).

3.4. Example 4: Sixth order non-linear BVP.

$$\begin{array}{c} w^{vi}(x) = \exp(-x)w^2(x) \qquad x\epsilon[0,1] \\ w(0) = 1, w''(0) = 1, w''''(0) = 1, \\ w(1) = e, w''(1) = e, w''''(1) = e \end{array} \tag{3. 19}$$

having exact solution $w(x) = \exp(x)$. Considering HPLM, the second order solution is obtained by determining $A = 1.000000014$, $B = 0.999999965$ and $C = 1.000000066$ as:

$$\begin{aligned} \tilde{w}(x) = \; & -1.09849 \times 10^8 + 4.27823 \times 10^7 e^{-2x} + 6.70663 \times 10^7 e^{-x} \\ & +3.21542 \times 10^7 x + 4.90861 \times 10^7 x e^{-2x} + 7.13906 \times 10^7 x e^{-x} \\ & -3.99368 \times 10^6 x^2 + 2.70287 \times 10^7 e^{-2x} x^2 + 2.743 \times 10^7 e^{-x} x^2 \\ & +264389 x^3 + 9.46789 \times 10^6 e^{-2x} x^3 + 6.10844 \times 10^6 e^{-x} x^3 \\ & -9389.45 x^4 + 2.35759 \times 10^6 e^{-2x} x^4 + 954196. e^{-x} x^4 + 144.449 x^5 \\ & +441776. e^{-2x} x^5 + 116230. e^{-x} x^5 + 64348.6 e^{-2x} x^6 + 11553.1 e^{-x} x^6 \\ & +7425.02 e^{-2x} x^7 + 918.445 e^{-x} x^7 + 685.517 e^{-2x} x^8 + 54.3684 e^{-x} x^8 \\ & +50.7766 e^{-2x} x^9 + 2.37847 e^{-x} x^9 + 3.00286 e^{-2x} x^{10} + 0.128264 e^{-x} x^{10} \\ & +0.13782 e^{-2x} x^{11} + 0.00497907 e^{-2x} x^{12} + 0.000128717 e^{-2x} x^{13} \\ & +2.17014 \times 10^{-6} e^{-2x} x^{14} + 1.80845 \times 10^{-8} e^{-2x} x^{15} \end{aligned} \tag{3. 20}$$

The numerical results for the solution (3. 20) using HPLM are shown in Table 4.

It can be clearly observed from Tables and Figures that HPLM provides more accuracy than any other mentioned analytical and numerical schemes.

x	Exact Solution	HPLM Solution	E* HPLM	E* HPM [28]	E* OHAM [5]	E* VIM [27]	E* ADM [35]
0.0	1.	1.	7.4E-9	0.	0.	0.	0.
0.1	1.10517	1.10517	5.3E-9	-1.2E-4	-9.4E-9	-1.2E-4	-1.2E-4
0.2	1.2214	1.2214	8.5E-9	-2.3E-4	-1.8E-8	-2.3E-4	-2.3E-4
0.3	1.34986	1.34986	-6.3E-9	-3.2E-4	-2.4E-8	-3.2E-4	-3.2E-4
0.4	1.49182	1.49182	6.7E-9	-3.8E-4	-2.9E-8	-3.8E-4	-3.8E-4
0.5	1.64872	1.64872	-9.5E-11	-4.0E-4	-3.0E-8	-4.0E-4	-4.0E-4
0.6	1.82212	1.82212	-7.5E-9	-3.9E-4	-2.8E-8	-3.9E-4	-3.9E-4
0.7	2.01375	2.01375	7.5E-9	-3.3E-4	-2.4E-8	-3.3E-4	-3.3E-4
0.8	2.22554	2.22554	2.7E-9	-2.4E-4	-1.7E-8	-2.4E-4	-2.4E-4
0.9	2.4596	2.4596	9.7E-9	-1.2E-4	-9.1E-9	-1.2E-4	-1.2E-4
1.0	2.71828	2.71828	6.7E-10	2.0E-9	2.5E-10	2.0E-9	2.0E-9

E* = Exact - Approx.

TABLE 4. Comparison of second order error of HPLM with various schemes for sixth order non-linear BVP (Section 3. 19).

4. CONCLUSION

This paper introduces a new scheme which provides more accurate results than the classical HPM, OHAM, VIMHP, ADM, and B-Spline and is applicable to a wide class of functional equations. We have successfully applied this approach to linear and non-linear BVPs and compared the results with various analytical and numerical schemes. These comparisons indicate that HPLM is more accurate than other mentioned analytical and numerical methods, hence, it will be more appealing for scientists and researchers to apply and extend its application to more complex problems arising in science and engineering.

REFERENCES

[1] G. Adomian, *Solving Frontier Problems in Physics: The Decomposition Method, vol 60 of Fundamental Theories of Physics*, 1994.

[2] G. Adomian, *A review of the decomposition method and some recent results for nonlinear equations,* Computer Math. Appl. **21**, No. 5 (1991) 101-127.

[3] J. Ali, *One Dimensional Differential Transform Method for Some Higher Order Boundary Value Problems in Finite Domain,* Int. J. Contemp. Math. Sciences **7**, No.5 (2012) 263-272.

[4] J. Ali, S. Islam, I. Ali and H. Khan, *Application of Optimal Homotopy Asymptotic Method to higher order boundary value problems,* Abstract and Applied Analysis (Article ID 401217), 2012.

[5] J. Ali, S. Islam, S. ul-Islam and G. Zaman, *The solution of multipoint boundary value problems by the Optimal Homotopy Asymptotic Method*, Computer Math.Appl. CAMWA-D-09-00585RI, 2009.

[6] J. Ali, S. Islam, H. Khan and G. Zaman, *Solution of a Parameterized Sixth-Order Boundary Value Problem by the Optimal Homotopy Asymptotic Method*, Proceedings of the Romanian Academy, Series A. **12**, No. 3 (2011) 167-172.

[7] J. Ali, S. Islam, S. I. A. Shah and H. Khan, *The Optimal Homotopy Asymptotic Method for the Solution of Fifth and Sixth Order Boundary Value Problems*, World Applied Sciences Journal **15**, No. 8 (2011) 1120-1126.

[8] H. Aminikhah, *The combined Laplace transform and new homotopy perturbation method for stiff systems of ODEs,* Appl. Math. Model **36**, (2012) 3638-3644.

[9] H. Aminikhah and M. Hemmatnezhad, *An effective modification of the homotopy perturbation method for stiff systems of ordinary differential equations,* Applied Mathematics Letters **24**, No. 9 (2011) 1502-1508.

[10] J. C. Butcher, *Numerical Methods for Ordinary Differential Equations,* Wiley, New York 2003.

[11] H. N. Caglar and S. H. Caglar, *The numerical solution of fifth-order boundary value problems with sixth-degree B-spline functions*, Applied Mathematics Letters **12**, (1999).

[12] S. Chandrasekhar, *Hydrodynamic hydromegnatic stability,* Clarendon Press, Oxford 1961.

[13] U. Filobello-Nino, H. Vazquez-Leal, B. Benhammouda, L. Hernandez-Martinez, C. Hoyos-Reyes, J. A. A. Perez-Sesma, V. M. Jimenez-Fernandez, D. Pereyra-Diaz, A. Marin-Hernandez, A. Diaz-Sanchez, J. Huerta-Chua and J. Cervantes-Perez, *Non-linear distribution Laplace transform-homotopy perturbation method,* SpringPlus **3**, No. 594 (2014).

[14] G. A. Glatzmaier, *Numerical simulations of stellar convection dynamics at the base of the convection zone*, Geophysical Fluid Dynamics **31**, No. 1-2 (1985) 137-150.

[15] J. H. He, *problems perturbation technique*, Computer Methods in Applied Mechanics and Engineering **170**, No. 3-4 (1999) 257262.

[16] J. H. He, *Variational approach to the sixth-order boundary value problems,* Appl. Math. Comput. **143**, No. 2-3 (2003) 537538.

[17] J. H. He, *A note on the homotopy perturbation method*, Geophysical Fluid Dynamics **14**, No. 2 (2010) 565-568.

[18] S. Islam and M. A. Khan, *A numerical method based on nonpolynomial sextic spline function for solution of special fifth order boundary value problems,* Appl. Math. Comput. **181**, (2006) 356361.

[19] S. Islam, I. A. Tirmizi, F. Haq and M. A. Khan, *Non-polynomial splines approach to the solution of sixthorder boundary-value problems,* Appl. Math. Comput. **195**, (2008) 270284.

[20] S. Islam, S. ul Haq and J. Ali, *Numerical solution of special 12th order boundary value problems using differential transform method*, Comm. Nonl. Sc. Nu. Sim. **14**, No. 4 (2009) 11321138.

[21] S. Liang and D. J. Jeffrey, *Approximate solutions to a parametrized sixth order boundary value problem*, Comput. Math. Appl. **59**, No. 1 (2010) 247253.

[22] S. Liao, *Beyond perturbation: Introduction to the homotopy analysis method*, Chapman and Hall/CRC, 2004.

[23] S. Liao, *Nmethod the homotopy analysis method: Some definitions and theorems,* Communications in Nonlinear Science and Numerical Simulation **14**, No. 4 (2008) 983997.

[24] M. Madani, M. Fathizadeh, Y. Khan Y and Ahmet, *On the coupling of the homotopy perturbation method and laplace transformation*, Mathematical and Computer Modeling **53**, (2011) 19371945.

[25] S. T. Mohyud-Din, A. Yildirim and M. M. Hosseini, *An iterative algorithm for fifth-order boundary value problems*, World Applied Sciences Journal **8**, No. 5 (2010) 531535.

[26] M. A. Noor and S. T. Mohyud-Din, *An efficient algorithm for solving fifth-order boundary value problems*, Mathematical and Computer Modeling **45**, (2007) 954964.

[27] M. A. Noor and S. T. Mohyud-Din, *A reliable approach for solving linear and nonlinear sixth-order boundary value problems*, International Journal of Computational and Applied Mathematics **2**, No. 2 (2007) 163172.

[28] M. A. Noor and S. T. Mohyud-Din, *Variational iteration method for solving sixth order boundary value problems*, 2007.

[29] M. A. Noor and S. T. Mohyud-Din, *Homotopy perturbation method for solving sixth-order boundary value problems*, Computer Math. Appl. **55**, (2008) 29532972.

[30] M. A. Noor and S. T. Mohyud-Din, *Variational iteration method for fifth-order boundary value problems using hes polynomials*, Mathematical Problems in Engineering 2008.

[31] M. Qayyum, H. Khan, M. T. Rahim and I. Ullah, *Analysis of unsteady axisymmetric squeezing fluid flow with slip and no-slip boundaries using OHAM*, Mathematical Problems in Engineering **2015**.

[32] M. Qayyum, H. Khan, M. T. Rahim and I. Ullah, *Modeling and analysis of unsteady axisymmetric squeezing fluid flow through porous medium channel with slip boundary*, PloS one **10**, No. 3 (2015).

[33] K. N. S. K. Viswanadham and P. M. Krishna, *Quintic b-spline galerkin method for fifth order boundary value problems*, ARPN Journal of Engineering and Applied Sciences **5**, No. 2 (2010).

[34] A. M. Wazwaz, *Approximate solutions to boundary value problems of higher-order by the modified decomposition method*, Computer Math, Appl. **40**, (2000) 679691.

[35] A. M. Wazwaz, *The numerical solution of fifth-order boundary value problems by decomposition method*, J. Comp. and Appl. Math. **136**, (2001) 259270.

[36] A. M. Wazwaz, *The numerical solution of sixth-order boundary value problems by the modified decomposition method*, Appl. Math. Comput. **118**, (2001) 311325.

[37] J. K. Zhou, *Differential transformation and its application for electrical circuits*, Huazhong University Press, Wuhan, China (in Chinese), 1986.